CONSTRUCTIVE THEORY OF FUNCTIONS, Varna 2002
(B. Bojanov, Ed.), DARBA, Sofia, 2003, pp. 320-326.

Characterization of Weighted K-functionals by New Moduli of Smoothness

Kamen G. Ivanov

Dedicated to Blagovest Sendov on the occasion of his 70th birthday.
Dedicated to the memory of Vassil Popov on the occasion of his 60th birthday

1. Introduction

In a number of approximation processes the approximation error is equivalent to (or is estimated by means of) an appropriate K-functional. This functional has generally the form

$$K(f,t) = K(f,t;X,Y,\mathcal{D}) = \inf\big\{\|f-g\|_X + t\|\mathcal{D}g\|_X \ : \ g \in Y \cap \mathcal{D}^{-1}(X)\big\},$$

where X is a Banach space, $\mathcal{D}$ is a differential operator of the form

$$\mathcal{D}g = \sum_{k=0}^{r} \varphi_k D^k g, \quad D = \frac{d}{dx}, \quad \varphi_k \in X, \ k = 0,\dots,r, \quad \varphi_r > 0 \ a.\ e,$$

$$\mathcal{D}^{-1}(X) = \{g \in X : \mathcal{D}g \in X\} \subset X$$

and Y is a dense subspace of X.

In this article we shall be concerned with spaces of univariate functions

$$X = L_p(w) = L_p(w)[a,b] = \big\{f : wf \in L_p[a,b]\big\}, \qquad L_p(1) = L_p,$$

where w is a Jacobean weight of the type $(x-a)^{\kappa_a}(b-x)^{\kappa_b}$ and L_p is equipped with the usual norm $\|\cdot\|_p$. By $\|\cdot\|_\infty$ we denote the uniform norm. As a main example for Y one may have in mind the largest possible space on which the infimum in the K-functional can be taken, i.e., $Y = AC_{loc}^{r-1}$ being the set of all functions with locally absolutely continuous derivatives of order up to $r-1$.

Examples of such K-functionals are:

- for the best weighted approximation of a function $f \in L_p(w)[-1,1]$, $1 \le p \le \infty$, by algebraic polynomials: $r \in \mathbb{N}$, $\mathcal{D} = \varphi^r D^r$ with $\varphi(x) = \sqrt{1-x^2}$;

- for Bernstein operators B_n: $[a,b]=[0,1]$, $p=\infty$, $r=2$, $\mathcal{D}=\varphi^2D^2$ with $\varphi(x)=\sqrt{x-x^2}$ and $Y=AC^1_{loc}[0,1]$;
- for Szász-Mirakjan operators S_n: $[a,b]=[0,\infty)$, $p=\infty$, $r=2$, $\mathcal{D}=\varphi^2D^2$ with $\varphi(x)=\sqrt{x}$ and $Y=AC^1_{loc}[0,\infty)$;
- for Kantorovich P_n and Durrmeyer M_n operators: $[a,b]=[0,1]$, $1\le p\le\infty$, $r=2$, $\mathcal{D}=D\phi D=\phi D^2+\phi' D$ with $\phi(x)=x-x^2$ and $Y=C^2[0,1]$.

There is a number of studies (see e.g. [2], [6], [8]) devoted to construction of moduli of functions that are equivalent to the weighted Peetre K-functionals of the type $K(f,t^r;L_p(w),AC^{r-1}_{loc},\varphi^rD^r)$.

An approach that differs from those in the above mentioned papers is developed in [3], [4], [7]. The approach has two aspects. From one side we study conditions on the triples $(X_1,Y_1,\mathcal{D}_1)$ and $(X_2,Y_2,\mathcal{D}_2)$ under which one can construct a linear operator $\mathcal{A}:X_1\to X_2$ such that

$$K(f,t;X_1,Y_1,\mathcal{D}_1)\sim K(\mathcal{A}f,t;X_2,Y_2,\mathcal{D}_2). \tag{1}$$

By $K_1(f,t)\sim K_2(f,t)$ we mean $c^{-1}K_1(f,t)\le K_2(f,t)\le cK_1(f,t)$, where c denotes a positive constant independent of f and t.

On the other side, for given $(X_1,Y_1,\mathcal{D}_1)$ we try to choose $(X_2,Y_2,\mathcal{D}_2)$ in (1) in such a way that the K-functional in the right-hand side has a known equivalent modulus $\Omega(F,t)$, i.e.,

$$K(F,t;X_2,Y_2,\mathcal{D}_2)\sim\Omega(F,t). \tag{2}$$

From (1) and (2) one gets $K(f,t;X_1,Y_1,\mathcal{D}_1)\sim\Omega(\mathcal{A}f,t)$. In order to make this equivalence effective for computations one has to require some additional properties of $\mathcal{A}$ as explicitness, simple form, easy to calculate for a given f, etc.

Constructions of operators $\mathcal{A}$ as in (1) for arbitrary r (the degree of the differential operator) were earlier done by B. Draganov in [3]. In this lecture we give a scheme which allows the described approach to be implemented for a variety of weighted Peetre K-functionals.

2. Construction of the Operator $\mathcal{A}$ in (1)

A sufficient condition for the validity of (1) is:

There exists a bounded linear operator $\mathcal{A}:X_1\to X_2$, which is invertible, its inverse $\mathcal{A}^{-1}:X_2\to X_1$ is also bounded, $\mathcal{A}(Y_1\cap\mathcal{D}_1^{-1}(X_1))=Y_2\cap\mathcal{D}_2^{-1}(X_2)$, and

(a) $\|\mathcal{D}_2\mathcal{A}f\|_{X_2}\le c\,\|\mathcal{D}_1f\|_{X_1}$ *for any* $f\in Y_1\cap\mathcal{D}_1^{-1}(X_1)$;

(b) $\|\mathcal{D}_1\mathcal{A}^{-1}F\|_{X_1}\le c\,\|\mathcal{D}_2F\|_{X_2}$ *for any* $F\in Y_2\cap\mathcal{D}_2^{-1}(X_2)$.

Banach inverse operator theorem can be applied in such circumstances, but the proof that the mapping is "onto" seems more difficult compared to the direct study of the inverse operator.

This sufficient condition looks rather restrictive. Nevertheless, such operators exist in a number of cases important for the applications. Note that the null spaces of the K-functionals in (1) have to be of equal dimensions and that $\mathcal{A}$ is an "one-to-one" correspondence between them.

The construction of $\mathcal{A}$ goes on the following scheme. First we observe in (1) that $\mathcal{A}$ can be represented as a composition of several operators solving simpler problems. Thus, we try to reduce the problem with a differential operator $\mathcal{D}$ in general form to the simpler differential operator $\mathcal{D} = \phi D^r$. For this purpose one can use the mapping

$$\begin{aligned}(Af)(x) = \Phi_r(x)f(x) + \sum_{i=1}^{r}(-1)^i\binom{r}{i}\int_{\xi}^{x}\frac{(x-y)^{i-1}}{(i-1)!}\Phi_r^{(i)}(y)f(y)\,dy \\ + \sum_{k=0}^{r-1}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\int_{\xi}^{x}\frac{(x-y)^{r-k+i-1}}{(r-k+i-1)!}\Phi_k^{(i)}(y)f(y)\,dy,\end{aligned} \tag{3}$$

which has the property $D^r(Af) = \sum_{k=0}^{r}\Phi_k D^k f$. The setting $\Phi_r = 1$; $\Phi_k = \varphi_k/\varphi_r$, $k < r$, in (3) gives a linear operator $A : X_1 \to X_2$ which is bounded together with its inverse for proper pairs (X_1, X_2) and possesses the property

$$\varphi_r D^r(Af) = \sum_{k=0}^{r}\varphi_k D^k f.$$

In order to show that such an approach can work in the next section we apply (3) with $r = 2$, $\Phi_2(x) = 1$, $\Phi_1(x) = (1-2x)/(x-x^2)$. We also demonstrate how one can treat a case when $A(Y_1) \neq Y_2$. The results from Section 3 are proved in [7].

In Section 4 we deal with $\mathcal{D}_1 = \varphi^r D^r$, $X_1 = L_p(w)[a,b]$, $1 \le p \le \infty$, the Jacobean weights φ and w and the largest possible space $Y_1 = AC_{loc}^{r-1}$. Using combinations of members of two families of operators we construct an operator $\mathcal{A}$ for which (1) is true when $(X_2, Y_2, \mathcal{D}_2) = (L_p, W_p^r, D^r)$. Hence in (2) one can take $\Omega(F,t) = \omega_r(F,t)_p$ – the usual r-th modulus of smoothness. The results from Section 4 are proved in [4].

The idea of using operators like $\mathcal{A}$ is not new. It can be traced back even before the invention of the K-functional. When comparing the best approximations by trigonometric polynomials and by algebraic polynomials several mathematicians used the mapping $(\mathcal{B}f)(y) = f(\cos y)$ in order to establish the so-called "effect of the end-points". It is well-known that this mapping solves the following problem in the case $r = 1$, $p = \infty$.

Problem. *Given $r \in \mathbb{N}$ and $1 \le p \le \infty$. Set $\varphi(x) = \sqrt{1-x^2}$. Find an operator $\mathcal{B} : L_p[-1,1] \to L_p[0,\pi]$ such that for every $t > 0$ and $f \in L_p[-1,1]$*

we have

$$\inf\{\|f-g\|_p+t^r\|\varphi^r g^{(r)}\|_p : g \in AC_{loc}^{r-1}\}\sim\inf\{\|\mathcal{B}f-G\|_p+t^r\|G^{(r)}\|_p : G \in W_p^r\}.$$

But this approach has also known difficulties when either $p < \infty$ or $r \geq 2$:

i) For $p < \infty$ we have an additional weight: $\|\mathcal{B}f\|_{p,[0,2\pi]} = \|wf\|_{p,[-1,1]}$ with $w(x) = (1-x^2)^{-\frac{1}{2p}}$.

ii) For $r \geq 2$ the r-th derivative of $\mathcal{B}g$ is not of the form $\varphi^r g^{(r)}$. For example, for $r = 2$ we have $(\mathcal{B}f)''(\arccos x) = (1-x^2)f''(x) - xf'(x)$.

In Section 4 we show that these difficulties can be overcomed. The partial case $w \equiv 1$ of Theorem 2 gives a solution of the above problem.

3. Change of the Differential Operator

The approximation errors of the Durrmeyer operators M_n and the Kantorovich operators P_n are characterized (in [1] and [5] respectively) as follows:

For every $n \in \mathbb{N}$ *and* $f \in L_p[0,1]$, $1 \leq p \leq \infty$, *we have with* $\phi(x) = x(1-x)$

$$\|f - M_n f\|_p \sim K(f, n^{-1}; L_p, C^2, D\phi D) \sim \|f - P_n f\|_p. \tag{4}$$

It is proved in [5] that for $1 < p \leq \infty$ we have

$$K(f, t; L_p, C^2, D\phi D) \sim K(f, t; L_p, AC_{loc}^1, \phi D^2) + \omega_1(f, t)_p. \tag{5}$$

The weighted moduli defined in [2] or [6] are equivalent to the K-functional in the right-hand side of (5). The results in the next section can be also applied to this functional. The equivalence (5) is NOT true for $p = 1$.

In order to treat the K-functional for $\mathcal{D} = D\phi D$ and $p = 1$ we utilize an operator $\mathcal{A}$ defined by

$$(\mathcal{A}f)(x) = f(x) + \int_{1/2}^{x} \left(\frac{x}{y^2} - \frac{1-x}{(1-y)^2}\right) f(y)\, dy, \qquad 0 < x < 1,$$

for $f \in L_1[0,1]$. The values of $\mathcal{A}f$ at $x = 0$ and $x = 1$ are defined by continuity when possible. The inverse operator is given by

$$(\mathcal{A}^{-1}f)(x) = f(x) - \int_{1/2}^{x} \left(\frac{1-2y}{y(1-y)} + 2\log\frac{x(1-y)}{(1-x)y}\right) f(y)\, dy,$$

As seen from the definitions $\mathcal{A} : L_p \to L_p$ is bounded for $1 \leq p \leq \infty$, while $\mathcal{A}^{-1} : L_p \to L_p$ is bounded only for $1 \leq p < \infty$ and $\mathcal{A}^{-1} : C \to C$ is unbounded.

One can easily see that $\phi(\mathcal{A}g)'' = \phi g'' + \phi' g'$ and $\mathcal{A}(AC^1_{loc}) = AC^1_{loc}$. Hence

$$K(f,t;L_p,AC^1_{loc},D\phi D) \sim K(\mathcal{A}f,t;L_p,AC^1_{loc},\phi D^2) \qquad \text{for } 1 \le p < \infty.$$

But $\mathcal{A}(C^2) \neq C^2$! In order to apply the scheme of Section 2 we take infimums on the C^2 subspaces $Z_1 = \{f \in C^2[0,1] : f'(0) = 0,\ f'(1) = 0\}$ and

$$Z_2 = \left\{ f \in C^2[0,1] : f(0) = 2\int_0^{1/2} f(y)\,dy,\ f(1) = 2\int_{1/2}^1 f(y)\,dy \right\}.$$

Z_1 and Z_2 are chosen so that $\mathcal{A}(Z_1) = Z_2$ and the respective K-functionals are close to those with Y_1 and Y_2, as shown in the next statement.

Theorem 1. *For every $t \in (0,1]$ and every $f \in L_1[0,1]$ we have*

$$\begin{aligned}K(f,t;L_1,C^2,D\phi D) &= K(f,t;L_1,Z_1,D\phi D)\\ &\sim K(\mathcal{A}f,t;L_1,Z_2,\phi D^2) \sim K(\mathcal{A}f,t;L_1,AC^1_{loc},\phi D^2) + t\omega_1(f,1)_1.\end{aligned}$$

From (4), Theorem 1 and Theorem 2 below we get

Corollary 1. *For every $n \in \mathbb{N}$ and every $f \in L_1[0,1]$ we have*

$$\|f - P_n f\|_1 \sim \|f - M_n f\|_1 \sim \omega_2(\mathcal{B}\mathcal{A}f, n^{-1/2})_1 + n^{-1}\omega_1(f,1)_1,$$

where $\mathcal{B}$ is the operator from Theorem 2 with $r = 2$, $p = 1$, $a = 0$, $b = 1$, $\lambda_0 = \lambda_1 = 1/2$, $\kappa_0 = \kappa_1 = 0$, $\xi = 1/2$.

4. Equivalence to Unweighted *K*-functionals

Let $r \in \mathbb{N}$. For a given finite interval $[a,b]$ let s be one of the points a or b (s stands for singularity), let e be the other end-point, and let ξ be a fixed point in $[a,b]$, $\xi \neq s$. For given $\rho,\sigma \in \mathbb{R}$, $\sigma \neq 0$, we set for every function f which is integrable on any $[c,d] \subset [a,b]$, $c \neq s \neq d$, and for every $x \in [a,b]$

$$\begin{aligned}(A(\rho)f)(x) &= (A(\rho;s,e;\xi)f)(x)\\ &= \left(\frac{x-s}{e-s}\right)^{\rho} f(x) + \frac{1}{e-s}\sum_{k=1}^{r} \alpha_{r,k}(\rho)\left(\frac{x-s}{e-s}\right)^{k-1}\int_{\xi}^{x}\left(\frac{y-s}{e-s}\right)^{-k+\rho} f(y)\,dy,\end{aligned}$$

$$\alpha_{r,k}(\rho) = \frac{(-1)^k}{(r-1)!}\binom{r-1}{k-1}\prod_{\nu=0}^{r-1}(\rho + r - k - \nu), \qquad k = 1,2,\dots,r,$$

and

$$(B(\sigma)f)(x) = (B(\sigma; s, e; \xi)f)(x) = f\left(s + (e-s)\left(\frac{x-s}{e-s}\right)^{\sigma}\right)$$

$$+\frac{1}{e-s}\sum_{k=2}^{r}\beta_{r,k}(\sigma)\left(\frac{x-s}{e-s}\right)^{k-1}\int_{\xi}^{x}\left(\frac{y-s}{e-s}\right)^{-k} f\left(s+(e-s)\left(\frac{y-s}{e-s}\right)^{\sigma}\right)dy,$$

$$\beta_{r,k}(\sigma) = \frac{(-1)^{r-k}}{(r-2)!}\binom{r-2}{k-2}\prod_{i=1}^{r-1}(k-1-i\sigma), \qquad k = 2, 3, \dots, r.$$

Some of the properties of operators $A(\rho)$ and $B(\sigma)$ for fixed r, s, e, ξ are:

a) $A(\rho)A(\rho') = A(\rho + \rho')$, i.e., $\{A(\rho)\}_{\rho\in\mathbb{R}}$ is a commutative group of operators with $A(0)$ as identity element and $A(\rho)^{-1} = A(-\rho)$;

b) $B(\sigma)B(\sigma') = B(\sigma\sigma')$, i.e., $\{B(\sigma)\}_{\sigma>0}$ and $\{B(\sigma)\}_{\sigma\neq 0}$ are commutative groups of operators with $B(1)$ as identity element and $B(\sigma)^{-1} = B(1/\sigma)$;

c) $A(\rho)$ and $B(\sigma)$ preserve the local smoothness of the functions;

d) $(A(\rho)g)^{(r)}(x) = \bar{x}^{\rho}g^{(r)}(x)$ and $(B(\sigma)g)^{(r)}(x) = \sigma^r\bar{x}^{r(\sigma-1)}g^{(r)}(s+(e-s)\bar{x}^{\sigma})$ a.e. for every $g \in AC_{loc}^{r-1}[a, b]$, where $\bar{x} = (x-s)/(e-s) \in (0, 1)$;

e) From d) we get $A(\rho)(\Pi_{r-1}) = \Pi_{r-1}$ and $B(\sigma)(\Pi_{r-1}) = \Pi_{r-1}$;

f) $A(\rho; a, b; \xi)A(\rho'; b, a; \xi) = A(\rho'; b, a; \xi)A(\rho; a, b; \xi)$.

An analogue of property f) with operators B is not true. Note that (3) with $\Phi_r(x) = \bar{x}^{\rho}$, $\Phi_i(x) \equiv 0$ for $i < r$ reduces to $A(\rho)$. In the next theorem we show how combinations of operators of type A and B can "clear" singularities of the weights φ and w in $K(f, t^r; L_p(w), AC_{loc}^{r-1}, \varphi^r D^r)$. The boundedness properties of these operators are governed by Hardy's inequality and its limitations reflects the restrictions on κ's and λ's.

Theorem 2. *Let $r \in \mathbb{N}$, $1 \le p \le \infty$, $\varphi(x) = (x-a)^{\lambda_a}(b-x)^{\lambda_b}$ with $\lambda_a, \lambda_b < 1$, $w(x) = (x-a)^{\kappa_a}(b-x)^{\kappa_b}$ with $-1/p < \kappa_a, \kappa_b$ if $p < \infty$ and $\kappa_a = \kappa_b = 0$ if $p = \infty$. For fixed $\xi \in (a, b)$, set*

$$\mathcal{B} = A(\rho_b; b, a; \xi)B(\sigma_b; b, a; \xi)A(\rho_a; a, b; \xi)B(\sigma_a; a, b; \xi),$$

$$\rho_a = \frac{\kappa_a + 1/p}{1-\lambda_a} - \frac{1}{p}, \quad \sigma_a = \frac{1}{1-\lambda_a}, \quad \rho_b = \frac{\kappa_b + 1/p}{1-\lambda_b} - \frac{1}{p}, \quad \sigma_b = \frac{1}{1-\lambda_b}.$$

Then, for every $t > 0$ and every $f \in L_p(w)[a, b]$, we have

$$K(f, t^r; L_p(w), AC_{loc}^{r-1}, \varphi^r D^r) \sim K(\mathcal{B}f, t^r; L_p, AC_{loc}^{r-1}, D^r) \sim \omega_r(\mathcal{B}f, t)_p.$$

On can extend the range of κ's and λ's for which statements like Theorem 2 are true by taking $\xi = e$ (the other end-point of the domain than the singularity treated by $A(\rho)$ or $B(\sigma)$). The approach also covers unbounded domains.

As general rules for applying operators $A(\rho)$ and $B(\sigma)$ we may say that:

- the operators allow separate treatment of the singularities at both ends of the domain;
- $A(\rho)$ "clears" a singularity in the weight w leaving φ untouched;
- $B(\sigma)$ "clears" a singularity in the weight φ of $\mathcal{D} = \varphi^r D^r$, but introduces an additional singularity at w.

References

[1] W. Chen, Z. Ditzian, and K. G. Ivanov, Strong converse inequality for the Bernstein-Durrmeyer operator, *J. Approx. Theory* **75**, 1 (1993), 25–43.

[2] Z. Ditzian and V. Totik, "Moduli of Smoothness", SSCM 9, Springer, New York, 1987.

[3] B. Draganov, New moduli for trigonometric polynomial approximation, *East J. Approx.* **8**, 4 (2002), 465–499.

[4] B. Draganov and K. G. Ivanov, A new characterization of weighted Peetre K-functionals, *Constr. Approx.*, submitted (2002).

[5] H. Gonska and X.-l. Zhou, The strong converse inequality for Bernstein-Kantorovich operators, *Comput. Math. Appl.* **30**, 3–6 (1995), 103–128.

[6] K. G. Ivanov, A characterization of weighted Peetre K-functionals, *J. Approx. Theory* **56**, 2 (1989), 185–211.

[7] K. G. Ivanov, A characterization theorem for the K-functional for Kantorovich and Durrmeyer operators, manuscript (2002).

[8] N. X. Ky, A method for characterization of weighted K-functionals, *Ann. Univ. Sci. Budapest.* **38** (1995), 147–152.

Kamen G. Ivanov
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
1113 Sofia
BULGARIA
E-mail: kamen@math.bas.bg