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On Estimations of the Norms of the Lagrange Interpolation Operators

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Let $C[-1,1]$ be the space of all real-valued continuous functions on $[-1,1]$. For given nodes, we consider the Langrange interpolation operator from $C[-1,1]$ with the supremum norm $\|\cdot\|_\infty$ to $C[-1,1]$ with the I norm $\|\cdot\|_I$, where $\|f\|_I = \sup_{J\subset[-1,1]} |\int_J f(x)\,dx|$, $f \in C[-1,1]$, and J is a subinterval of $[-1,1]$. In this note, some estimations of norms of the Lagrange interpolation operators are shown.

1. Introduction

Polynomial approximation has a long history and lays the foundation of approximation theory. And, it has been furnishing important problems. In this note, a problem on interpolation by polynomials is discussed.

Let $C[-1,1]$ be the space of all real-valued continuous functions on $[-1,1]$ and Π_n the subspace of $C[-1,1]$ which consists of polynomials of degree at most n. For given nodes $-1 \le x_0 < x_1 < \cdots < x_n \le 1$ and any $f(x) \in C[-1,1]$, we define the polynomials

$$\ell_i(x) = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}, \qquad i = 0,\dots,n,$$

and a linear operator from $C[-1,1]$ to $C[-1,1]$ by

$$L_n(f)(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x).$$

$\ell_i(x)$ and L_n are called the *fundamental polynomials* for the nodes $x_0,\dots,x_n$ and the *Lagrange Interpolation Operator*, respectively. If $C[-1,1]$ is endowed with the supremum norm and we consider L_n from $(C[-1,1], \|\cdot\|_\infty)$ to $(C[-1,1], \|\cdot\|_\infty)$, then the following propositions are well-known:

(i) The norm $\|L_n\|_\infty$ equals $\|\sum_{i=0}^{n} |\ell_i(x)|\,\|_\infty$.

(ii) The estimates $\frac{2^n}{4n(n-1)} \le \|L_n\|_\infty \le 2^n$ hold for the nodes $x_i = -1 + \frac{2}{n}i$, $i = 0, 1, \dots, n,$.

(iii) The conjectures of Bernstein and Erdös were proven by Kilgore [2] and de Boor and Pinkus [1] in 1978.

The purpose of this note is to consider L_n from $(C[-1,1], \|\cdot\|_\infty)$ to $(C[-1,1], \|\cdot\|_I)$, where $\|f\|_I = \sup_{J\subset[-1,1]} |\int_J f(x)\,dx|$, $f \in C[-1,1]$, and J is an subinterval of $[-1,1]$. And we shall find nodes such that

$$\|L_n\|_I = \{\|L_n(f)(x)\|_I \,:\, \|f\|_\infty \le 1,\ f \in C[-1,1]\}$$

is as small as possible and shall estimate $\|L_n\|_I$.

2. The Lagrange Interpolation Operators from $(C[-1,1], \|\cdot\|_\infty)$ to $(C[-1,1], \|\cdot\|_I)$

First we present results corresponding to (i) and (ii) in Introduction.

Proposition 1 (Kitahara, Okada, and Sakamori [4, Theorem 7]). *For any given $n+1$ nodes,*

$$\|L_n\|_I = \sup_{J\subset[-1,1]} \sum_{i=0}^{n} \left| \int_J \ell_i(x)\,dx \right|,$$

where J denotes a subinterval of $[-1,1]$.

Proposition 2. *For the nodes $x_i = -1 + \frac{2}{n}i$, $i = 0, 1, \dots, n$,*

$$\frac{2^n}{16n^3} < \|L_n\|_I \le 2^{n+1}.$$

The proof is similar to that of (ii).

Since $L_n(1) = 1$, we clearly have $\|L_n\|_I \ge 2$. In general,

$$\|L_n\|_I = \max_{|\sigma_i|=1, i=0,1,\dots,n} \left\| \sum_{i=0}^{n} \sigma_i \ell_i(x) \right\|_I.$$

By this fact, we can show necessary and sufficient conditions that $\|L_i\|_I = 2$, $i = 1, 2$. The readers can verify the following proposition without difficulty.

Proposition 3. *The following assertions are true:*

(a) *For nodes $-1 \le x_0 < x_1 \le 1$, $\|L_1\|_I = 2$ if and ony if $x_1 - x_0 \ge \frac{1}{2}$ and $0 \le \left|\dfrac{x_0 + x_1}{2}\right| \le -1 + \sqrt{2(x_1 - x_0)}$.*

(b) *For nodes $-1 \le x_0 < x_1 < x_2 \le 1$ with $-x_0 = x_2$ and $x_1 = 0$, $\|L_2\|_I = 2$ if and only if $\frac{1}{\sqrt{3}} \le x_2 \le 1$.*

By Proposition 3, we are led to the problem whether there exist nodes $x_0, \ldots, x_n \in [-1,1]$ satisfying $\|L_n\|_I = 2$ for each $n \in \mathbb{N}$.

It is well-known that, for any function $f \in C[-1,1]$ and any given nodes $-1 \leq x_0 < \cdots < x_n \leq 1$,

$$f(x) - L_n(f)(x) = (x - x_0)\cdots(x - x_n)f[x_0, \ldots, x_n, x], \qquad x \in [-1,1],$$

where $f[x_0, \ldots, x_n, x]$ denotes the divided difference of order $(n+1)$ of f with respect to the points $x_0, \ldots, x_n$ and x. If $f(x)$ is sufficiently smooth and the sign of $f(x) - L_n(f)(x)$ is equal to the sign of $(x - x_0)\cdots(x - x_n)$ on $[-1,1]$, then

$$\|f(x) - L_n(f)(x)\|_I \leq \max_{x \in [-1,1]} |f[x_0, \ldots, x_n, x]| \cdot \|(x - x_0)\cdots(x - x_n)\|_I.$$

Hence, the nodes $x_0, \ldots, x_n$ which satisfy

$$\|(x - x_0)\cdots(x - x_n)\|_I = \inf_{-1 \leq y_0 < y_1 < \cdots < y_n \leq 1} \|(x - y_0)\cdots(x - y_n)\|_I \qquad (1)$$

seem to be nearly optimal. Since the zeros of the Chebyshev polynomial U_{n+1} of degree $(n+1)$ of the second kind satisfy (1) by Theorem B$'$ in Kitahara, Kuri and Sakamori [3], there is possibility that the nodes $u_k = \cos\frac{k\pi}{n+2}$, $k = 1, \ldots, n+1$ are optimal.

For the nodes u_k, $k = 1, \ldots, n+1$, we will give estimations of the I norms of the Lagrange interpolation operators L_n in theoretical and numerical approaches.

2.1. Theoretical Approach

Since $|\int_J \ell_k(x)\,dx| \leq \|\ell_k(x)\|_I$ for all subintervals $J \subset [-1,1]$, the following inequality holds in general:

$$\|L_n\|_I = \sup_{J \subset [-1,1]} \sum_{k=1}^{n+1} \left|\int_J \ell_k(x)\,dx\right| \leq \sum_{k=1}^{n+1} \|\ell_k(x)\|_I.$$

Using this inequality, we get the estimation $\|L_n\|_I < 8$. Before showing this, we need the following two results.

Lemma 1. *For the nodes $u_k = \cos\frac{k}{n+2}\pi$, $k = 1, \ldots, n+1$, which lie at the zeros of U_{n+1}, we have*

$$\|\ell_k(x)\|_I = \int_{u_{k+1}}^{u_{k-1}} \ell_k(x)\,dx, \qquad k = 1, \ldots, n+1,$$

where $u_0 = 1$, $u_{n+2} = -1$.

Proof. For any fixed $k = 1, \ldots, n+1$, we set $S_j = |\int_{u_j}^{u_{j-1}} \ell_k(x)\,dx|$, $j = 1, \ldots, n+2$. If we prove that

$$S_k > S_{k-1} > \cdots > S_1 \tag{2}$$

and

$$S_{k+1} > S_{k+2} > \cdots > S_{n+2}, \tag{3}$$

since $\ell_k(x)$ changes signs only at $u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_{n+1}$ and $\ell_k(x) > 0$, $x \in (u_{k+1}, u_{k-1})$, we would obtain $\|\ell_k(x)\|_I = \int_{u_{k+1}}^{u_{k-1}} \ell_k(x)\,dx$. It is sufficient to show that (2) holds. (3) is verified in an analogous way.

Noting that $\ell_k(x) = c\,\frac{U_{n+1}}{x-u_k}$ for some constant c, we have

$$\begin{aligned} S_j &= \left|\int_{u_j}^{u_{j-1}} \ell_k(x)\,dx\right| = \left| c \int_{\frac{j\pi}{n+2}}^{\frac{j-1}{n+2}\pi} \frac{\sin(n+2)\theta}{\sin\theta\left(\cos\theta - \cos\frac{k\pi}{n+2}\right)}(-\sin\theta)\,d\theta \right| \\ &= \left| c \int_{(j-1)\pi}^{j\pi} \frac{\sin\phi}{\cos\frac{\phi}{n+2} - \cos\frac{k\pi}{n+2}} \cdot \frac{1}{n+2}\,d\phi \right| \qquad ((n+2)\theta = \phi) \\ &= \left| \frac{c}{n+2} \int_0^{\pi} \frac{\sin\xi}{\cos\frac{\xi+(j-1)\pi}{n+2} - \cos\frac{k\pi}{n+2}}\,d\xi \right| \qquad (\phi = \xi + (j-1)\pi). \end{aligned}$$

For $1 \le j \le k-1$, taking into account the estimates $0 < \frac{\xi+(j-1)\pi}{n+2} < \frac{\xi+j\pi}{n+2} < \frac{k\pi}{n+2}$ for all $\xi \in (0,\pi)$, we obtain

$$\left| \frac{c}{n+2} \int_0^{\pi} \frac{\sin\xi}{\cos\frac{\xi+(j-1)\pi}{n+2} - \cos\frac{k\pi}{n+2}}\,d\xi \right| < \left| \frac{c}{n+2} \int_0^{\pi} \frac{\sin\xi}{\cos\frac{\xi+j\pi}{n+2} - \cos\frac{k\pi}{n+2}}\,d\xi \right|. \quad \square$$

Let $p_0(x), \ldots, p_{n+1}(x)$ be a system of orthogonal polynomials on $[a,b]$ with respect to a weight function $w(x)$ and let the interpolation nodes be at the zeros $x_1, \ldots, x_{n+1}$ of $p_{n+1}(x)$. Then, it is well-known that each $\ell_k(x)$, $k = 1, \ldots, n+1$, is expressed as

$$\ell_k(x) = w_k \sum_{j=0}^{n} \frac{p_j(x_k)p_j(x)}{\lambda_j},$$

where $\lambda_j = \int_a^b p_j^2(x)w(x)\,dx$, $\frac{1}{w_k} = \frac{\mu_n}{\mu_{n+1}\lambda_n}\,p_n(x_k)p'_{n+1}(x_k)$ and μ_n denotes the coefficient of x^n in $p_n(x)$ (see Mori [5, p. 147]).

Applying this result to $U_0(x), \ldots, U_{n+1}(x)$, we verify the following without difficulty.

Lemma 2. *If the interpolation nodes coincide with the zeros $u_k = \cos\frac{k\pi}{n+2}$, $k = 1, \ldots, n+1$, of U_{n+1}, then each $\ell_k(x)$, $k = 1, \ldots, n+1$, is expressed as*

$$\ell_k(x) = \frac{2}{n+2} \sin\frac{k\pi}{n+2} \sum_{j=0}^{n} \sin\frac{(j+1)k\pi}{n+2}\,U_j(x).$$

Now we are in position to give an estimation of $\|L_n\|_I$.

Proposition 4. *If the nodes are at the zeros of* U_{n+1}, *then* $\|L_n\| < 8$.

Proof. Noting that

$$\int_{u_{k+1}}^{u_{k-1}} U_j(x)\,dx = \int_{\frac{k-1}{n+2}\pi}^{\frac{k+1}{n+2}\pi} \sin(j+1)\theta\,d\theta = \frac{2}{j+1}\sin\frac{(j+1)k\pi}{n+2}\sin\frac{(j+1)\pi}{n+2},$$

from Lemma 1 and Lemma 2, we obtain, for each $k = 1, \dots, n+1$,

$$\|\ell_k\|_I = \int_{u_{k+1}}^{u_{k-1}} \ell_k(x)\,dx = \frac{4}{n+2}\sin\frac{k\pi}{n+2}\sum_{j=0}^{n}\frac{1}{j+1}\sin^2\frac{(j+1)k\pi}{n+2}\sin\frac{(j+1)\pi}{n+2}.$$

Hence, we have

$$\|L_n\|_I \le \sum_{k=1}^{n+1}\|\ell_k\|_I = \frac{4}{n+2}\sum_{k=1}^{n+1}\sin\frac{k\pi}{n+2}\sum_{j=0}^{n}\frac{1}{j+1}\sin^2\frac{(j+1)k\pi}{n+2}\sin\frac{(j+1)\pi}{n+2}.$$

Furthermore, using

$$\frac{4}{n+2}\sum_{k=1}^{n+1}\sin\frac{k\pi}{n+2} = \frac{4}{n+2}\cdot\frac{\sin\frac{\pi}{n+2}}{1-\cos\frac{\pi}{n+2}} = \frac{4}{n+2}\cdot\frac{1+\cos\frac{\pi}{n+2}}{\sin\frac{\pi}{n+2}}$$

and

$$\begin{aligned}\sum_{j=0}^{n}\frac{1}{j+1}\sin^2\frac{(j+1)k\pi}{n+2}\sin\frac{(j+1)\pi}{n+2} &< \sum_{j=0}^{n}\frac{1}{j+1}\sin\frac{(j+1)\pi}{n+2}\\ &< \sum_{j=0}^{n}\frac{1}{j+1}\cdot\frac{(j+1)\pi}{n+2} < \pi,\end{aligned}$$

we obtain

$$\|L_n\|_I < \frac{4\pi}{n+2}\cdot\frac{1+\cos\frac{\pi}{n+2}}{\sin\frac{\pi}{n+2}} < 8,$$

because the function $g(\theta) = \theta\,\frac{1+\cos\theta}{\sin\theta}$ is monotone decreasing on $(0, \frac{\pi}{2}]$ and $\lim_{\theta\to+0} g(\theta) = 2$. □

Remark. Using *Maple V*, we get numerically

$$\frac{4}{n+2}\sum_{k=1}^{n+1}\sin\frac{k\pi}{n+2}\sum_{j=0}^{n}\frac{1}{j+1}\sin\frac{(j+1)k\pi}{n+2}\sin\frac{(j+1)\pi}{n+2} < 2.4,$$

for $n = 5, 10, 20, 30, 40, 50$. This makes us think that there must be much better estimations of $\sum_{k=1}^{n+1}\|\ell_k(x)\|_I$ than that of Proposition 4.

2.2. Numerical Approach

By using *Maple V*, we obtain approximate values of $\|L_n\|_I$ with respect to two types of nodes: the equidistant points and the zeros of the Chebyshev polynomials of the second kind. The numerical results are given in the table below.

number of nodes	equidistance	Chebyshev 2nd
5	7.6	2
10	55.3381	2
15	2135.16	2
20	24648.7	2

Finally, we state the following problem.

Problem 1. *Let n be any nonnegative integer and let the zeros of $U_{n+1}(x)$ be the nodes. Is it true that $\|L_n\|_I = 2$?*

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References

[1] C. de Boor and A. Pinkus, Proof of the conjectures of Bernstein and Erdös concerning the optimal nodes for polynomial interpolation, *J. Approx. Theory* **24** (1978), 289–303.

[2] T. A. Kilgore, A characterization of the Lagrange interpolating projection with minimal Tchebycheff norm, *J. Approx. Theory* **24** (1978), 273–288.

[3] K. Kitahara, K. Kuri, and Y. Sakamori, On a characterization of Chebyshev polynomial, *East J. Approx.* **4** (1998), 565–568.

[4] K. Kitahara, T. Okada, and Y. Sakamori, Some problems on polynomial approximation, *in* "Unsolved Problems on Mathematics for the 21st Century" (J. M. Abe and S. Tanaka, Eds.), pp. 211–221, IOS Press, Amsterdam, 2001.

[5] M. Mori, "Suchi Kaiseki" 2nd Ed. ("Numerical Analysis" 2nd Ed.), Kyouritu Publ. Co., Tokyo, 2002. [In Japanese]

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