A Pompeiu-Type Mean-Value Theorem and Divided Differences

Mircea Ivan and Ulrich Abel

1. Introduction and Preliminary Results

In [1] Pompeiu gave the following variant of the Lagrange mean value theorem.

**Theorem 1** (Pompeiu, [1]). Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\), differentiable on \((a, b)\) and \(0 \not\in [a, b]\). Then there exists a point \( c \in (a, b) \) such that

\[
\frac{a f(b) - b f(a)}{a - b} = f(c) - c f'(c).
\]

A geometric interpretation of Theorem 1 is given in Figure 1.

![Figure 1: The graph of the Taylor polynomial \( T_1(f; c) \), the graph of the Lagrange interpolating polynomial \( L_1[a, b; f] \) and the \(Oy\) axis intersect in the same point.](image)

Another Pompeiu-type mean-value theorem is given in [2].

**Theorem 2** (Ivan, [2]). Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\). If \( f \) has no roots in \((a, b)\) and \( f(a) \neq f(b) \), then there exists a point \( c \in (a, b) \) such that

\[
\frac{a f(b) - b f(a)}{f(b) - f(a)} = c - \frac{f(c)}{f'(c)}.
\]
We point out that $f'$ can have roots in $(a, b)$. A geometric interpretation of Theorem 2 is given in Figure 2.

![Figure 2](image-url)

**Figure 2**: The graph of the Taylor polynomial $T_1(f; c)$, the graph of the Lagrange interpolating polynomial $L_1[a; b; f]$ and the Ox axis intersect in the same point.

In what follows we consider the points $a \leq x_0 < \cdots < x_n \leq b$, $n \geq 1$. Let $f : [a, b] \to \mathbb{R}$. As usual, we denote by $L(x_0, \ldots, x_n; f)$ the Lagrange polynomial interpolating $f$ at the points $x_0, \ldots, x_n$. We have

$$L(x_0, \ldots, x_n; f)(x) = \sum_{i=0}^{n} f(x_i) \prod_{j=0}^{n} \frac{x-x_j}{x_i-x_j}.$$ 

The divided difference of $f$ on the distinct knots $x_0, \ldots, x_n$ is defined to be the coefficient of $x^n$ in $L(x_0, \ldots, x_n; f)$ and is denoted by $[x_0, \ldots, x_n; f]$.

Let us denote by $D_n[a, b]$ the set of all functions $f$ continuous on $[a, b]$ and possessing a derivative of order $n$ on $(a, b)$.

If $f$ has a derivative of order $n$ at a point $c$, we denote by $T_n(f; c)$ the Taylor polynomial of degree $n$ associated with $f$ at $c$,

$$T_n(f; c)(x) := \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(x-c)^i.$$ 

Recall also a well-known extension of Lagrange’s Mean Value Theorem to the case of divided differences.

**Proposition 1** (Cauchy, [3, p. 36]). If $f \in D_n[a, b]$, then there exists $c \in (a, b)$ such that

$$[x_0, \ldots, x_n; f] = \frac{f^{(n)}(c)}{n!}.$$ 

Among the many other extensions of the Pompeiu’s Theorem we focus on that of Stamate [4].
Theorem 3 (Stamate, [4]). If \( f \in \mathcal{D}_n[a, b] \) and \( 0 \notin [a, b] \), then there exists a point \( c \in (a, b) \) such that

\[
\sum_{i=0}^{n} f(x_i) \prod_{j=0, j \neq i}^{n} \frac{x_j}{x_j - x_i} = \sum_{i=0}^{n} (-1)^i c^i \frac{f^{(i)}(c)}{i!}.
\]  

(1)

Stamate proved his theorem by applying Proposition 1 to \( \varphi(t) = t^n f(1/t) \) for \( t_i = \frac{1}{x_i}, \ i = 0, 1, \ldots, n \). Since

\[
L(x_0, \ldots, x_n; f)(0) = \sum_{i=0}^{n} f(x_i) \prod_{j=0, j \neq i}^{n} \frac{x_j}{x_j - x_i}, \quad T_n(f; c)(0) := \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!} (-c)^i.
\]

we can rewrite Theorem 3 in the form:

**Theorem 4** (Reformulation of Stamate’s Theorem 3). If \( 0 \notin [a, b] \) and \( f \in \mathcal{D}_n[a, b] \), then there exists a point \( c \in (a, b) \) such that

\[
L_n(x_0, \ldots, x_n; f)(0) = T_n(f; c)(0).
\]  

(2)

Let \( k \in \{0, \ldots, n\} \). Szasz [5] obtained the following theorem.

**Theorem 5** (Szasz, [5]). If \( f \) is continuous on \( [a, b] \) and possesses a derivative of order \( k \) on \( (a, b) \), then there exist points \( x_{k,0}, \ldots, x_{k,n-k} \), such that

\[
L^{(k)}(x_0, \ldots, x_n; f) = L(x_{k,0}, \ldots, x_{k,n-k}; f^{(k)}).
\]  

(3)

We note that, by using (3) and Stamate’s Theorem 3, Szasz obtained mean-value type formulas for all coefficients of the Lagrange polynomial.

**Problem 1.** Let \( f \in \mathcal{D}_n[a, b] \) and \( p \in \mathbb{R} \setminus [a, b] \). Find \( q \in \mathbb{R} \) such that there exists \( c \in (a, b) \) with \( q = T_n(f; c)(p) \).

Of course, for some \( q \), Problem 1 has no solution (see Figure 4).

2. Main Results

Let \( k \in \{0, \ldots, n\} \). The following is a generalization of Stamate’s Theorem 3.

**Theorem 6.** If \( f \in \mathcal{D}_n[a, b] \) and \( p \in \mathbb{R} \setminus [a, b] \), then there exists a point \( c \in (a, b) \) such that

\[
T_{n-k}(f^{(k)}; c)(p) = \left(L_n[x_0, \ldots, x_n; f]\right)^{(k)}(p).
\]  

(4)
Proof. Let \( x_{k,0}, \ldots, x_{k,n-k} \) be given by Theorem 5. Since the degree of the polynomial \((L_n[x_0, \ldots, x_n; f])^{(k)}(p)\) is at most \(n-k\), we get
\[
\binom{n}{k} \Bigg[ p, x_{k,0}, \ldots, x_{k,n-k}; (L_n[x_0, \ldots, x_n; f])^{(k)} \Bigg] = 0,
\]
where \( x_{0,k} := x_k \ (k = 0, \ldots, n) \). We have:
\[
0 = \binom{n}{k} \left[ p, x_{k,0}, \ldots, x_{k,n-k}; (L_n[x_0, \ldots, x_n; f])^{(k)} \right] \\
= \left[ x_{k,0}, \ldots, x_{k,n-k}; \frac{(L_n[x_0, \ldots, x_n; f])^{(k)}(t) - (L_n[x_0, \ldots, x_n; f])^{(k)}(p)}{t-p} \right] \\
= \left[ x_{k,0}, \ldots, x_{k,n-k}; \frac{f^{(k)}(t) - (L_n[x_0, \ldots, x_n; f])^{(k)}(p)}{t-p} \right].
\]
By Proposition 1, there exists \( c \in (a, b) \) such that
\[
\frac{1}{n!} \left( \frac{d}{dt} \right)^{n-k} \left( \frac{f^{(k)}(t) - (L_n[x_0, \ldots, x_n; f])^{(k)}(p)}{t-p} \right) \bigg|_{t=c} = 0.
\]
Application of Leibniz’s derivation rule yields
\[
\sum_{i=0}^{n-k} \binom{n-k}{i} \left( f^{(k)}(t) - (L_n[x_0, \ldots, x_n; f])^{(k)}(p) \right)^{(i)} \left( \frac{1}{t-p} \right)^{(n-k-i)} \bigg|_{t=c} = 0,
\]
hence
\[
(L_n[x_0, \ldots, x_n; f])^{(k)}(p) = \sum_{i=0}^{n-k} f^{(k+i)}(c) \frac{(p-c)^i}{i!} = T_{n-k}(f^{(k)}; c)(p).
\]

The following consequence of Theorem 6 gives a sufficient condition related to Problem 1.

**Corollary 1.** If \( f \in D_n[a, b] \), then there exists \( c \in (a, b) \) such that
\[
L_n(x_0, \ldots, x_n; f)(p) = T_n(f; c)(p).
\]
(For a geometric interpretation of Corollary 1, see Figure 3.)

Note that, in the special case of \( f = L_n(x_0, \ldots, x_n; g) \), since \( f \) is a polynomial of degree at most \( n \), we obtain
\[
T_n(f, x) = f, \quad L_n(x_0, \ldots, x_n; f)(p) = f(p) = T_n(f, x)(p), \quad \forall x \in (a, b).
\]
Remark. The following results are particular cases of Theorem 6:

- Theorem 1 of Pompeiu \((n = 1, \ k = 0, \ x_0 = a, \ x_1 = b, \ p = 0)\);
- Theorem 2 \((n = 1, \ k = 0, \ x_0 = a, \ x_1 = b, \ p = \frac{b f(a) - a f(b)}{f(a) - f(b)})\);
- Theorem 4 of Stamate \((k = 0, \ p = 0)\);
- The mean value formulas for the coefficients of the Lagrange Polynomial \(L_n(x_0, \ldots, x_n; f)\) obtained by Szasz \((p = 0)\).

**Figure 3:** For \(q = L_n[x_0, \ldots, x_n; f](0)\), there exists a point \(c \in (a, b)\) such that \(T_n(f, c)(0) = q\).

**Figure 4:** There exists no \(c \in (a, b)\) such that \(T_1(f; c)(p) = q\).

References


Mircea Ivan  
Department of Mathematics  
Technical University of Cluj-Napoca  
C. Daicoviciu Street 15  
3400 Cluj-Napoca  
ROMANIA  
E-mail: mircea.ivan@math.utcluj.ro

Ulrich Abel  
Fachbereich MND  
Fachhochschule Giessen–Friedberg  
University of Applied Sciences  
Wilhelm–Leuschner–Straße 13  
D-61169 Friedberg  
GERMANY  
E-mail: Ulrich.Abel@mnd.fh-friedberg.de