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Vector Variants of Some Approximation Theorems of Korovkin and of Sendov and Popov

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Dedicated to Academician Bl. Sendov on his 70th anniversary

1. Introduction. Higher Order Convexity

Let $(E, \langle \cdot, \cdot \rangle)$ be an Euclidean space with the norm denoted by $\| \cdot \|$. Let $I = [a, b]$, $a < b$, be a real interval. Let $C(I, E)$ be the space of continuous functions, endowed with the sup-norm denoted by $\| \cdot \|_I$. If $\varphi : I \to \mathbb{R}$ and $w \in F$, we denote by φw the function $(\varphi w)(x) = \varphi(x) w$. Consider the monomial functions $e_j(x) := x^j,\ x \in I,\ j = 0, 1, \dots$

If $L : C(I, E) \to C(I, E)$ is a linear operator, we denote

$$\|L\| := \sup_{\|f\|_I \le 1} \|L(f)\|_I.$$

For any $f \in C(I, E)$ and any distinct points $y_1, \dots, y_p$ of I, we denote by $[f; y_1, \dots, y_p]$, the divided difference of f at these points.

Definition 1 ([2]). *A function $f : I \to E$ is called* convex of order k, *$k \ge -1$, if for any strictly ordered (increasing or decreasing) points $x_1, \dots, x_{k+3}$ of I we have*

$$\langle [f; x_1, \dots, x_{k+2}], [f; x_2, \dots, x_{k+3}] \rangle \ge 0, \tag{1}$$

or in an equivalent mode,

$$\|[f; x_1, \dots, x_{k+2}] + t[f; x_2, \dots, x_{k+3}]\| \ge \|[f; x_1, \dots, x_{k+2}]\|, \tag{2}$$

for all $t > 0$. Denote by $K^k(I, E)$ the space of convex functions of order k.

Remark ([2]). In the case $E=\mathbb{R}$, $f\in K^k(I,\mathbb{R})$ if and only if f or $-f$ is convex in the usual sense of order k.

This type of convexity is studied in [2]. In this paper we used only the cases $k=-1$ and $k=0$, when the convexity generalizes the positivity and the monotonicity.

Definition 2. *A linear operator $L : C(I,E)\to C^{k+1}(I,E)$, $k\geq -1$, is called* convex of order k, *if for any $f\in C^{k+1}(I,E)$ such that $f^{(k+1)}\in K^{-1}(I,E)$, we have $(L(f))^{(k+1)}\in K^{-1}(I,E)$. In the case $k=-1$ the operator L is called* positive.

2. A Korovkin Type Theorem

Lemma 1. *Let $\{L_n\}_n$ be a sequence of positive linear operators, $L_n : C(I,E)\to C(I,E)$ with the property that*

$$\lim_{n\to\infty}\|L_n(e_jw)-e_jw\|_I=0,\qquad \textit{for all } w\in E,\ \textit{and } j=0,1. \tag{3}$$

Then, there is a positive integer n_0 such that $\|L_n\|\leq 8$, for any $n\geq n_0$.

Proof. Let us denote $\rho := \min\left\{\frac{1}{240},\frac{1}{120}\cdot\frac{b-a}{1+|a|}\right\}$. Since the space E is finite dimensional it follows that the limit in (3) is uniform with respect to w, $\|w\|=1$, and j. In other words, there is a natural number n_0 such that

$$\|L_n(e_jw)-e_jw\|_I<\rho,\qquad \text{for all } w\in E,\ \|w\|=1,\ j=0,1,\ n\geq n_0.$$

Let us fix a natural number $n\geq n_0$. Let $f\in C(I,E)$, $\|f\|_I\leq 1$. Suppose that $\|L_n(f)\|_I\geq 8$. Let $z\in I$ be such that $\|L_n(f,z)\|=\|L_n(f)\|_I$. We distinguish between two cases.

Case 1. For all $x\in I$ we have $\|L_n(f,x)-L_n(f,z)\|<\frac{1}{32}\|L_n(f)\|_I$. It follows that $\|L_n(f,x)-L_n(f,y)\|\leq\frac{1}{16}\|L_n(f)\|_I$, for all $x,y\in I$ and also $\|L_n(f,x)\|\geq\frac{31}{32}\|L_n(f)\|_I$, for all $x\in I$. Define

$$w:=\frac{L_n(f,a)}{\|L_n(f,a)\|},\quad \mu:=\frac{4}{5}\|L_n(f,a)\|,\quad g:=\frac{\mu}{2(b-a)}\big((2b-3a)e_0+e_1\big)w-f.$$

We show that $g\in K^{-1}(I,E)$. Let $x,y\in I$ and $t>0$. Note that $\mu>6$. We have successively:

$$\begin{aligned}\|g(x)+tg(y)\|-\|g(x)\| &\geq t\left(\|g(x)\|-\|g(x)-g(y)\|\right)\\ &\geq t\Big[\mu\Big(1+\frac{x-a}{2(b-a)}\Big)\|w\|-2\|f(x)\|-\|f(y)\|-\mu\,\frac{|y-x|}{2(b-a)}\,\|w\|\Big]\\ &\geq t\Big(\frac{1}{2}\mu-3\Big)\geq 0.\end{aligned}$$

Since L_n is positive it follows that $L_n(g) \in K^{-1}(I,E)$. But, on the other hand,

$$\begin{aligned}
&\|L_n(g,a)+L_n(g,b)\|-\|L_n(g,a)\| \\
&\le \left\|\frac{5}{2}\mu w - L_n(f,a)-L_n(f,b)\right\| - \|\mu w - L_n(f,a)\| \\
&\quad + \mu\Big[\Big(3+\frac{3|a|}{2(b-a)}\Big)\|L_n(e_0 w)-e_0 w\|_I + \frac{3}{2(b-a)}\|L_n(e_1 w)-e_1 w\|_I\Big] \\
&\le \|L_n(f,a)-L_n(f,b)\| - \frac{1}{4}\,\mu + \frac{\mu}{40} \\
&\le \frac{1}{16}\|L_n(f)\|_I - \frac{9}{50}\|L_n(f,a)\| < 0,
\end{aligned}$$

a contradiction.

Case 2. There exists $x \in I$ such that $\|L_n(f,x)-L_n(f,z)\| \ge \frac{1}{32}\|L_n(f)\|_I$. Define

$$\mu := \frac{1}{5}\|L_n(f,x)+4L_n(f,z)\|, \quad w := \frac{1}{5\mu}(L_n(f,x)+4L_n(f,z)), \quad g := \mu e_0 w - f.$$

We have

$$\mu > \frac{4}{5}\|L_n(f,z)\| - \frac{1}{5}\|L_n(f,x)\| \ge \frac{3}{5}\|L_n(f)\|_I \ge \frac{24}{5}.$$

For any $x_1, x_2 \in I$ and $t > 0$ we have:

$$\begin{aligned}
\|g(x_1)+t\,g(x_2)\| - \|g(x_1)\| &\ge t\left(\|g(x_1)\| - \|g(x_1)-g(x_2)\|\right) \\
&\ge t\left(\mu\|w\| - 2\|f(x_1)\| - \|f(x_2)\|\right) \\
&\ge t(\mu-3) \ge 0.
\end{aligned}$$

Hence $g \in K^{-1}(I,E)$. Then $L_n(g) \in K^{-1}(I,E)$. But

$$\begin{aligned}
&\|L_n(g,x)+4L_n(g,x)\| - \|L_n(g,x)\| \\
&\qquad \le 6\mu\|L_n(e_0 w)-e_0 w\|_I - \|\mu w - L_n(f,x)\| \\
&\qquad < 6\mu\rho - \frac{4}{5}\|L_n(f,z)-L_n(f,x)\| \le \|L_n(f)\|_I\Big(6\rho - \frac{1}{40}\Big) \le 0,
\end{aligned}$$

and we arrived again at a contradiction. Therefore $\|L_n(f)\|_I < 8$. The lemma is proved. □

Theorem 1. *Let $\{L_n\}_n$ be a sequence of positive linear operators $L_n : C(I,E) \to C(I,E)$ with the property*

$$\lim_{n\to\infty} \|L_n(e_j w) - e_j w\|_I = 0, \qquad \text{for all } w \in E \text{ and } j = 0,1,2.$$

Then, for any $f \in C(I,E)$ we have

$$\lim_{n\to\infty} \|L_n(f) - f\|_I = 0. \tag{4}$$

Proof. First, we consider the particular case where $f \in C^2(I,E)$. Let n_0 be the integer assured by Lemma 1. Let us fix a point $x \in I$ and set $v := f'(x)$. By Taylor's formula,

$$f(y) = f(x) + (y-x)v + \int_x^y (y-u)f''(u)\,du, \qquad y \in I.$$

Now we choose a number μ such that

$$\mu > \max\Big\{\|f''\|_I,\ \frac{16}{(b-a)^2}\big(11\|f\|_I + (b-a)\|v\| + 1\big)\Big\}.$$

For any $w \in E$ with $\|w\| = 1$ consider the functions

$$h_w := e_0 f(x) + (e_1 - xe_0)v + \mu(e_1 - xe_0)^2 w \qquad \text{and} \qquad g_w := h_w - f.$$

We claim that $g_w \in K^{-1}(I,E)$. Indeed, let $y_1, y_2 \in I$. Consider first the case when $y_1 \neq x$, $y_2 \neq x$. For $i = 1,2$, put

$$T_i := (y_i - x)^{-2} \int_x^{y_i} (y_i - u)f''(u)\,du.$$

Clearly $\|T_i\| \leq \frac{1}{2}\|f''\|_I$. One obtains

$$\begin{aligned}\langle\, g(y_1)\,, g(y_2)\rangle &= (y_1-x)^2(y_2-x)^2\big[\mu^2 - \mu\,(\langle\, w\,, T_1\rangle + \langle\, w\,, T_2\rangle) + \langle\, T_1\,, T_2\rangle\big] \\ &= (y_1-x)^2(y_2-x)^2\Big[\mu^2 - \frac{1}{2}\mu\|f''\|_I - \frac{1}{4}\|f''\|_I^2\Big] \geq 0.\end{aligned}$$

For $y_1 = x$ or $y_2 = x$ the inequality above is immediate. Consequently, $L_n(g_w)$ is positive.

Let $\varepsilon > 0$. From the hypothesis there exists $n_x \in \mathbb{N}$, $n_x \geq n_0$, such that

$$\|L_n(h_w) - h_w\| < \frac{\varepsilon}{4}, \qquad \text{for all } w \in E,\ \|w\| = 1,\ n \geq n_x.$$

Suppose that there is $n \geq n_x$ such that $\|L_n(f,x) - f(x)\| \geq \frac{\varepsilon}{2}$. Consider, for a choice that $x \leq \frac{1}{2}(a+b)$. Choose

$$w := \frac{L_n(f,x) - f(x)}{\|L_n(f,x) - f(x)\|}, \quad \text{and put} \quad \lambda := \frac{\|L_n(f,x) - f(x)\|}{\mu(b-x)^2}.$$

Note that $h_w(x) = f(x)$ and

$$\lambda \leq \frac{\|f\|_I + \|L_n(f)\|_I}{\mu(b-x)^2} \leq \frac{9\,\|f\|_I}{\mu(b-x)^2} < 1.$$

It follows:

$$
\begin{aligned}
&\|L_n(g_w,b)+L_n(g_w,x)\|-\|L_n(g_w,b)\| \\
&\quad\le \|\lambda L_n(g_w,b)+L_n(g_w,x)\|-\lambda\|L_n(g_w,b)\| \\
&\quad\le 2\lambda\|L_n(g_w,b)-g_w(b)\|+\|L_n(h_w,x)-h_w(x)\| \\
&\qquad +\|\lambda g_w(b)-L_n(f,x)+f(x)\|-\lambda\|g_w(b)\| \\
&\quad\le 2\lambda((b-x)\|v\|+\|f(x)\|+\|f(b)\|) \\
&\qquad +2\lambda\big(\|L_n(h_w,b)-h_w(b)\|+\|L_n(f,b)\|+\|f(b)\|\big)+\frac{\varepsilon}{4}-\lambda(b-x)^2\mu \\
&\quad\le 2\lambda\Big(\frac{\varepsilon}{4}+11\|f\|_I+(b-a)\|v\|\Big)+\frac{\varepsilon}{4}-\lambda(b-x)^2\mu \\
&\quad< \frac{\varepsilon}{4}-\frac{1}{2}\lambda(b-x)^2\mu\le 0.
\end{aligned}
$$

The contradiction that we obtained prove that $\|L_n(f,x)-f(x)\|<\frac{\varepsilon}{2}$. From the continuity of the function $L_n(f)-f$ it follows that there is a neighbourhood V_x of x such that

$$\|L_n(f,y)-f(y)\|<\varepsilon, \qquad \text{for all } y\in I\cap V_x,\ n\ge n_x.$$

Since I is compact, there are the points $x_1,\dots,x_m\in I$ such that $I\subset V_{x_1}\cup\dots\cup V_{x_m}$. Set $n_\varepsilon:=\max\{n_{x_1},\dots,n_{x_m}\}$. Then we have $\|L_n(f)-f\|_I<\varepsilon$, for $n\ge n_\varepsilon$.

Consider now the general case, when $f\in C(I,E)$. Let $0<\varepsilon<1$. Choose $\tilde f\in C^2(I,E)$, such that $\|f-\tilde f\|_I<\frac{\varepsilon}{18}$. There is $n_\varepsilon\in\mathbb{N}$, $n_\varepsilon>n_0$, such that $\|L_n(\tilde f)-\tilde f\|<\frac{\varepsilon}{2}$, for $n\ge n_\varepsilon$. Then, for such integers n we have

$$
\begin{aligned}
\|L_n(f)-f\|_I &\le \|L_n(f-\tilde f)\|_I+\|L_n(\tilde f)-\tilde f\|_I+\|\tilde f-f\|_I \\
&\le 9\,\|f-\tilde f\|_I+\|L_n(\tilde f)-\tilde f\|_I<\varepsilon. \quad \square
\end{aligned}
$$

3. Simultaneous Approximation

In this section we give a generalization of the theorem of Sendov and Popov on simultaneous approximation. The idea of the proof is the same as in [1].

Lemma 2. *If* $a,b,v\in E$ *are such that* $\|a\|=\|b\|=\|v\|=1$ *and* $\langle a,b\rangle\le 0$ *then* $\max\{\langle a,v\rangle,\langle b,v\rangle\}\ge-\frac{1}{\sqrt2}$.

Proof. Let $m:=\dim E$. Suppose that $m\ge 2$ and $a\ne -b$, since otherwise, lemma is obvious. Let $\{u_1,\dots,u_m\}$ be an orthonormal base such that $u_1=\frac{a+b}{\|a+b\|}$, $u_2=\frac{a-b}{\|a-b\|}$. We have the representation $v=\sum_{i=1}^m\lambda_iu_i$, where $\sum_{i=1}^m\lambda_i^2=1$.

We get

$$
\begin{aligned}
&\max\{\langle a, v\rangle, \langle b, v\rangle\}\\
&\quad = \max\left\{\frac{\lambda_1(1+\langle a, b\rangle)}{\|a+b\|} + \frac{\lambda_2(1-\langle a, b\rangle)}{\|a-b\|}, \frac{\lambda_1(\langle a, b\rangle+1)}{\|a+b\|} + \frac{\lambda_2(\langle a, b\rangle-1)}{\|a-b\|}\right\}\\
&\quad = \frac{\lambda_1(1+\langle a, b\rangle)}{\|a+b\|} + \frac{|\lambda_2|(1-\langle a, b\rangle)}{\|a-b\|}.
\end{aligned}
$$

In this expression, let consider a, b be fixed and v variable. The minimum value is obtained for $\lambda_1 = -1$ and $\lambda_i = 0$, for $2 \le i \le m$ and it is equal to

$$
-\frac{1+\langle a, b\rangle}{\|a+b\|} = -\frac{1}{\sqrt{2}} \cdot \sqrt{1+\langle a, b\rangle} \ge e - \frac{1}{\sqrt{2}}. \quad \square
$$

Lemma 3. *If the sequence of functions $\{f_n\}_n$, $f_n \in C^1(I, E)$ is uniformly convergent on I to the function $f \in C^1(I, E)$, and if $f_n' \in K^0(I, E)$, $n \in \mathbb{N}$, then for any subinterval $[c, d] \subset (a, b)$, the sequence $\{f_n'\}_n$ is uniformly convergent on $[c, d]$ to f'.*

Proof. Suppose the contrary. Then there are a number $\lambda > 0$, a sequence $(x_k)_k$, $x_k \in [c, d]$ and a subsequence of natural numbers $(n_k)_k$ such that

$$
\|f'(x_k) - (f_{n_k})'(x_k)\| > \lambda, \qquad k \in \mathbb{N}.
$$

There is $\delta_1 > 0$ such that $\|f'(x) - f'(y)\| < \frac{\lambda}{4}$, for all $x, y \in I$, $|x-y| < \delta_1$. Put $\delta := \min\{\delta_1, c-a, b-d\}$ and $\rho := \frac{1}{8}\lambda\delta$. Let k be fixed such that $\|f - f_{n_k}\|_{[a,b]} < \rho$. Put $g := f_{n_k}$ and

$$
T_1 := \int_{x_k-\delta}^{x_k} \big(g'(t) - g'(x_k)\big)\, dt, \qquad T_2 := \int_{x_k}^{x_k+\delta} \big(g'(t) - g'(x_k)\big)\, dt.
$$

Since $g' \in K^0(I, E)$ it follows that $\langle g'(t_1) - g'(x_k), g(t_2) - g(x_k)\rangle \le 0$, for all $t_1 \in [a, x_k)$, $t_2 \in (x_k, b]$. If we approximate the integrals T_1 and T_2 by Riemann sums we get from above that

$$
\langle T_1, T_2\rangle \le 0.
$$

First consider that $T_i \ne 0$, $i = 1, 2$. Define

$$
\alpha := \frac{T_1}{\|T_1\|}, \qquad \beta := \frac{T_2}{\|T_2\|}, \qquad v := \frac{u}{\|u\|},
$$

where $u := \delta\big(g'(x_k) - f'(x_k)\big)$. By Lemma 2 we have $\max\{\langle \alpha, v\rangle, \langle \beta, v\rangle\} \ge$

$-\frac{1}{\sqrt{2}}$. Suppose, for a choice that $\langle \beta , v \rangle \geq -\frac{1}{\sqrt{2}}$. We have successively

$$\|g(x_k+\delta)-f(x_k+\delta)\| = \Big\|g(x_k)+\int_{x_k}^{x_k+\delta} g'(t)\,dt - f(x_k) - \int_{x_k}^{x_k+\delta} f'(t)\,dt\Big\|$$
$$\geq \Big\|\int_{x_k}^{x_k+\delta}(g'(t)-f'(t))\,dt\Big\| - \rho$$
$$\geq \Big\|\int_{x_k}^{x_k+\delta}(g'(t)-f'(x_k))\,dt\Big\| - \Big\|\int_{x_k}^{x_k+\delta}(f'(x_k)-f'(t))\,dt\Big\| - \rho$$
$$\geq \Big\|\int_{x_k}^{x_k+\delta}(g'(t)-f'(x_k))\,dt\Big\| - 3\rho = \|T_2+u\| - 3\rho$$
$$= \sqrt{\|T_2\|^2+\|u\|^2+2<T_2,\, u>} - 3\rho \geq \sqrt{\|T_2\|^2+\|u\|^2-\sqrt{2}\|T_2\|\,\|u\|} - 3\rho$$
$$\geq \frac{1}{\sqrt{2}}\|u\| - 3\rho > \frac{1}{\sqrt{2}}\,\lambda\delta - 3\rho > \rho.$$

Therefore we obtained a contradiction. In the case $T_2 = 0$, then like as above we obtain $\|g(x_k+\delta)-f(x_k+\delta)\| \geq \|u\| - 3\rho > \rho$. Contradiction. Lemma is proved. □

Theorem 2. *Let $\{L_n\}_n$ be a sequence of linear operators $L_n : C(I,E) \to C^{r+1}(I,E)$, $r \geq 1$, with the properties:*

1) L_n are convex of order k for $-1 \leq k \leq r$.

2) $\lim\limits_{n\to\infty} \|L(e_j w) - e_j w\|_{[a,b]} = 0$, for all $w \in E$ and $0 \leq j \leq r+2$.

Then, for any subinterval $[c,d] \subset (a,b)$, we have

$$\lim_{n\to\infty} \|\big(L_n(f)\big)^{(r)} - f^{(r)}\|_{[c,d]} = 0, \qquad \text{for all } f \in C^r(I,E). \tag{5}$$

Proof. Fix a subinterval $[c,d] \subset [a,b]$. For $0 \leq k \leq r$, consider the points $c_k := a + \frac{k}{r}(c-a)$ and $d_k := b - \frac{k}{r}(b-d)$. We prove by induction with respect to $0 \leq k \leq r$ the following relations

$$\lim_{n\to\infty} \|\big(L_n(e_j w)\big)^{(k)} - (e_j w)^{(k)}\|_{[c_k,d_k]} = 0, \tag{6}$$

for all $w \in E$, $0 \leq j \leq r+2$. For $k = 0$ relations (6) are assured by the condition 2) of the theorem. Suppose now that relations (6) are true for $k < r$ and prove it for $k+1$. Fix w and j. One can obtain immediately that $(e_j w)^{(k+2)} \in K^{-1}(I,E)$. Since L_n is an operator which is convex of order $k+1$, then $\big(L_n(e_j w)\big)^{(k+2)} \in K^{-1}(I,E)$. It follows that $\big(L_n(e_j w)\big)^{(k+1)} \in K^0(I,E)$. Indeed, let us denote $g := \big(L_n(e_j w)\big)^{(k+1)}$ and take the points $x_1 < x_2 < x_3$ of I. For any $s \in [x_1,x_2]$ and $t \in [x_2,x_3]$ we have $\langle g'(s), g'(t) \rangle \geq 0$. Then, by approximation of the integrals $\int\limits_{x_1}^{x_2} g'(s)\,ds$ and $\int\limits_{x_2}^{x_3} g'(t)\,dt$ by Riemann sums, we get $\langle \int\limits_{x_1}^{x_2} g'(s)\,ds\,, \int\limits_{x_2}^{x_3} g'(t)\,dt \rangle \geq 0$, that is $\langle g(x_2) - g(x_1)\,, g(x_3) - g(x_2) \rangle \geq 0$. Then using Lemma 3, we get relation (6) for $k+1$.

Now, consider the operators

$$U(g,x) := \int_a^x \frac{(x-t)^{r-1}}{(r-1)!} g(t)\,dt, \;\; g \in C(I,E), \; x \in I; \;\; R_n := (L_n \circ U)^{(r)}, \; n \in \mathbb{N}.$$

If $f \in C^r(I,E)$ and $n \in \mathbb{N}$ we have

$$\big(L_n(f)\big)^{(r)} = \sum_{p=0}^{r-1} \sum_{j=0}^{p} \binom{p}{j} \frac{(-a)^{p-j}}{p!} \big(L_n(e_j f^{(p)}(a))\big)^{(r)} + R_n(f^{(r)}). \tag{7}$$

From relations (7) and (6), (for $k = r$ and $0 \leq j \leq r-1$), it follows that in order to prove the theorem it suffices to show that

$$\lim_{n\to\infty} \|R_n(f^{(r)}) - f^{(r)}\|_{[c,d]} = 0.$$

For this we apply Theorem 1. Since $\big(U(g)\big)^{(r)} = g$ for any $g \in C(I,E)$ and L_n is convex of order $r-1$, it follows that the operator R_n is positive. It remains to show that

$$\lim_{n\to\infty} \|R_n(e_j w) - e_j w\|_{[c,d]} = 0, \qquad \text{for all } w \in E \text{ and } j = 0,1,2. \tag{8}$$

For such $w \in E$ and j we have, after a short calculus, that

$$R_n(e_j w) = \Big[L_n\Big(\Big(\sum_{i=0}^{j} \frac{i!}{(r+i)!} \binom{j}{i} a^{j-i} (e_1 - a e_0)^{r+i}\Big) w\Big)\Big]^{(r)}. \tag{9}$$

Now from (9) we can remark that relations (6) imply relations (8). □

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