CONSTRUCTIVE THEORY OF FUNCTIONS, Varna 2002
(B. Bojanov, Ed.), DARBA, Sofia, 2003, pp. 295-301.

Commutativity of Durrmeyer-Type Modifications of Meyer-König and Zeller and Baskakov-Operators

Margareta Heilmann

In this paper we prove the commutativity of Durrmeyer-type modifications of the Meyer-König and Zeller and Baskakov operators, respectively, by using a connection between these operators and methods for the proof which we already used in [8] and [9], respectively.

1. Introduction and Definition of the Operators

A nice and useful property of Durrmeyer-type modifications of classical positive linear operators is their commutativity. For the Bernstein-Durrmeyer operators this was shown by Ditzian and Ivanov [6] using the representation of the operators in terms of their eigenfunctions which were derived in [5]. In [8] and [9], respectively, the author proved the commutativity for Durrmeyer modifications of Baskakov-type operators with different methods. It was an open problem to show the desired property also for Durrmeyer-type modifications of the Meyer-König and Zeller operators defined in different ways by Chen [4] and Abel and Gupta [1]. Some nice attempts were made by Ellend [7]. We now present a definition of Durrmeyer-type modifications of the Meyer-König and Zeller operators which generalizes different variants in the literature.

Definition 1. *Let $l \in \mathbb{N}_0$. For $f \in L_1[0,1]$, $n \in \mathbb{N}$, we define the MKZD-operators (Durrmeyer-type modifications of the Meyer-König and Zeller operators) by*

$$(M_{n,l}f)(x) = \sum_{k=0}^{\infty} m_{n,k}(x)(n+l)\int_0^1 m_{n+l,k}(t)(1-t)^{-2}f(t)\,dt\,, \qquad x \in [0,1)\,,$$

where the weights are given by $m_{n,k}(x) = \binom{n+k}{k}x^k(1-x)^{n+1}$.

We want to point out that for $l = 2$ the definition coincides with the operators introduced by Chen [4], and for $l = 1$ with the operators considered by Abel and Gupta [1]. Important for the proof of our commutativity result is the connection of the MKZD-operators to Durrmeyer-type modifications of the Baskakov-operator which also depend on the paprameter $l \in \mathbb{N}_0$. They are defined as follows.

Definition 2. *Let $l \in \mathbb{N}_0$. For $f \in L_p[0,\infty)$, $1 \le p \le \infty$, $n \in \mathbb{N}$, let*

$$(B_{n,l}f)(x) = \sum_{k=0}^{\infty} b_{n,k}(x)(n-1+l)\int_0^{\infty} b_{n+l,k}(t)f(t)\,dt\,, \qquad x \in [0,\infty)\,,$$

where the weights are given by $b_{n,k}(x) = \binom{n-1+k}{k}x^k(1+x)^{-(n+k)}$.

Remark 1. The operators given in Definitions 1 and 2 are connected in the following way (for the corresponding result for the classical Meyer-König and Zeller and the Baskakov-operators see [11, p. 226]). Let $\sigma : [0,1) \longrightarrow \mathbb{R}_0^+$, $\sigma(x) = \frac{x}{1-x}$, $g(y) = f(\sigma^{-1}(y))$, $\sigma^{-1}(y) = \frac{y}{1+y}$. Then it can be verified that

$$(M_{n,l}f)(x) = (B_{n+1,l}g)(\sigma(x))\,. \tag{1}$$

2. Commutativity Results

We first show the commutativity result for the Durrmeyer-type modifications of the Baskakov operators. The proof follows the lines of [8, §4] and [9], respectively.

Theorem 1. *Let $f \in L_p[0,\infty)$, $1 \le p \le \infty$. Then*

$$B_{n+1,l}(B_{m+1,\lambda}f) = B_{m+1,\lambda}(B_{n+1,l}f) \quad \Longleftrightarrow \quad l = \lambda\,.$$

Proof. We first point out the main steps of our proof.

Step 1. For $x \in [0,\infty)$ we show the integral representations

$$B_{n+1,l}(B_{m+1,\lambda}f)(x) \;=\; \int_0^{\infty} f(y)G_{n,l;m,\lambda}(x,y)\,dy \tag{2}$$

with the kernel

$$G_{n,l;m,\lambda}(x,y) = \frac{(n+l)(m+\lambda)}{(n+l)!\,m!}(n+m+l)! \times \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} b_{n+1,k}(x)b_{m+1+\lambda,j}(y)\binom{k+j}{j}\frac{(n+l+k)!\,(m+j)!}{(n+m+l+1+k+j)!}$$

and

$$B_{m+1,\lambda}(B_{n+1,l}f)(x) \;=\; \int_0^{\infty} f(y)G_{m,\lambda;n,l}(x,y)\,dy \tag{3}$$

with the kernel

$$G_{m,\lambda;n,l}(x,y) = \frac{(n+l)(m+\lambda)}{n!\,(m+\lambda)!}(n+m+\lambda)! \times \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} b_{m+1,k}(x) b_{n+1+l,j}(y) \binom{k+j}{j} \frac{(m+\lambda+k)!\,(n+j)!}{(n+m+\lambda+1+k+j)!}\,.$$

Step 2. We consider the kernels as functions of two complex variables and prove that they are holomorphic for $(x,y) \in G \times G$, $G = \{z \in \mathbb{C} : \mathrm{Re}\,(z) > -\frac{1}{2}\}$.

Step 3. By considering Taylor expansions of the kernels we show that they are equal in an open neighbourhood of $(0,0)$.

Step 4. We finish our proof by using the identity theorem for analytic functions from which we derive equality of the kernels for all $x, y \in [0, \infty)$.

Proof of Step 1. Interchanging the order of integration and summation, using a corollary of Lebesgue's dominated convergence theorem [10, Corollary 12.33], we derive

$$B_{n+1,l}(B_{m+1,\lambda} f)(x) = \int_0^\infty (n+l)(m+\lambda) \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} b_{n+1,k}(x) b_{m+1+\lambda,j}(y) \int_0^\infty b_{n+1+l,k}(t) b_{m+1,j}(t)\,dt\,dy\,. \tag{4}$$

From the definition of $b_{n+1+l,k}$ and $b_{m+1,j}$ we get

$$b_{n+1+l,k}(t) b_{m+1,j}(t) = \binom{k+j}{j} \frac{(n+l+k)!}{(n+l)!} \times \frac{(m+j)!}{m!} \cdot \frac{(n+m+l+1)!}{(n+m+l+1+k+j)!}\, b_{n+m+l+2,k+j}(t)$$

and

$$\int_0^\infty b_{n+m+l+2,k+j}(t)\,dt = \frac{1}{n+m+l+1}\,.$$

Putting this in (4) we derive the integral representation (2). The analogous result (3) follows in the same way.

Proof of Step 2. We now consider the kernels as functions of two complex variables $(x,y) \in G \times G$, where $G = \{z \in \mathbb{C} : \mathrm{Re(z)} > -\frac{1}{2}\}$.

Using Stirling's formula (see [2, (3.9]) we get

$$
\begin{aligned}
&(m+\lambda)\binom{k+j}{j}\frac{\Gamma(n+l+k+1)\Gamma(m+j+1)}{\Gamma(n+m+l+k+j+2)}\cdot\frac{\Gamma(n+m+l+1)}{\Gamma(n+l)\Gamma(m+1)}\\
&\quad\le (m+\lambda)\mathrm{e}^3\underbrace{\frac{\binom{k+j}{j}(n+l+k+1)^k(m+j+1)^j}{(n+m+l+k+j+2)^{k+j}}}_{\le 1}\\
&\qquad\times\underbrace{\left\{\frac{(n+l+k+1)(m+j+1)(n+m+l+1)}{(n+m+l+k+j+2)(n+l)(m+1)}\right\}^{-1/2}}_{\le 1}\\
&\qquad\times\underbrace{\frac{(n+l+k+1)^{n+l+1}(m+j+1)^{m+1}}{(n+m+l+k+j+2)^{n+m+l+2}}}_{\le 1}\cdot\frac{(n+m+l+1)^{n+m+l+1}}{(n+l)^{n+l}(m+1)^{m+1}}\,.
\end{aligned}
$$

Hence, for $(x,y)\in G\times G$, by using the generalized binomial formula,

$$
\begin{aligned}
&|G_{n,l;m,\lambda}(x,y)|\le(m+\lambda)\mathrm{e}^3\frac{(n+m+l+1)^{n+m+l+1}}{(n+l)^{n+l}(m+1)^{m+1}}\\
&\times\underbrace{\sum_{k=0}^{\infty}\binom{n+k}{k}|x|^k|1+x|^{-(n+1+k)}}_{=(|1+x|-|x|)^{-(n+1)}}\underbrace{\sum_{j=0}^{\infty}\binom{m+\lambda+j}{j}|y|^k|1+y|^{-(m+\lambda+1+j)}}_{=(|1+y|-|y|)^{-(m+\lambda+1)}}\,.
\end{aligned}
$$

Thus $G_{n,l;m,\lambda}$ is absolutely uniformly convergent in a closed neighbourhood of every $(x,y)\in G\times G$. As $b_{n+1,k}$ and $b_{m+1,\lambda}$ are holomorphic in G we get from the theorem of convergence by Weierstraß and Hartog's theorem (see [3]) that $G_{n,l;m,\lambda}$ is holomorphic in $G\times G$. The same holds true for $G_{m,\lambda;n,l}$. Thus, at every point $(x,y)\in G\times G$ the kernels can be represented by their power series being convergent in a certain neighbourhood of (x,y).

Proof of Step 3. Now, let again x,y be real. The Taylor expansions of the kernels are convergent in a certain neighbourhood $U(0,0)$ of the point $(0,0)$ and coincide with the functions there. Thus, for $(x,y)\in U(0,0)$, we have

$$
G_{n,l;m,\lambda}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{1}{r!\,s!}x^ry^s\left\{\frac{\partial^{r+s}}{\partial x^r\partial y^s}G_{n,l;m,\lambda}(x,y)\right\}(0,0)
$$

and

$$
G_{m,\lambda;n,l}(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{1}{r!\,s!}x^ry^s\left\{\frac{\partial^{r+s}}{\partial x^r\partial y^s}G_{m,\lambda;n,l}(x,y)\right\}(0,0)\,,
$$

respectively. So we have to show that the partial derivatives are equal. Since

$$\left\{\frac{\partial^r}{\partial x^r} b_{n+1,k}(x)\right\}(0) = \begin{cases} (-1)^{r+k}\binom{r}{k}\frac{(n+r)!}{n!}, & k \le r \\ 0, & k > r \end{cases},$$

$$\left\{\frac{\partial^s}{\partial y^s} b_{m+1+\lambda,j}(y)\right\}(0) = \begin{cases} (-1)^{s+j}\binom{s}{j}\frac{(m+\lambda+s)!}{(m+\lambda)!}, & j \le s \\ 0, & j > s \end{cases},$$

we get together with the definition of $G_{n,l;m,\lambda}$ that

$$\begin{aligned} &\left\{\frac{\partial^{r+s}}{\partial x^r \partial y^s} G_{n,l;m,\lambda}(x,y)\right\}(0,0) \\ &\quad = \frac{(n+r)!}{(n+l-1)!\,n!} \cdot \frac{(m+\lambda+s)!}{(m+\lambda-1)!\,m!}(n+m+l)!(-1)^{r+s} \\ &\quad\quad \times \sum_{k=0}^{r}\binom{r}{k}(-1)^k \sum_{j=0}^{s}\binom{s}{j}(-1)^j\binom{k+j}{j}\frac{(n+l+k)!\,(m+j)!}{(n+m+l+1+k+j)!}\,. \end{aligned} \tag{5}$$

With the definition of the β-function we have

$$\frac{(n+l+k)!\,(m+j)!}{(n+m+l+1+k+j)!} = \beta(n+l+k+1, m+j+1) = \int_0^1 x^{n+l+k}(1-x)^{m+j}\,dx\,.$$

So we get

$$\begin{aligned} &\sum_{k=0}^{r}\binom{r}{k}(-1)^k \sum_{j=0}^{s}\binom{s}{j}(-1)^j\binom{k+j}{j}\frac{(n+l+k)!\,(m+j)!}{(n+m+l+1+k+j)!} \\ &\quad = \int_0^1 x^{n+l}(1-x)^m \sum_{k=0}^{r}\binom{r}{k}(-1)^k\frac{1}{k!}x^k \sum_{j=0}^{s}\binom{s}{j}(-1)^j\frac{(k+j)!}{j!}(1-x)^j\,dx\,. \end{aligned} \tag{6}$$

From [8, (4.8), p. 53–54] we use the identity

$$\sum_{k=0}^{r}\binom{r}{k}(-1)^k\frac{1}{k!}x^k \sum_{j=0}^{s}\binom{s}{j}(-1)^j\frac{(k+j)!}{j!}(1-x)^j = \binom{r+s}{s}x^s(1-x)^r\,.$$

Putting this in (6) gives

$$\begin{aligned} &\sum_{k=0}^{r}\binom{r}{k}(-1)^k \sum_{j=0}^{s}\binom{s}{j}(-1)^j\binom{k+j}{j}\frac{(n+l+k)!\,(m+j)!}{(n+m+l+1+k+j)!} \\ &\quad = \binom{r+s}{s}\int_0^1 x^{n+l+s}(1-x)^{m+r}\,dx = \binom{r+s}{s}\frac{(n+l+s)!\,(m+r)!}{(n+m+r+s+l+1)!}\,. \end{aligned}$$

So, together with (5), we have proved that

$$\begin{aligned} &\left\{\frac{\partial^{r+s}}{\partial x^r \partial y^s} G_{n,l;m,\lambda}(x,y)\right\}(0,0) \\ &\quad = \frac{(n+r)!\,(n+l+s)!\,(m+r)!\,(m+\lambda+s)!\,(n+m+l)!}{(n+l-1)!\,n!\,(m+\lambda-1)!\,m!\,(n+m+r+s+l+1)!}(-1)^{r+s}\binom{r+s}{s}\,. \end{aligned}$$

In the same way as above we get

$$\left\{\frac{\partial^{r+s}}{\partial x^r \partial y^s} G_{m,\lambda;n,l}(x,y)\right\}(0,0)$$
$$= \frac{(n+r)!\,(n+l+s)!\,(m+r)!\,(m+\lambda+s)!\,(n+m+\lambda)!}{(n+l-1)!\,n!\,(m+\lambda-1)!\,m!\,(n+m+r+s+\lambda+1)!}(-1)^{r+s}\binom{r+s}{s}.$$

Thus the kernels are equal in $U(0,0)$ if and only if $l=\lambda$.

Proof of Step 4. Since $G_{n,l;m,\lambda}(x,y) = G_{m,\lambda;n,l}(x,y)$ in $U(0,0)$ for $l=\lambda$ and the kernels are analytic for all $x,y \in (-1/2,\infty)$, we can conclude the equality of the kernels for all $x,y \in [0,\infty)$ if and only if $l=\lambda$ by the identity theorem for analytic functions (see [3]). Together with the integral representations proved in the first step we get the commutativity of the operators if and only if $l=\lambda$.

For $1 \le p < \infty$ let

$$L_{p,w}[0,1] = \left\{ f \,:\, \Big(\int_0^1 |f(t)|^p (1-t)^{-2}\,dt\Big)^{1/p} < \infty \right\}.$$

Then, using the notation in Remark 1, we have that $f \in L_{p,w}[0,1]$ implies $g \in L_p[0,\infty)$, $1 \le p < \infty$, and $f \in L_\infty[0,1]$ implies $g \in L_\infty[0,\infty)$. So together with equation (1) we get from Theorem 1:

Corollary 1. *Let $f \in L_{p,w}[0,1]$, $1 \le p < \infty$, or $f \in L_\infty[0,1]$. Then*

$$M_{n,l}(M_{m,\lambda}f) = M_{m,\lambda}(M_{n,l}f) \quad \Longleftrightarrow \quad l=\lambda\,.$$

References

[1] U. Abel and V. Gupta, The rate of convergence by a new type of Meyer-König and Zeller operators, *Fasc. Math.*, to appear.

[2] E. Artin, "The Gamma Function", Holt, Rinehart and Winston, 1964.

[3] H. Cartan, "Elementare Theorie der analytischen Funktionen einer oder mehrerer komplexen Veränderlichen", BI-Hochschultaschenbücher, 1966.

[4] W. Chen, On the inegral type Meyer-König and Zeller operators, *Approx. Theory Appl.* **2**, 3 (1986), 7–18.

[5] M. M. Derriennic, Sur l'approximation des fonctions intégrables sur $[0,1]$ par des polynômes de Berstein modifiés, *J. Approx. Theory* **31** (1981), 325–343.

[6] Z. Ditzian and K. Ivanov, Bernstein-type operators and their derivatives, *J. Approx. Theory* **56**, 1 (1989), 72–90.

[7] M. Ellend, "Das Approximationsverfahren der Meyer-König und Zeller-Operatoren vom Durrmeyer-Typ", Diplomarbeit, Universität Dortmund, 1994.

[8] M. Heilmann, "Approximation auf $[0, \infty)$ durch das Verfahren der Operatoren vom Baskakov-Durrmeyer Typ", Dissertation, Universität Dortmund, 1987.

[9] M. Heilmann, Commutativity of operators from Baskakov-Durrmeyer type, *in* "Constructive Theory of Functions '87" (Bl. Sendov, P. Petrushev, K. Ivanov, and R. Maleev, Eds.), pp. 197–206, Publishing House of BAS, Sofia, 1988.

[10] E. Hewitt and K. Stromberg, "Real and Abstract Analysis", Springer-Verlag, 1969.

[11] V. Totik, Uniform approximation by Baskakov and Meyer-König and Zeller operators, *Period. Math. Hungar.* **14**, 3-4 (1983), 209–228.

Margareta Heilmann

Department of Mathematics
University of Wuppertal
Gaußstraße 20
D-42097 Wuppertal
GERMANY
E-mail: `heilmann@math.uni-wuppertal.de`