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Jackson Order of Approximation by Riesz Means for Freud Weights

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1. Introduction

Using the well-known Jackson theorem, proved by Ditzian and Lubinsky [2], we give an explicit form of the order of approximation by Riesz means of a function in some weighted Lipschitz classes.

We have to note that such a theorem appears at first in Joó's paper [4] in 1991 on the Riesz means with parameter $\frac{1}{2}$, and when the function itself is in the Lipschitz class $Lip\,(\alpha, p)_w$. Lemma 1 appeared also in that paper at first. We give an elementary proof of it. Using the theory of selfadjoint differential operators, statements 5.8 and 6.6 of Ditzian [1] give that lemma in general case, but this technique does not work for Freud weights.

We will use the following notations.

Definition 1. *$w(x) = e^{-Q(x)}$ is a Freud weight, if $Q : \mathbb{R} \longrightarrow \mathbb{R}$ is an even continuous function, Q'' is continuous and $Q' > 0$ in $(0, \infty)$. Furthermore, for some $1 < A < B$,*

$$A \leq \frac{\big(xQ'(x)\big)'}{Q'(x)} \leq B, \qquad x \in (0, \infty).$$

Definition 2. *The Riesz means of a function f are*

$$R_n^\nu(f, x) = \sum_{k=0}^{n} \left(1 - \frac{k^\nu}{(n+1)^\nu}\right) a_k\, p_k(w^2, x),$$

where $\nu > 0$, $p_k(w^2, x)$ are the orthonormal polynomials with respect to w^2, a_k are the Fourier coefficients of f, if they exist.

The de la Vallée Poussin means of a function f are defined by

$$\vartheta_n^\nu(f, x) = \frac{1}{2^\nu - 1} \sum_{k=n+1}^{2n+1} \frac{(k+1)^\nu - k^\nu}{(n+1)^\nu}\, S_k(f, x),$$

where $S_k(f, x)$ are the Fourier partial sums of f, if they exist.

We shall use the so-called Mhaskar-Rahmanov-Saff number, which is the solution of the equation

$$u = \frac{2}{\pi} \int_0^1 a_u t \, Q'(a_u t)(1-t^2)^{-1/2} \, dt,$$

and the abbreviation

$$a'_u := \frac{\partial a_u}{\partial u}.$$

The modulus of smoothness we use is defined by

$$\omega_{r,p}(f,w,t) := \sup_{0<h\le t} \|w\Delta_h^r(f,x,\mathbb{R})\|_{L_p(I_h)} + \inf_{p\in P_{r-1}} \|(f-p)w\|_{L_p(\mathbb{R}\setminus I_t)},$$

where I_h is an interval depending on h, and $\Delta_h^r(f,x,\mathbb{R})$ is the r-th symmetric difference of f.

The weighted best approximation of a function f is denoted by

$$E_n(f)_{w,p} := \inf_{p\in P_n} \|(f-p)w\|_{L_p(\mathbb{R})}.$$

2. The Result

Theorem 1. *Let $w = e^{-Q}$ be a Freud weight, and let us assume further that $(x^2Q'')' = O(xQ'')$. If $1 \le p \le \infty$, $\alpha < \min\{r, \frac{AB}{B-A}\}$, $\nu > \alpha(1-\frac{1}{B})$, and $[\nu \ne \alpha+1, \alpha+2;$ if $\alpha \ge 1$, then $\nu < \alpha+2]$, or instead of $[\dots]$ we can assume that $[(x^2Q'')' \sim xQ'']$. With these assumptions we have*

$$\|(R_n^\nu(f) - f)w\|_{L_p(\mathbb{R})} = O\left(\left(\frac{a_n}{n}\right)^\alpha\right)$$

if and only if

$$\omega_{r,p}(f,w,t) = O(t^\alpha).$$

Lemma 1. *Let $w = e^{-Q}$ be a Freud weight, $fw \in L_p(\mathbb{R})$, $1 \le p \le \infty$, and let $\nu > 0$ be arbitrary. Then*

$$\|(R_n^\nu(f) - f)w\|_{L_p(\mathbb{R})} \le \frac{C}{(n+1)^\nu} \sum_{k=0}^{n} (k+1)^{\nu-1} E_k(f)_{w,p}.$$

Proof. For a $\nu > 0$ we have

$$(n+1)^\nu R_n^\nu(f) = S_0 + (2^\nu - 1) \sum_{k=0}^{m-1} 2^{k\nu} \vartheta_{2^k-1}^\nu(f) + \sum_{k=2^m}^{n} ((k+1)^\nu - k^\nu) S_k(f),$$

that is,

$$R_n^\nu(f)-f=\frac{1}{(n+1)^\nu}\Bigg\{S_0-f+(2^\nu-1)\sum_{k=0}^{m-1}2^{k\nu}(\vartheta_{2^k-1}^\nu(f)-f)$$
$$+\big((n+1)^\nu-2^{m\nu}\big)\left(\sum_{k=2^m}^{n}\frac{(k+1)^\nu-k^\nu}{(n+1)^\nu-2^{m\nu}}S_k-f\right)\Bigg\}.$$

We use here the de la Vallée Poussin means because they have the reconstructing property, that is, $\vartheta_n^\nu(p_k)=p_k$ if $p_k\in P_k$ is a polynomial of degree k, $k\le n$, and the operator $T_n: L_{p,w}(\mathbb{R})\longrightarrow L_{p,w}(\mathbb{R})$; $f\to\vartheta_n^\nu(f)$, is bounded. That is

$$\|\vartheta_n^\nu(f)w\|_{L_p(\mathbb{R})}\le C\|fw\|_{L_p(\mathbb{R})},$$

where C does not depend on n and f. The last inequality follows immediately from Freud's theorem on strong $(C,1)$ means [3]. These two properties together yield

$$\|(\vartheta_n^\nu(f)-f)w\|_{L_p(\mathbb{R})}\le CE_n(f)_{w,p}.$$

Similarly, the reminder term

$$M_{2^m}^n:=\sum_{k=2^m}^{n}\frac{(k+1)^\nu-k^\nu}{(n+1)^\nu-2^{m\nu}}S_k$$

also has the reconstructing property, and if we choose m such that

$$c\,2^{m\nu}<(n+1)^\nu-2^{m\nu}$$

for an arbitrary but fixed c, then we get as before that

$$\|(M_{2^m}^n(f)-f)w\|_{L_p(\mathbb{R})}\le CE_{2^m}(f)_{w,p}.$$

Using further the elementary fact that

$$2^{k\nu}E_{2^k-1}(f)_{p,w}\le C\sum_{j=2^{k-1}}^{2^k-1}(j+1)^{\nu-1}E_j(f)_{p,w},$$

and

$$\big((n+1)^\nu-2^{m\nu}\big)E_{2^m}(f)_{p,w}\le C2^{m\nu}E_{2^m}(f)_{p,w},$$

we get the statement of the lemma.

Lemma 2 ([2, Corollary 1.6]).

$$E_k(f)_{w,p}=O\left(\left(\frac{a_k}{k}\right)^\alpha\right)\qquad \textit{if and only if}\quad \omega_{r,p}(f,w,t)=O\left(t^\alpha\right),$$

where $\alpha<r$.

Lemma 3 ([5, Lemma 5.2]).

$$\frac{1}{B} \le \frac{ua'_u}{a_u} \le \frac{1}{A}, \qquad u \in (0,\infty),$$
$$u^{1/B} \le \frac{a_u}{a_1} \le u^{1/A}, \qquad u \in [1,\infty).$$

Lemma 4. *If $w = e^{-Q}$ is a Freud weight with $(x^2Q'')' = O(xQ'')$, then*

$$|a''_u| = O\left(\frac{a_u}{u^2}\right).$$

Proof. After twice differentiation of the defining equality of a_u we obtain

$$-a''_u \int_0^1 \frac{t}{\sqrt{1-t^2}}\big(Q'(a_u t) + a_u t Q''(a_u t)\big)dt$$
$$= \int_0^1 \frac{t^2(a'_u)^2}{\sqrt{1-t^2}}\big(2Q''(a_u t) + a_u t Q'''(a_u t)\big)dt,$$

and thus

$$-\frac{a''_u}{a'_u}\, u = \frac{2}{\pi}\, u \int_0^1 \frac{t^2(a'_u)^2}{\sqrt{1-t^2}}\big(2Q''(a_u t) + a_u t Q'''(a_u t)\big)dt.$$

Hence,

$$\left|\frac{a''_u}{a'_u}\, u\right| \le C\frac{1}{u}\cdot\frac{u^2(a'_u)^2}{a_u^2}\int_0^1 \frac{a_u t}{\sqrt{1-t^2}}\big(a_u t Q''(a_u t)\big)dt \le C,$$

and the last inequality follows from the previous lemma and the definition of a_u.

Lemma 5. *Under the assumptions of Theorem 1 we have*

$$\frac{1}{(n+1)^\nu}\sum_{k=0}^{n}(k+1)^{\nu-1}\left(\frac{a_{k+1}}{k+1}\right)^\alpha = O\left(\left(\frac{a_n}{n}\right)^\alpha\right).$$

We have to note that if $Q(x) = x^\gamma$, then $a_n = n^{1/\gamma}$, and the statement is trivial, but if e.g. $a_u = \left(\frac{u^b}{\ln u}\right)^{1/\alpha}$, $\alpha > 0$, and $\nu = \alpha - b$, then a_u is monotone increasing, $\frac{a_u}{u} \sim a'_u$ ($a_u < u^{1/B}, B > 1$ does not hold), but $\int_1^n a_u^\alpha u^{\nu-\alpha-1}du \ne O(a_n^\alpha n^{\nu-\alpha})$.

Proof. Let

$$F(u) = u^{\nu-\alpha-1}a_u^\alpha.$$

If we can decompose $F(u) = F_1(u) + F_2(u)$ so that $f_1(u) := F'_1(u)$ is monotone increasing and tends to infinity, and $f_2(u) := F'_2(u)$ is monotone and tends to zero, then we could use Euler's summation formula which yields that

$$\frac{1}{n^\nu}\sum_{k=1}^{n+1} k^{\nu-1}\left(\frac{a_k}{k}\right)^\alpha = \frac{1}{n^\nu}\left\{\frac{F(1)+F(n+1)}{2} + \int_1^{n+1} F(u)\,du \right.$$
$$\left. + O(f_1(n+1)) + O(f_2(n+1))\right\}. \quad (1)$$

For giving such a decomposition we first compute the second derivative of $F(u)$:

$$\begin{aligned} F''(u) &= (\nu-\alpha-1)(\nu-\alpha-2)u^{\nu-\alpha-3}a_u^\alpha + \alpha(\alpha-1)u^{\nu-\alpha-1}a_u^{\alpha-2}(a'_u)^2 \\ &\quad - \alpha u^{\nu-\alpha-1}a_u^{\alpha-1}a''_{u,-} + 2\alpha(\nu-\alpha-1)u^{\nu-\alpha-2}a_u^{\alpha-1}a'_u \\ &\quad + \alpha u^{\nu-\alpha-1}a_u^{\alpha-1}a''_{u,+} \\ &= g_1(u)+g_2(u)+g_3(u)+g_4(u)+g_5(u). \end{aligned}$$

Now we have to distinguish some cases. In addition to the assumptions of the theorem let:

Case 1. If $\nu < \alpha+1,\ \alpha \le 1$, then

$$\begin{aligned} f'_1(u) &= g_1(u)+g_5(u), \\ f'_2(u) &= g_2(u)+g_3(u)+g_4(u). \end{aligned}$$

Case 2. If $\alpha+1 \le \nu < \alpha+2,\ \alpha \le 1$, then

$$\begin{aligned} f'_1(u) &= g_4(u)+g_5(u), \\ f'_2(u) &= g_1(u)+g_2(u)+g_3(u). \end{aligned}$$

Case 3. If $\nu \ge \alpha+2,\ \alpha \le 1$, then

$$\begin{aligned} f'_1(u) &= g_1(u)+g_4(u)+g_5(u), \\ f'_2(u) &= g_2(u)+g_3(u). \end{aligned}$$

Case 4. If $\nu < \alpha+1,\ \alpha > 1$, then

$$\begin{aligned} f'_1(u) &= g_1(u)+g_2(u)+g_5(u), \\ f'_2(u) &= g_3(u)+g_4(u). \end{aligned}$$

Case 5. If $\alpha+1 \le \nu < \alpha+2,\ \alpha > 1$, then

$$\begin{aligned} f'_1(u) &= g_2(u)+g_4(u)+g_5(u), \\ f'_2(u) &= g_1(u)+g_3(u). \end{aligned}$$

Case 6. If $\nu \ge \alpha+2,\ \alpha > 1$, then

$$\begin{aligned} f'_1(u) &= g_1(u)+g_2(u)+g_4(u)+g_5(u), \\ f'_2(u) &= g_3(u). \end{aligned}$$

(The last case could occur only when $(x^2Q''(x))' \sim xQ''(x)$.) Since the functions f_1 and f_2 are monotone, adding a constant we can achieve that they tend to zero or infinity. Now we have to estimate the right-hand side of (1).

$$\frac{1}{n^\nu} \cdot \frac{F(1)+F(n+1)}{2} = \frac{1}{n^\nu}\left(a_1^\alpha + a_{n+1}^\alpha (n+1)^{\nu-\alpha-1}\right) = o\left(\left(\frac{a_n}{n}\right)^\alpha\right). \tag{2}$$

The last equality follows from Lemma 3 (b) under the assumption that $\nu > \alpha(1-\frac{1}{B})$.

By the assumption on ν and Lemma 3 again, and after integration by parts, we have

$$\begin{aligned}\frac{1}{n^\nu}\int_1^{n+1} F(u)\,du &\sim \frac{1}{n^\nu}\int_1^{n+1} u^{\nu-\alpha} a_u^{\alpha-1} a_u'\,du \\ &\sim \frac{1}{n^\nu}\left\{a_{n+1}^\alpha (n+1)^{\nu-\alpha} - a_1^\alpha - (\nu-\alpha)\int_1^{n+1} F(u)\,du\right\},\end{aligned}$$

that is,

$$\frac{1}{n^\nu}\int_1^{n+1} F(u)\,du = O\left(\left(\frac{a_n}{n}\right)^\alpha\right). \tag{3}$$

Now we have to estimate only the reminder terms. Lemma 3 (a) and Lemma 4 yield

$$|g_i(u)| = O(u^{\nu-\alpha-3} a_u^\alpha),$$

which implies

$$|f_i(n)| \le K + C\int_1^n a_u^\alpha u^{\nu-\alpha-3}\,du = K + CI_n.$$

If $\nu > \alpha(1-\frac{1}{B}) + 2$, as before, with an integration by parts we get

$$\left(1 + \frac{B}{\alpha}(\nu-\alpha-2)\right) I_n \le \frac{B}{\alpha}\left(a_n^\alpha n^{\nu-\alpha-2} - a_1^\alpha\right),$$

that is, the assumption on ν and Lemma 3 yield that the coefficient of I_n is positive, and

$$\frac{1}{n^\nu}|f_i(n)| = o\left(\left(\frac{a_n}{n}\right)^\alpha\right). \tag{4}$$

If $\alpha(1-\frac{1}{B}) < \nu \le \alpha(1-\frac{1}{A}) + 2$, then

$$I_n \le \int_1^n x^{\alpha/A+\nu-\alpha-3}\,dx \le c\ln n,$$

and by the first inequality on ν we get the same estimation of the reminder term as before. (We have to note that this case exists according to the assumption of the theorem on α.)

If $\alpha(1-\frac{1}{B}) < \alpha(1-\frac{1}{A}) + 2 < \nu \le \alpha(1-\frac{1}{B}) + 2$, we have

$$I_n \le \int_1^n a_u^\alpha u^{\nu-\alpha-1}\,du = J_n,$$

and in the same way we get that

$$\frac{1}{n^\nu} J_n = O\left(\left(\frac{a_n}{n}\right)^\alpha\right).$$

Therefore

$$\frac{1}{n^\nu}|f_i(n)| = O\left(\left(\frac{a_n}{n}\right)^\alpha\right). \tag{5}$$

Now, taking into account (1)–(5), we finish the proof of Lemma 5.

Proof of Theorem 1. The proof follows immediately from Lemma 2 and Lemma 5.

References

[1] Z. Ditzian, Fractional derivatives and best approximation, *Acta Math. Hung.* **81**, 4 (1998), 323–348.

[2] Z. Ditzian and D. S. Lubinsky, Jackson and smoothness theorems for Freud weights in L_p, $(0 < p \le \infty)$, *Constr. Approx.* **13** (1997), 99–152.

[3] G. Freud, On polynomial approximation with the weight $\exp(-\frac{1}{2}x^{2k})$, *Acta Math. Hung.* **24**, 3-4 (1973), 363–371.

[4] I. Joó, Saturation theorems for Hermite-Fourier series, *Acta Math. Hung.* **57**, 1-2 (1991), 169–179.

[5] A. L. Levin and D. S. Lubinsky, Christoffel functions, orthogonal polynomials, and Nevai's conjecture for Freud weights, *Constr. Approx.* **8** (1992), 463–535.

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