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A Weighted Markov's Inequality on the Real Line

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We present an exact Markov's inequality for the k-th derivative of a weighted polynomial of the form $u(x) = e^{-x^2}p(x)$, where $p(x)$ is an algebraic polynomial having only real zeros.

1. Introduction

We consider Markov type inequalities for weighted polynomials on the real line of the form

$$\|(wp)'\| \le c\,\|wp\|,$$

where w is a weight on $\mathbb{R}$, p is a polynomial of degree not exceeding n, c is a constant and $\|\cdot\|$ is the sup-norm on $\mathbb{R}$.

An exact Markov's inequality for the weight $\mu(x) = e^{-x^2}$ was proved by Li, Mohapatra, and Rodrigues [2]. Their result is as follows:

If p is an algebraic polynomial of degree not exceeding n, then

$$\|(\mu p)'\| \le \|(\mu \hat{T}_n)'\| \cdot \|\mu p\|,$$

where $\hat{T}_n$ is the weighted Chebyshev polynomial, normalized by the condition $\|\mu \hat{T}_n\| = 1$.

In [3] we proved that the same inequality holds for any k-th derivative, provided that the polynomial has only real zeros.

Our approach is based on a method for derivation of estimates for functionals in the set of algebraic polynomials, having only real zeros, proposed by Bojanov and Rahman [1].

In this note we summarize the main results of [3].

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2. Main Results

Let $\mathcal{U}_n$ be the set of all generalized polynomials of the form

$$u(x) = \mu(x) p_n(x),$$

where $p_n(x)$ is an algebraic polynomial of degree n, which has n simple zeros in $(-\infty, \infty)$.

First, we consider a problem about interpolation at extremal points for polynomials from $\mathcal{U}_n$. We prove the following.

Theorem 1. *Given positive numbers $h_0, \dots, h_n$, there exist a unique polynomial $u \in \mathcal{U}_n$ and a unique set of points $t_0 < \cdots < t_n$ such that*

$$\begin{aligned} u(t_k) &= (-1)^{n-k} h_k, \quad \text{for } k = 0, \dots, n, \\ u'(t_k) &= 0, \qquad\qquad\quad \text{for } k = 0, \dots, n. \end{aligned} \tag{1}$$

Since every $u \in \mathcal{U}_n$ has exactly $n+1$ extremal points $t_0 < \cdots < t_n$, Theorem 1 shows that the parameters $h_i(u) := |u(t_i)|$, $i = 0, \dots, n$, determine u uniquely (up to multiplication by -1).

Next we investigate the polynomials from $\mathcal{U}_n$ depending on their local extrema $h_0, \dots, h_n$.

Let

$$H = \{\mathbf{h} = (h_0, \dots, h_n) : h_i > 0 \ , \ i = 0, \dots, n\}.$$

Given $\mathbf{h} \in H$, we shall denote by

$$u(\mathbf{h}; x) = c(\mathbf{h}) \mu(x) \prod_{j=1}^{n} (x - x_j(\mathbf{h})) \ \in \mathcal{U}_n$$

the unique solution of the problem (1).

We consider (1) as a system of $2n+2$ equations

$$F_i(\mathbf{h}; X) = 0, \qquad i = 1, \dots, 2n+2, \tag{2}$$

in $2n+2$ unknowns

$$X = (c, x_1, \dots, x_n, t_0, t_1, \dots, t_n).$$

We order the equations in the following way:

$$\begin{aligned} F_i(\mathbf{h}; X) &:= u(t_{i-1}) - (-1)^{n-i+1} h_{i-1}, \quad && i = 1, \dots, n+1, \\ F_i(\mathbf{h}; X) &:= u'(t_{i-n-2}), && i = n+2, \dots, 2n+2. \end{aligned}$$

Let $J(\mathbf{h}; X)$ be the Jacobian matrix of the system (2).

Lemma 1. *For every $\mathbf{h} \in H$, $\det J(\mathbf{h}; X) \neq 0$ at any solution of* (2).

The next lemma contains an explicit expression for the derivative of $u'(\mathbf{h};x)$ with respect to h_k.

Let

$$g_k(x) = g_k(\mathbf{h};x) := -\frac{u'(\mathbf{h};x)}{x-t_k}, \qquad k = 0,\dots,n.$$

Lemma 2. *For every* $\mathbf{h} \in H$ *and* $x \in \mathbb{R}$,

$$\frac{\partial}{\partial h_k} u'(\mathbf{h};x) = \frac{g_k'(x)}{|g_k(t_k)|}, \qquad k = 0,\dots,n.$$

In what follows, we shall denote by

$$\xi_0(\mathbf{h}) < \xi_1(\mathbf{h}) < \cdots < \xi_{n+1}(\mathbf{h})$$

the extremal points of $u'(\mathbf{h};x)$.

Lemma 3. *For each* $i = 0,\dots,n+1$ *the quantity* $|u'(\mathbf{h};\xi_i(\mathbf{h}))|$ *is a strictly increasing function of* $h_0,\dots,h_n$ *in* H.

Theorem 2. *Let* u_1 *and* u_2 *be polynomials from* $\mathcal{U}_n$. *Suppose that*

$$0 < h_i(u_1) \le h_i(u_2), \qquad \text{for } i = 0,\dots,n.$$

Then for every natural number k, *the inequalities*

$$0 < h_j(u_1^{(k)}) \le h_j(u_2^{(k)}), \qquad j = 0,\dots,n+k, \tag{3}$$

hold. In particular

$$\|u_1^{(k)}\| \le \|u_2^{(k)}\|. \tag{4}$$

Moreover, the equality in (3) *(for some* j*) and* (4) *is attained if and only if* $h_i(u_1) = h_i(u_2)$ *for all* $i = 0,\dots,n$.

Consequently, the norm of the k-th derivative of a polynomial from $\mathcal{U}_n$ is a strictly increasing function of $h_0,\dots,h_n$.

Let $\overline{\mathcal{U}}_n$ be the set of all generalized polynomials of the form

$$u(x) = \mu(x)p_n(x),$$

where $p_n(x)$ is an algebraic polynomial of n-th degree which has n real (possibly multiple) zeros.

Denote by $u_*(x)$ the polynomial from Theorem 1, corresponding to $\mathbf{h} = (1,1,\dots,1)$. Using Theorem 2 and some variational arguments, we prove the following exact Markov inequality in the set $\overline{\mathcal{U}}_n$.

Theorem 3. *Let* u *belongs to* $\overline{\mathcal{U}}_n$. *Then for every natural number* k *the inequality*

$$\|u^{(k)}\| \le \|{u_*}^{(k)}\| \cdot \|u\|$$

holds. The equality is attained if and only if $u = cu_*$, $c \in \mathbb{R}$.

References

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