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A Pompeiu-Type Mean-Value Theorem and Divided Differences

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1. Introduction and Preliminary Results

In [1] Pompeiu gave the following variant of the Lagrange mean value theorem.

Theorem 1 (Pompeiu, [1]). *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b) and $0 \notin [a,b]$. Then there exists a point $c \in (a,b)$ such that*

$$\frac{a\,f(b) - b\,f(a)}{a-b} = f(c) - c\,f'(c).$$

A geometric interpretation of Theorem 1 is given in Figure 1.

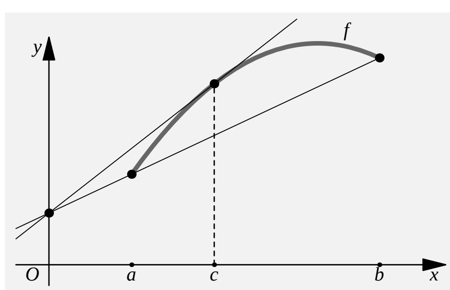

Figure 1: The graph of the Taylor polynomial $T_1(f;c)$, the graph of the Lagrange interpolating polynomial $L_1[a,b;\,f]$ and the Oy axis intersect in the same point.

Another Pompeiu-type mean-value theorem is given in [2].

Theorem 2 (Ivan, [2]). *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If f has no roots in (a,b) and $f(a) \neq f(b)$, then there exists a point $c \in (a,b)$ such that*

$$\frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = c - \frac{f(c)}{f'(c)}.$$

We point out that f' can have roots in (a,b). A geometric interpretation of Theorem 2 is given in Figure 2.

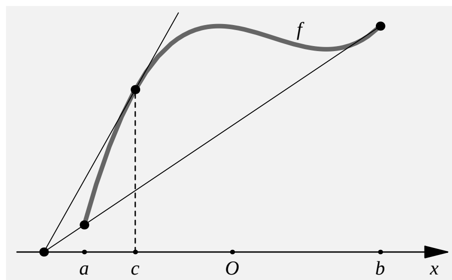

Figure 2: The graph of the Taylor polynomial $T_1(f;c)$, the graph of the Lagrange interpolating polynomial $L_1[a,b;\, f]$ and the Ox axis intersect in the same point.

In what follows we consider the points $a \le x_0 < \cdots < x_n \le b$, $n \ge 1$. Let $f : [a,b] \to \mathbb{R}$. As usual, we denote by $L(x_0,\dots,x_n;\, f)$ the Lagrange polynomial interpolating f at the points $x_0,\dots,x_n$. We have

$$L(x_0,\dots,x_n;f)(x) = \sum_{i=0}^{n} f(x_i) \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j}.$$

The divided difference of f on the distinct knots $x_0,\dots,x_n$ is defined to be the coefficient of x^n in $L(x_0,\dots,x_n;\, f)$ and is denoted by $[x_0,\dots,x_n;\, f]$.

Let us denote by $\mathcal{D}_n[a,b]$ the set of all functions f continuous on $[a,b]$ and possessing a derivative of order n on (a,b).

If f has a derivative of order n at a point c, we denote by $T_n(f;c)$ the Taylor polynomial of degree n associated with f at c,

$$T_n(f;c)(x) := \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(x-c)^i.$$

Recall also a well-known extension of Lagrange's Mean Value Theorem to the case of divided differences.

Proposition 1 (Cauchy, [3, p. 36]). *If $f \in \mathcal{D}_n[a,b]$, then there exists $c \in (a,b)$ such that*

$$[x_0,\dots,x_n;\, f] = \frac{f^{(n)}(c)}{n!}.$$

Among the many other extensions of the Pompeiu's Theorem we focus on that of Stamate [4].

Theorem 3 (Stamate, [4]). *If $f \in \mathcal{D}_n[a,b]$ and $0 \notin [a,b]$, then there exists a point $c \in (a,b)$ such that*

$$\sum_{i=0}^{n} f(x_i) \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x_j}{x_j - x_i} = \sum_{i=0}^{n} (-1)^i c^i \frac{f^{(i)}(c)}{i!}. \tag{1}$$

Stamate proved his theorem by applying Proposition 1 to $\varphi(t) = t^n f(1/t)$ for $t_i = \frac{1}{x_i}$, $i = 0, 1, \ldots, n$. Since

$$L(x_0, \ldots, x_n; f)(0) = \sum_{i=0}^{n} f(x_i) \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x_j}{x_j - x_i}, \quad T_n(f;c)(0) := \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!}(-c)^i.$$

we can rewrite Theorem 3 in the form:

Theorem 4 (Reformulation of Stamate's Theorem 3). *If $0 \notin [a,b]$ and $f \in \mathcal{D}_n[a,b]$, then there exists a point $c \in (a,b)$ such that*

$$L_n(x_0, \ldots x_n; f)(0) = T_n(f;c)(0). \tag{2}$$

Let $k \in \{0, \ldots, n\}$. Szasz [5] obtained the following theorem.

Theorem 5 (Szasz, [5]). *If f is continuous on $[a,b]$ and possesses a derivative of order k on (a,b), then there exist points $x_{k,0}, \ldots, x_{k,n-k}$, such that*

$$L^{(k)}(x_0, \ldots, x_n; f) = L(x_{k,0}, \ldots, x_{k,n-k}; f^{(k)}). \tag{3}$$

We note that, by using (3) and Stamate's Theorem 3, Szasz obtained mean-value type formulas for all coefficients of the Lagrange polynomial.

Problem 1. *Let $f \in \mathcal{D}_n[a,b]$ and $p \in \mathbb{R} \setminus [a,b]$. Find $q \in \mathbb{R}$ such that there exists $c \in (a,b)$ with $q = T_n(f;c)(p)$.*

Of course, for some q, Problem 1 has no solution (see Figure 4).

2. Main Results

Let $k \in \{0, \ldots, n\}$. The following is a generalization of Stamate's Theorem 3.

Theorem 6. *If $f \in \mathcal{D}_n[a,b]$ and $p \in \mathbb{R} \setminus [a,b]$, then there exists a point $c \in (a,b)$ such that*

$$T_{n-k}(f^{(k)}; c)(p) = \big(L_n[x_0, \ldots, x_n; f]\big)^{(k)}(p). \tag{4}$$

Proof. Let $x_{k,0}, \dots x_{k,n-k}$ be given by Theorem 5. Since the degree of the polynomial $\left(L_n[x_0, \dots, x_n;\, f]\right)^{(k)}(p)$ is at most $n-k$, we get

$$\left[p, x_{k,0}, \dots x_{k,n-k};\; \left(L_n[x_0, \dots, x_n;\, f]\right)^{(k)}\right] = 0,$$

where $x_{0,k} := x_k$ $(k = 0, \dots, n)$. We have:

$$\begin{aligned}
0 &= \left[p, x_{k,0}, \dots, x_{k,n-k}; \left(L_n[x_0, \dots, x_n;\, f]\right)^{(k)}\right] \\
&= \left[x_{k,0}, \dots, x_{k,n-k}; \frac{\left(L_n[x_0, \dots, x_n;\, f]\right)^{(k)}(t) - \left(L_n[x_0, \dots, x_n;\, f]\right)^{(k)}(p)}{t-p}\right]_t \\
&= \left[x_{k,0}, \dots x_{k,n-k}; \frac{L_n[x_{k,0}, \dots, x_{k,n-k};\, f^{(k)}](t) - \left(L_n[x_0, \dots, x_n;\, f]\right)^{(k)}(p)}{t-p}\right]_t \\
&= \left[x_{k,0}, \dots, x_{k,n-k}; \frac{f^{(k)}(t) - \left(L_n[x_0, \dots, x_n;\, f]\right)^{(k)}(p)}{t-p}\right]_t .
\end{aligned}$$

By Proposition 1, there exists $c \in (a,b)$ such that

$$\frac{1}{n!}\left(\frac{d}{dt}\right)^{n-k}\left(\frac{f^{(k)}(t) - \left(L_n[x_0, \dots, x_n;\, f]\right)^{(k)}(p)}{t-p}\right)\Bigg|_{t=c} = 0.$$

Application of Leibniz's derivation rule yields

$$\sum_{i=0}^{n-k}\binom{n-k}{i}\left(f^{(k)}(t) - \left(L_n[x_0, \dots, x_n;\, f]\right)^{(k)}(p)\right)^{(i)}\left(\frac{1}{t-p}\right)^{(n-k-i)}\Bigg|_{t=c} = 0,$$

hence

$$\left(L_n[x_0, \dots, x_n;\, f]\right)^{(k)}(p) = \sum_{i=0}^{n-k}\frac{f^{(k+i)}(c)}{i!}(p-c)^i = T_{n-k}(f^{(k)}; c)(p).$$

□

The following consequence of Theorem 6 gives a sufficient condition related to Problem 1.

Corollary 1. *If $f \in \mathcal{D}_n[a,b]$, then there exists $c \in (a,b)$ such that*

$$L_n(x_0, \dots x_n; f)(p) = T_n(f; c)(p).$$

(For a geometric interpretation of Corollary 1, see Figure 3.)

Note that, in the special case of $f = L_n(x_0, \dots x_n; g)$, since f is a polynomial of degree at most n, we obtain

$$T_n(f,x) = f, \quad L_n(x_0, \dots x_n; f)(p) = f(p) = T_n(f,x)(p), \qquad \forall x \in (a,b).$$

Remark. The following results are particular cases of Theorem 6:

- Theorem 1 of Pompeiu ($n = 1,\ k = 0,\ x_0 = a,\ x_1 = b,\ p = 0$);
- Theorem 2 ($n = 1,\ k = 0,\ x_0 = a,\ x_1 = b,\ p = \frac{b\,f(a) - a\,f(b)}{f(a) - f(b)}$);
- Theorem 4 of Stamate ($k = 0,\ p = 0$);
- The mean value formulas for the coefficients of the Lagrange Polynomial $L_n(x_0, \ldots, x_n;\ f)$ obtained by Szasz ($p = 0$).

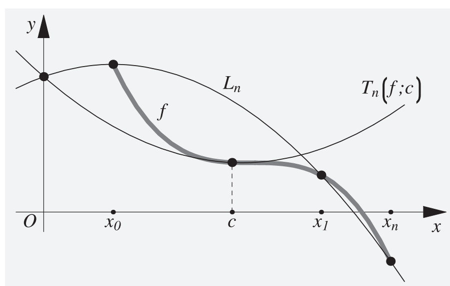

Figure 3: For $q = L_n[x_0, \ldots, x_n;\ f](0)$, there exists a point $c \in (a, b)$ such that $T_n(f, c)(0) = q$.

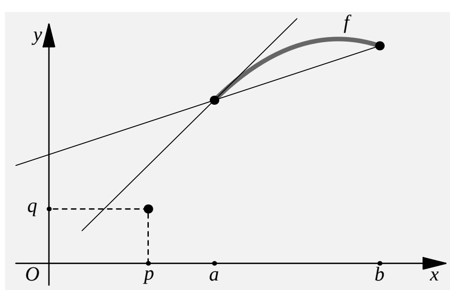

Figure 4: There exists no $c \in (a, b)$ such that $T_1(f; c)(p) = q$.

References

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