CONSTRUCTIVE THEORY OF FUNCTIONS, Varna 2002
(B. Bojanov, Ed.), DARBA, Sofia, 2003, pp. 327-332.

Constructing Polynomial Surfaces from Vector Subdivision Schemes

Kurt Jetter and Georg Zimmermann

Dedicated to Professor Blagovest Sendov on his seventieth birthday.

For a given matrix mask and the corresponding vector subdivision scheme, we give a recursive algorithm to determine the input polynomial vector sequence generating a polynomial surface.

1. Introduction

We deal with a two-scale equation

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha}} \boldsymbol{A}_{\boldsymbol{\alpha}}\, \boldsymbol{\Phi}(2\boldsymbol{x}-\boldsymbol{\alpha})\,, \tag{1}$$

with $\boldsymbol{\Phi} = (\phi_1, \ldots, \phi_n)' \,:\, \mathbb{R}^d \to \mathbb{R}^n$ a column vector of continuous, compactly supported functions, and $\boldsymbol{A} = (\boldsymbol{A}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{Z}^d}$ a matrix mask, *i.e.*, a matrix sequence of $(n{\times}n)$-matrices. Throughout the paper, $\sum_{\boldsymbol{\alpha}}$ is short for summation over the entire lattice $\mathbb{Z}^d$, but we assume that the mask is finitely supported, *i.e.*, $\boldsymbol{A}_{\boldsymbol{\alpha}} \neq \boldsymbol{0}$ only for finitely many $\boldsymbol{\alpha} \in \mathbb{Z}^d$.

In order to construct, or approximately evaluate, surfaces of type

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha}} \boldsymbol{\Lambda}_{\boldsymbol{\alpha}}\, \boldsymbol{\Phi}(\boldsymbol{x}-\boldsymbol{\alpha}) = \sum_{\boldsymbol{\alpha}} \sum_{i=1}^{n} \lambda_{\boldsymbol{\alpha},i}\, \phi_i(\boldsymbol{x}-\boldsymbol{\alpha}) \tag{2}$$

with

$$\boldsymbol{\Lambda} = (\boldsymbol{\Lambda}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{Z}^d} = (\lambda_{\boldsymbol{\alpha},1}, \cdots, \lambda_{\boldsymbol{\alpha},n})_{\boldsymbol{\alpha}\in\mathbb{Z}^d}$$

a sequence of <u>row</u> vectors, one can use stationary subdivision as follows:

$$\boldsymbol{\Lambda}^{(0)} := \boldsymbol{\Lambda}\,, \qquad \text{and}$$
$$\boldsymbol{\Lambda}^{(k)} := \mathcal{S}\boldsymbol{\Lambda}^{(k-1)}\,, \quad k = 1, 2, \ldots$$

Here, $\mathcal{S} = \mathcal{S}_{\boldsymbol{A}}$ is the subdivision operator mapping an input (row) vector sequence $\boldsymbol{d} = (\boldsymbol{d}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{Z}^d}$ onto the vector sequence $\mathcal{S}\boldsymbol{d}$ according to the rule

$$(\mathcal{S}\boldsymbol{d})_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}} \boldsymbol{d}_{\boldsymbol{\beta}}\, \boldsymbol{A}_{\boldsymbol{\alpha}-2\boldsymbol{\beta}}\,, \quad \boldsymbol{\alpha}\in\mathbb{Z}^d\,.$$

Whence,

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha}} \boldsymbol{\Lambda}^{(k)}_{\boldsymbol{\alpha}}\, \boldsymbol{\Phi}(2^k\boldsymbol{x}-\boldsymbol{\alpha})\,, \quad k=0,1,\dots\,, \tag{3}$$

and the vectors $\boldsymbol{\Lambda}^{(k)} = (\boldsymbol{\Lambda}^{(k)}_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{Z}^d}$ carry information about the function values of f at dyadic scaled lattice points.

In this short note we deal with the generation of polynomial surfaces $f(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{k}}$ in terms of finding the associated input vector sequence $\boldsymbol{\Lambda} =: \boldsymbol{d}^{(\boldsymbol{k})}$ in (2). The problem reduces to the question of finding polynomial eigenvector sequences of the subdivision operator,

$$\mathcal{S}\,\boldsymbol{d}^{(\boldsymbol{k})} = \frac{1}{2^{|\boldsymbol{k}|}}\,\boldsymbol{d}^{(\boldsymbol{k})}\,, \tag{4}$$

where $\boldsymbol{d}^{(\boldsymbol{k})}$ has coordinate degree $\boldsymbol{k}$. It is obvious that the set $L\subset\mathbb{N}^d$ of all such $\boldsymbol{k}$ is of 'lower' type, *i.e.*, $\boldsymbol{k}\in L$ implies $\boldsymbol{k}'\in L$ for all $\boldsymbol{0}\le\boldsymbol{k}'\le\boldsymbol{k}$. Here, and in what follows, we use usual multiindex notation. The motivation for all this comes from the following simple observation: From (3) we see that

$$f(\frac{\boldsymbol{x}}{2}) = \sum_{\boldsymbol{\alpha}} (\mathcal{S}\boldsymbol{\Lambda})_{\boldsymbol{\alpha}}\, \boldsymbol{\Phi}(\boldsymbol{x}-\boldsymbol{\alpha})\,,$$

while (2) and the homogeneity condition for the monomial $f(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{k}}$ yield

$$f(\frac{\boldsymbol{x}}{2}) = \frac{1}{2^{|\boldsymbol{k}|}} \sum_{\boldsymbol{\alpha}} \boldsymbol{\Lambda}_{\boldsymbol{\alpha}}\, \boldsymbol{\Phi}(\boldsymbol{x}-\boldsymbol{\alpha})\,.$$

Thus, at least in case of a 'stable' vector $\boldsymbol{\Phi}$, the homogeneity condition for $\boldsymbol{x}^{\boldsymbol{k}}$ is equivalent to (4). Further motivation may be taken from the analysis of polynomial reproduction as given, *e.g.*, in [4] or [5].

The construction of these polynomial sequences is based on properties of the scaled matrix symbol

$$\boldsymbol{H}(\boldsymbol{\xi}) := \frac{1}{2^d}\widehat{\boldsymbol{A}}(\boldsymbol{\xi}) = \frac{1}{2^d}\sum_{\boldsymbol{\alpha}} \boldsymbol{A}_{\boldsymbol{\alpha}}\, e^{-i\boldsymbol{\alpha}\cdot\boldsymbol{\xi}} \tag{5}$$

which is a multiple of the Fourier transform of the matrix mask. We will also refer to the submasks

$$\boldsymbol{B}_{\boldsymbol{e}} := \sum_{\boldsymbol{\beta}} \boldsymbol{A}_{\boldsymbol{e}-2\boldsymbol{\beta}}\,, \quad \boldsymbol{e}\in E\,, \tag{6}$$

where E denotes the set of the 2^d corners of the unit cube $[0,1]^d$ (which are the canonical representers of the cosets of $\mathbb{Z}^d/2\mathbb{Z}^d$). Throughout the paper, we make the

Standard Assumption: *The common left eigenspace of the submasks $\boldsymbol{B_e}$, $\boldsymbol{e} \in E$, for the common eigenvalue 1, has dimension one.*

Equivalently, the system of equations

$$\boldsymbol{v}\Big(\boldsymbol{H}(\pi\boldsymbol{e}) - \delta_{\boldsymbol{e},\boldsymbol{0}}\,\boldsymbol{I}\Big) = \boldsymbol{0}\,, \quad \boldsymbol{e} \in E\,, \tag{7}$$

has a unique solution $\boldsymbol{v_0} \neq \boldsymbol{0}$, up to a scalar factor. This in turn is equivalent to the fact that the constant sequence $\boldsymbol{d}^{(\boldsymbol{0})} = (\boldsymbol{v_0})_{\boldsymbol{\alpha}\in\mathbb{Z}^d}$ satisfies the eigenvector equation (4) for $\boldsymbol{k} = \boldsymbol{0}$, and we shall see that $\boldsymbol{v_0}$ is the starting vector for a recursive computation of the polynomial solutions for (4).

2. The Algorithm

For the solution of (4), we make the Ansatz

$$\boldsymbol{d}^{(\boldsymbol{k})}_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{0}\le\boldsymbol{j}\le\boldsymbol{k}} \binom{\boldsymbol{k}}{\boldsymbol{j}} \boldsymbol{w}_{\boldsymbol{k},\boldsymbol{j}}\, \boldsymbol{\alpha}^{\boldsymbol{k}-\boldsymbol{j}}\,, \quad \boldsymbol{\alpha} \in \mathbb{Z}^d\,,$$

which in the Fourier transform domain may be expressed as

$$\widehat{\boldsymbol{d}^{(\boldsymbol{k})}}(\boldsymbol{\xi}) = \sum_{\boldsymbol{0}\le\boldsymbol{j}\le\boldsymbol{k}} \binom{\boldsymbol{k}}{\boldsymbol{j}} \boldsymbol{w}_{\boldsymbol{k},\boldsymbol{j}}\; i^{|\boldsymbol{k}-\boldsymbol{j}|}\, \big(D^{\boldsymbol{k}-\boldsymbol{j}}\delta\big)(\boldsymbol{\xi})\,.$$

Here, we have used the fact that the Fourier transform of the monomial sequence $(\boldsymbol{\alpha}^{\boldsymbol{k}-\boldsymbol{j}})_{\boldsymbol{\alpha}\in\mathbb{Z}^d}$ is given by $i^{|\boldsymbol{k}-\boldsymbol{j}|}\, D^{\boldsymbol{k}-\boldsymbol{j}}\delta$. Making use of the identities

$$\big(\widehat{\mathcal{S}_{\boldsymbol{A}}\boldsymbol{d}}\big)(\boldsymbol{\xi}) = \widehat{\boldsymbol{d}}(2\boldsymbol{\xi})\,\widehat{\boldsymbol{A}}(\boldsymbol{\xi})\,,$$

$$\big(D^{\boldsymbol{n}}\delta\big)(2\boldsymbol{\xi}) = 2^{-|\boldsymbol{n}|}\,\frac{1}{2^d}\sum_{\boldsymbol{e}\in E}\big(D^{\boldsymbol{n}}\delta_{\pi\boldsymbol{e}}\big)(\boldsymbol{\xi})\,,$$

$$\big(D^{\boldsymbol{n}}\delta_{\boldsymbol{\xi}_0}\big)(\boldsymbol{\xi})\,f(\boldsymbol{\xi}) = \sum_{\boldsymbol{0}\le\boldsymbol{\nu}\le\boldsymbol{n}}\binom{\boldsymbol{n}}{\boldsymbol{\nu}}(-1)^{|\boldsymbol{n}-\boldsymbol{\nu}|}\,\big(D^{\boldsymbol{n}-\boldsymbol{\nu}}f\big)(\boldsymbol{\xi}_0)\,\big(D^{\boldsymbol{\nu}}\delta_{\boldsymbol{\xi}_0}\big)(\boldsymbol{\xi})\,,$$

we obtain

$$\big(\widehat{\mathcal{S}_{\boldsymbol{A}}\boldsymbol{d}^{(\boldsymbol{k})}}\big)(\boldsymbol{\xi}) = 2^{-|\boldsymbol{k}|}\sum_{\boldsymbol{e}\in E}\sum_{\boldsymbol{0}\le\boldsymbol{j}\le\boldsymbol{k}}\binom{\boldsymbol{k}}{\boldsymbol{j}}\,\widetilde{\boldsymbol{w}}_{\boldsymbol{k},\boldsymbol{j},\boldsymbol{e}}\; i^{|\boldsymbol{k}-\boldsymbol{j}|}\,\big(D^{\boldsymbol{k}-\boldsymbol{j}}\delta_{\pi\boldsymbol{e}}\big)(\boldsymbol{\xi})$$

with

$$\widetilde{\boldsymbol{w}}_{\boldsymbol{k},\boldsymbol{j},\boldsymbol{e}} = \sum_{\boldsymbol{0}\le\boldsymbol{\ell}\le\boldsymbol{j}}\binom{\boldsymbol{j}}{\boldsymbol{\ell}}\,2^{|\boldsymbol{j}-\boldsymbol{\ell}|}\,\boldsymbol{w}_{\boldsymbol{k},\boldsymbol{j}-\boldsymbol{\ell}}\; i^{-|\boldsymbol{\ell}|}\,\big(D^{\boldsymbol{\ell}}\boldsymbol{H}\big)(\pi\boldsymbol{e})\,.$$

Thus, (4) is equivalent to the recursive family of linear equation systems

$$\begin{aligned} & \boldsymbol{w}_{\boldsymbol{k},\boldsymbol{j}} \Big(\boldsymbol{H}(\pi \boldsymbol{e}) - \delta_{\boldsymbol{e},\boldsymbol{0}} \frac{1}{2^{|\boldsymbol{j}|}} \boldsymbol{I} \Big) \\ & \qquad = - \sum_{\substack{\boldsymbol{0} \le \boldsymbol{\ell} \le \boldsymbol{j} \\ \boldsymbol{0} \ne \boldsymbol{\ell}}} \binom{\boldsymbol{j}}{\boldsymbol{\ell}} \boldsymbol{w}_{\boldsymbol{k},\boldsymbol{j}-\boldsymbol{\ell}} \, (2i)^{-|\boldsymbol{\ell}|} \, \big(D^{\boldsymbol{\ell}} \boldsymbol{H} \big)(\pi \boldsymbol{e}) \,, \quad \boldsymbol{e} \in E \,, \end{aligned} \tag{8}$$

for $\boldsymbol{0} \le \boldsymbol{j} \le \boldsymbol{k}$. A more detailed derivation (with a slightly modified 'Ansatz') can be found in [6]. Similar ideas can also be found in [7].

Note that these recursions do **not** depend on $\boldsymbol{k}$, except as termination index! In particular, for $\boldsymbol{j} = \boldsymbol{0}$, this is just (7), so by the Standard Assumption, we have (w.l.o.g.) $\boldsymbol{w}_{\boldsymbol{k},\boldsymbol{0}} = \boldsymbol{v}_{\boldsymbol{0}}$. So it makes sense to denote the solutions of (8) by

$$\boldsymbol{v}_{\boldsymbol{j}} = \boldsymbol{w}_{\boldsymbol{k},\boldsymbol{j}} \qquad \text{for all } \boldsymbol{k},$$

and we obtain

$$\boldsymbol{d}^{(\boldsymbol{k})}_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{0} \le \boldsymbol{j} \le \boldsymbol{k}} \binom{\boldsymbol{k}}{\boldsymbol{j}} \boldsymbol{v}_{\boldsymbol{j}} \, \boldsymbol{\alpha}^{\boldsymbol{k}-\boldsymbol{j}} \,, \quad \boldsymbol{\alpha} \in \mathbb{Z}^d \,.$$

Maximal values for $\boldsymbol{k}$ can thus be determined by running the recursion (8) until it stops.

Naturally, the next question is about uniqueness of the $\boldsymbol{v}_{\boldsymbol{j}}$. The Standard Assumption ensures (essential) uniqueness of $\boldsymbol{v}_{\boldsymbol{0}}$, but it might happen that the system (8) has nonunique solutions from some $\boldsymbol{j} \ne \boldsymbol{0}$ on.

One possibility is to replace the Standard Assumption by the somewhat stronger condition that the system of linear equations

$$\boldsymbol{v} \, \boldsymbol{H}(\pi \boldsymbol{e}) = \boldsymbol{0} \,, \quad \boldsymbol{e} \in E \setminus \{\boldsymbol{0}\} \,,$$

has a unique solution $\boldsymbol{v}_{\boldsymbol{0}} \ne \boldsymbol{0}$, up to a scalar factor, and that this solution also satisfies

$$\boldsymbol{v}_{\boldsymbol{0}} \big(\boldsymbol{H}(\boldsymbol{0}) - \boldsymbol{I} \big) = \boldsymbol{0} \,.$$

This ensures that a vector in the joint kernel of the matrices appearing on the left hand side of (8) must be of the form $\lambda \, \boldsymbol{v}_{\boldsymbol{0}}$, with $\lambda = 0$ for $\boldsymbol{j} \ne \boldsymbol{0}$. Therefore, the solution of (8) is unique, if it exists.

Once we have determined the polynomial eigensequence $\boldsymbol{d}^{(\boldsymbol{k})}$, assuming convergence of the subdivision scheme and due to the Standard Assumption, we know that there is a scalar valued function $f_{\boldsymbol{k}}$ such that

$$\lim_{n \to \infty} \big\| f_{\boldsymbol{k}} \big|_n \, \boldsymbol{v}_{\boldsymbol{0}} - \mathcal{S}^n \boldsymbol{d}^{(\boldsymbol{k})} \big\|_\infty = 0 \,. \tag{9}$$

Here, $f_{\boldsymbol{k}}\big|_n$ denotes the sequence $\big(f_{\boldsymbol{k}}(\frac{\boldsymbol{\alpha}}{2^n}) \big)_{\boldsymbol{\alpha} \in \mathbb{Z}^d}$, and the convergence in (9) is uniform on compact sets. Since

$$\mathcal{S}^n \boldsymbol{d}^{(\boldsymbol{k})} = 2^{-|\boldsymbol{k}| n} \, \boldsymbol{d}^{(\boldsymbol{k})} \,,$$

given a dyadic point

$$\boldsymbol{x} = \frac{\boldsymbol{\beta}}{2^m} = \frac{2^{n-m}\boldsymbol{\beta}}{2^n}\,,$$

say, we find

$$\begin{aligned} f_{\boldsymbol{k}}(\boldsymbol{x})\,\boldsymbol{v_0} &= \lim_{n\to\infty} \big(\mathcal{S}^n \boldsymbol{d}^{(\boldsymbol{k})}\big)_{2^{n-m}\boldsymbol{\beta}} \\ &= \lim_{n\to\infty} 2^{-|\boldsymbol{k}|n} \sum_{\boldsymbol{0}\le\boldsymbol{j}\le\boldsymbol{k}} \binom{\boldsymbol{k}}{\boldsymbol{j}}\, \boldsymbol{v_j}\,\big(2^n\,\frac{\boldsymbol{\beta}}{2^m}\big)^{\boldsymbol{k}-\boldsymbol{j}} = \big(\frac{\boldsymbol{\beta}}{2^m}\big)^{\boldsymbol{k}}\,\boldsymbol{v_0}\,, \end{aligned}$$

whence $f_{\boldsymbol{k}}(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{k}}$.

3. An Example

As an example, we start with the refinement mask for the piecewise linear splines on the four-directional mesh as studied, *e.g.*, in [1]. Obviously, this scheme reproduces linear functions, but no polynomials of higher degree. In an attempt to increase the degree of reproduced polynomials, we introduce two parameters ϑ and ω in such a way as to preserve the inherent symmetry of the mask and then run our algorithm in order to determine optimal values for the parameters. The mask with parameters is

$$\begin{pmatrix}
0 & \begin{pmatrix}-\vartheta & 0\\ 0 & 0\end{pmatrix} & 0 & \begin{pmatrix}-\omega & 0\\ 0 & -\vartheta\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & -\vartheta\end{pmatrix} & \begin{pmatrix}-\vartheta & 0\\ 0 & 0\end{pmatrix} & 0 \\
\begin{pmatrix}-\vartheta & 0\\ 0 & 0\end{pmatrix} & 0 & \begin{pmatrix}\vartheta & -\vartheta\\ 0 & -\omega\end{pmatrix} & \begin{pmatrix}0 & -\vartheta\\ 0 & \vartheta\end{pmatrix} & \begin{pmatrix}\vartheta & 0\\ 0 & \vartheta\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & -\omega\end{pmatrix} & \begin{pmatrix}-\vartheta & 0\\ 0 & 0\end{pmatrix} \\
0 & \begin{pmatrix}\vartheta & -\omega\\ 0 & -\vartheta\end{pmatrix} & \begin{pmatrix}0 & \vartheta\\ 0 & \vartheta\end{pmatrix} & \begin{pmatrix}\frac12+\omega & \vartheta\\ 0 & \frac12+\omega\end{pmatrix} & \begin{pmatrix}0 & -\omega\\ 1 & \frac12+\omega\end{pmatrix} & \begin{pmatrix}\vartheta & 0\\ 0 & \vartheta\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & -\vartheta\end{pmatrix} \\
\begin{pmatrix}-\omega & -\vartheta\\ 0 & 0\end{pmatrix} & \begin{pmatrix}0 & \vartheta\\ 0 & -\vartheta\end{pmatrix} & \begin{pmatrix}\frac12+\omega & \frac12+\omega\\ 0 & \vartheta\end{pmatrix} & \begin{pmatrix}1 & \frac12+\omega\\ 0 & \frac12+\omega\end{pmatrix} & \begin{pmatrix}\frac12+\omega & \vartheta\\ 0 & \frac12+\omega\end{pmatrix} & \begin{pmatrix}0 & -\vartheta\\ 0 & \vartheta\end{pmatrix} & \begin{pmatrix}-\omega & 0\\ 0 & -\vartheta\end{pmatrix} \\
\begin{pmatrix}0 & -\vartheta\\ 0 & 0\end{pmatrix} & \begin{pmatrix}\vartheta & \vartheta\\ 0 & 0\end{pmatrix} & \begin{pmatrix}0 & \frac12+\omega\\ 0 & -\omega\end{pmatrix} & \begin{pmatrix}\frac12+\omega & \frac12+\omega\\ 0 & \vartheta\end{pmatrix} & \begin{pmatrix}0 & \vartheta\\ 0 & \vartheta\end{pmatrix} & \begin{pmatrix}\vartheta & -\vartheta\\ 0 & -\omega\end{pmatrix} & 0 \\
\begin{pmatrix}-\vartheta & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}0 & -\omega\\ 0 & 0\end{pmatrix} & \begin{pmatrix}\vartheta & \vartheta\\ 0 & 0\end{pmatrix} & \begin{pmatrix}0 & \vartheta\\ 0 & -\vartheta\end{pmatrix} & \begin{pmatrix}\vartheta & -\omega\\ 0 & -\vartheta\end{pmatrix} & 0 & \begin{pmatrix}-\vartheta & 0\\ 0 & 0\end{pmatrix} \\
0 & \begin{pmatrix}-\vartheta & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}0 & -\vartheta\\ 0 & 0\end{pmatrix} & \begin{pmatrix}-\omega & -\vartheta\\ 0 & 0\end{pmatrix} & 0 & \begin{pmatrix}-\vartheta & 0\\ 0 & 0\end{pmatrix} & 0
\end{pmatrix}
\begin{matrix} \\ \\ \\ \leftarrow \alpha_2{=}0 \\ \\ \\ \\ \end{matrix}\,.$$

$$\uparrow$$
$$\alpha_1{=}0$$

Our algorithm shows that for arbitrary ϑ and ω, this scheme still reproduces linear polynomials; each of the three polynomials of total degree 2 requires $\omega = \frac{1}{16} - 2\,\vartheta$, and under this condition, the algorithm shows that the scheme reproduces polynomials up to total degree 3, but no monomials of higher order. The vectors $\boldsymbol{v_j}$ are

$$\boldsymbol{v_0} = \big(\,1\,,\,1\,\big) \quad \text{and} \quad \boldsymbol{v_j} = \big(\,0\,,\,2^{-|\boldsymbol{j}|}\,\big)\,, \;\; 1 \le |\boldsymbol{j}| \le 3\,,$$

independent of the choice for ϑ. We obtain

$$\boldsymbol{d}_{\boldsymbol{\alpha}}^{(\boldsymbol{k})} = \big(\,\boldsymbol{\alpha}^{\boldsymbol{k}}\,,\,(\boldsymbol{\alpha} + (\tfrac{1}{2},\tfrac{1}{2}))^{\boldsymbol{k}}\,\big) \quad \text{for } 0 \le |\boldsymbol{k}| \le 3\,,$$

since the scheme is interpolating on the set $\mathbb{Z}^d \cup (\mathbb{Z}^d + (\frac{1}{2},\frac{1}{2}))$.

It is worth noting that for $\vartheta = 0$, *i.e.*, $\omega = \frac{1}{16}$, the mask shows a remarkable connection with the "four point scheme" described, *e.g.*, in [2] and [3].

References

[1] C. Conti and K. Jetter, A new subdivision method for bivariate splines on the four-direction mesh, *J. Comput. Appl. Math.* **119** (2000), 81–96.

[2] S. Dubuc, Interpolation through an iterative scheme, *J. Math. Anal. Appl.* **114** (1986), 185–204.

[3] N. Dyn, J. A. Gregory, and D. Levin, A four-point interpolatory subdivision scheme for curve design, *Comput. Aided Geom. Design* **4** (1987), 257–268.

[4] N. Dyn and D. Levin, Matrix subdivision-analysis by factorization, *in* "Approximation Theory" (B. D. Bojanov, Ed.), pp. 187–211, DARBA, Sofia, 2002.

[5] B. Han, Approximation properties and construction of Hermite interpolants and biorthogonal multiwavelets, *J. Approx. Theory* **110** (2001), 18–53.

[6] K. Jetter and G. Zimmermann, Polynomial reproduction in subdivision, *Adv. Comput. Math.*, to appear.

[7] Q.-T. Jiang, Multivariate matrix refinable functions with arbitrary matrix dilation, *Trans. Amer. Math. Soc.* **351** (1999), 2407–2438.

Kurt Jetter, Georg Zimmermann

Institut für Angewandte Mathematik und Statistik
Universität Hohenheim
D–70593 Stuttgart
GERMANY
E-mail: kjetter@uni-hohenheim.de, gzim@uni-hohenheim.de