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Convergence of Generalized Bernstein Polynomials Based on the q-Integers

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In 1912 Bernstein [1] gave his famous proof of the Weierstrass Approximation Theorem based on some probabilistic reasoning. For any function $f : [0,1] \to \mathbb{R}$ he defined the following polynomials

$$B_n(f;x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}, \qquad n = 1,2,\ldots \tag{1}$$

and proved that if f is continuous on $[0,1]$, then the sequence $B_n(f;x)$ is uniformly convergent to f on $[0,1]$. The polynomials (1) are called now *Bernstein polynomials*. They have been studied in a great number of papers.

A rich variety of generalizations of Bernstein polynomials is known (cf. [2]). One more generalization was suggested by Phillips [3] in 1997, who considered Bernstein polynomials based on the q-integers. In case $q = 1$ these polynomials coincide with the classical ones. For $q \neq 1$ one gets a new class of polynomials having interesting properties. Generalized Bernstein polynomials based on the q-integers were studied by Phillips, Goodman, and Oruç in [3] – [7]. In particular, Phillips obtained analogs of Bernstein's and Voronovskaya's theorems for generalized Bernstein polynomials. A survey of these results is presented in [8].

We present new results concerning convergence properties of generalized Bernstein polynomials based on the q-integers in case $q \in (0,1)$. Our results show that these properties are significantly different from those in the classical case.

To present our results we need the following definitions. Let $q > 0$. For any $n = 0,1,2,\ldots$ the *q-integer* $[n]_q$ and the *q-factorial* $[n]_q!$ are defined as

$$[n]_q := 1 + q + \ldots + q^{n-1}, \quad [n]_q! := [1]_q[2]_q \ldots [n]_q, \qquad n = 1,2,\ldots,$$

$$[0]_q := 0, \qquad [0]_q! := 1.$$

For integers $0 \le k \le n$ the *q-binomial* or the *Gaussian coefficient* is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Definition 1 (Phillips [3]). *Let* $f : [0,1] \to \mathbb{R}$. *The generalized Bernstein polynomial based on the q-integers is defined by*

$$B_n(f,q;x) := \sum_{k=0}^{n} f\left([k]_q/[n]_q\right) p_{nk}(q;x), \tag{2}$$

where

$$p_{nk}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k} (1-q^s x), \qquad n = 1,2,\ldots$$

(An empty product is taken to be equal to 1.) Note that for $q = 1$ the polynomials (2) coincide with the classical Bernstein polynomials: $B_n(f,1;x) = B_n(f;x)$. Phillips obtained the following analog of Bernstein's Theorem.

Theorem A ([3]). *Let a sequence* $\{q_n\}$ *satisfy* $0 < q_n < 1$ *and* $q_n \to 1$ *as* $n \to \infty$. *If* $f \in C[0,1]$, *then*

$$B_n(f,q_n;x) \rightrightarrows f(x) \quad [x \in [0,1]\,;\ n \to \infty].$$

(The expression $g_n(x) \rightrightarrows g(x)$ $[x \in E\,;\ n \to \infty]$ means convergence of g_n to g uniformly on E as $n \to \infty$.)

It is shown in [3] that

$$B_n(at+b,q;x) = ax+b, \qquad q > 0,\ n = 1,2,\ldots \tag{3}$$

Further, it follows directly from (2) that

$$B_n(f,q;0) = f(0), \quad B_n(f,q;1) = f(1), \qquad q > 0,\ n = 1,2,\ldots \tag{4}$$

Formulae (3) and (4) show that the generalized Bernstein polynomials reproduce linear functions and possess the endpoint interpolation property, similarly to the classical ones. The question arises about approximating properties of the sequence $\{B_n(f,q;x)\}$. Direct calculations (cf. [3, formula (13)]) show that

$$B_n(t^2,q;x) = x^2 + \frac{x(1-x)}{[n]_q},$$

and hence for $q \in (0,1)$ we get

$$B_n(t^2,q;x) \rightrightarrows x^2 + (1-q)x(1-x) \neq x^2 \quad [x \in [0,1]\,;\ n \to \infty].$$

Therefore, in general, the sequence $\{B_n(f,q;x)\}$ is not an approximating one for the function f.

To investigate the problem of convergence for $\{B_n(f,q;x)\}$, consider the limits

$$\lim_{n\to\infty} \frac{[k]_q}{[n]_q} = 1 - q^k, \qquad k = 0,1,2,\ldots$$

and

$$\lim_{n\to\infty} p_{nk}(q;x) = \frac{x^k}{(1-q)^k[k]_q!}\prod_{s=0}^{\infty}(1-q^s x) =: p_{\infty k}(q;x).$$

Note that the functions $p_{\infty k}(q;x)$ are transcendental entire functions. Obviously $p_{\infty k}(q;x) \geq 0$ for $x \in [0,1]$ and by Euler's Identity we have

$$\sum_{k=0}^{\infty} p_{\infty k}(q;x) = 1, \qquad x \in [0,1).$$

For $f : [0,1] \to \mathbb{R}$ we set

$$B_\infty(f,q;x) = \begin{cases} \sum_{k=0}^{\infty} f(1-q^k)p_{\infty k}(q;x), & \text{if } x \in [0,1) \\ f(1), & \text{if } x = 1. \end{cases} \tag{5}$$

Note that the function $B_\infty(f,q;x)$ is well-defined on $[0,1]$, whenever the function $f(x)$ is bounded on the interval.

Our main results on convergence are Theorems 1–3 below. Let us point out to the fact that in Theorem 1 the parameter q remains fixed.

Theorem 1. *Let $0 < q < 1$ and $0 < \alpha < 1$. If $f \in C[0,1]$, then*

$$B_n(f,q;x) \rightrightarrows B_\infty(f,q;x) \quad [(q,x) \in [\alpha,1] \times [0,1]\,;\ n \to \infty].$$

Therefore, for any $f \in C[0,1]$ the sequence $\{B_n(f,q;x)\}_{n=1}^{\infty}$ is uniformly convergent on $[0,1]$, but, in general, the limit function is not equal to f. However, the following statement is true.

Theorem 2. *If $f \in C[0,1]$, then*

$$B_\infty(f,q;x) \rightrightarrows f(x) \quad [x \in [0,1]\,;\ q \uparrow 1].$$

Evidently, Theorem A follows from Theorems 1 and 2.

Suppose that $q \in (0,1)$ is fixed. Let $B_n(f,q;x) \rightrightarrows f(x)$ $[x \in [0,1]\,;\ n \to \infty]$. What can be said about the function f? The following theorem provides an exhaustive answer to this question.

Theorem 3. *Let $f \in C[0,1]$. Then $B_\infty(f,q;x) = f(x)$ for all $x \in [0,1]$ if and only if $f(x) = ax + b$, where a and b are constants.*

This means that the sequence $\{B_n(f,q;x)\}$ is not an approximating one for the function $f(x)$ unless $f(x)$ is a *linear function.*

It is natural to ask how properties of $f(x)$ affect the limit function $B_\infty(f,q;x)$. The following theorem treats analytic properties of $B_\infty(f,q;x)$.

Theorem 4. i) *For any* $f \in C[0,1]$ *the function* $B_\infty(f,q;x)$ *is continuous on* $[0,1]$ *and analytic in the unit disk* $\{x \in \mathbb{C} : |x| < 1\}$.

ii) *If* f *satisfies the Lipschitz condition at* 1*, then* $B_\infty(f,q;x)$ *is differentiable from the left at* 1.

iii) *If* f *is a polynomial, then* $B_\infty(f,q;x)$ *is a polynomial, and* $\deg B_\infty(f,q;x) = \deg f$.

Remark. In general $B_\infty(f,q;x)$ may not be differentiable at 1. For example, let $f \in C[0,1]$, $f(0) = f(1) = 0$, and $f(1-q^k) = 1/k$, $k = 1,2,\ldots$ Direct calculations show that

$$\lim_{h \to 0^+} h^{-1}[B_\infty(f,q;1) - B_\infty(f,q;1-h)] = \infty.$$

It follows from (5) that the function $B_\infty(f,q;x)$ depends only on the values of f at the points $\{1-q^k\}$, $k = 0,1,2,\ldots$ In fact, the following unicity theorem is true.

Theorem 5. $B_\infty(f,q;x) = B_\infty(g,q;x)$ *for* $x \in [0,1]$ *if and only if* $f(1-q^k) = g(1-q^k)$ *for all* $k = 0,1,2\ldots$

In particular, there exist functions f different from polynomials such that $B_\infty(f,q;x)$ is a polynomial. However, the following statement holds.

Theorem 6. *Let* $f \in C[0,1]$ *and let there exists a sequence* $q_j \uparrow 1$ *such that* $B_\infty(f,q_j;x)$ *is a polynomial of degree* $\le m$ *for each* q_j, $j = 1,2,\ldots$ *Then* f *is a polynomial of degree* $\le m$.

The proofs of these theorems are given in [9].

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