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Properties of Discrete Rational Orthonormal Systems

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The Malmquist-Takenaka systems $(\Phi_n, n \in \mathbb{N}^*)$ form an orthonormal system on the unit circle $\mathbb{T}$. The restriction of the finite collection $(\Phi_n, n = 1, \dots, N)$ to a subset $\mathbb{T}_N^a = \{e^{i\tau_1}, \dots, e^{i\tau_N}\}$ of $\mathbb{T}$ is a discrete orthonormal system with respect to the scalar product $\langle\cdot,\cdot\rangle_N$. It is shown that $(e^{i\tau_1}, \dots, e^{i\tau_N})$ is a stationary point for the potential function.

1. Introduction

In control theory the Malmquist-Takenaka systems $(\Phi_n, n \in \mathbb{N}^*)$ [5], are often used to identify the transfer function of the system ([1], [2], [3]). This orthonormal system is generated by a sequence $a = (a_1, a_2, \dots)$ of complex numbers of the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and can be expressed by the Blaschke-functions

$$B_b(z) := \frac{z - b}{1 - \bar{b}z} \qquad (b \in \mathbb{D},\ z \in \mathbb{C}).$$

Namely (see [4]), the systems $\Phi_n,\ (n \in \mathbb{N}^*)$ in question are defined by

$$\Phi_1(z) := \frac{\sqrt{1 - |a_1|^2}}{1 - \overline{a_1}z},$$
$$\Phi_n(z) := \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a_n}z} \prod_{k=1}^{n-1} B_{a_k}(z) \qquad (z \in \mathbb{D},\ n = 2, 3, \dots).$$

The Malmquist-Takenaka functions form an orthonormal system on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, i.e.,

$$\langle \Phi_n, \Phi_m \rangle := \frac{1}{2\pi} \int_0^{2\pi} \Phi_n(e^{it})\, \overline{\Phi_m(e^{it})}\, dt = \delta_{mn},$$

where δ_{mn} is the Kronecker symbol (see [3], [4]).

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In the special case $a_0 = a_1 = \ldots = 0$ we reobtain the trigonometric system. If $a_0 = a_1 = \ldots = a$, then $\Phi_n = L_n^a$ $(n \in \mathbb{N}^*)$ is the discrete Laguerre-system, and if $a_{2k-1} = a,\ a_{2k} = b$ $(k \in N^*)$, then Φ_n $(n \in \mathbb{N}^*)$ is the Kautz-system investigated in [2].

If we have an orthonormal system with respect to a continuous scalar product on the unit circle, in natural way arise the following questions:

- How to choose the nodal points on the unit circle and how to modify the continuous scalar product so that the orthonormal property of the system with respect to the discrete scalar product over the set of nodal points to be valid?
- What kind of properties has this set of nodal points?

The answer of the first question (see [2], [6]) can be summarized in the following way: If $b \in \mathbb{D}$, then B_b is an 1-1 map on $\mathbb{D}$ and on $\mathbb{T}$, respectively. Moreover (see [1]), B_b can be written in the form

$$B_b(e^{it}) = e^{i\beta_b(t)} \qquad (t \in \mathbb{R},\ b = re^{i\tau} \in \mathbb{D}),$$

where

$$\beta_b(t) := \tau + \gamma_s(t-\tau), \qquad \gamma_s(t) := 2\arctan(s \tan \frac{t}{2}), \quad (t \in [-\pi, \pi),\ s := \frac{1+r}{1-r})$$

and γ_s is extended to $\mathbb{R}$ by $\gamma_s(t + 2\pi) = 2\pi + \gamma_s(t)$ $(t \in \mathbb{R})$.

Thus the product $\prod_{j=1}^{N} B_{a_j}$ is of the form

$$\prod_{j=1}^{N} B_{a_j}(e^{it}) = e^{i(\beta_{a_1}(t)+\ldots+\beta_{a_N}(t))}.$$

This implies that the solution of the equation

$$\frac{z - a_1}{1 - \overline{a_1}z} \cdot \frac{z - a_2}{1 - \overline{a_2}z} \cdots \frac{z - a_N}{1 - \overline{a_N}z} = 1$$

can be written as

$$w_k := e^{i\tau_k}, \quad \tau_k := \theta_N^{-1}(2\pi(k-1)/N) \qquad (k = 1, \ldots, N),$$

where θ_N^{-1} is the inverse of the function

$$\theta_N(t) := \frac{1}{N}\big(\beta_{a_1}(t) + \ldots + \beta_{a_N}(t)\big) \qquad (t \in \mathbb{R}).$$

We set

$$\mathbb{T}_N := \mathbb{T}_N^a := \big\{w_k = e^{i\tau_k} : \tau_k = \theta_N^{-1}(2\pi(k-1)/N),\ k = 1, \ldots, N\big\}$$

$(N = 1, 2, \ldots)$ and introduce the weight function ρ_N by

$$\frac{1}{\rho_N(z)} := \sum_{k=1}^{N} \frac{1 - |a_k|^2}{|1 - \overline{a_k}z|^2} \qquad (z \in \mathbb{T},\ N = 1, 2, \ldots).$$

Theorem 1. *The finite collection of Φ_n $(1 \le n \le N)$ form a discrete orthonormal system with respect to the scalar product*

$$\langle F, G\rangle_N := \sum_{z\in\mathbb{T}_N} F(z)\,\overline{G(z)}\,\rho_N(z),$$

namely,

$$\langle \Phi_n, \Phi_m\rangle_N = \delta_{mn} \qquad (1 \le m, n \le N).$$

The second question was studied in [6], where the following equilibrium property was proved: For any complex number $z \in \mathbb{C}$ set $z^* := 1/\overline{z}$ and introduce the polynomials

$$\omega_1 := \prod_{k=1}^{N}(z - a_k), \qquad \omega_2 := \prod_{k=1}^{N}(1 - \overline{a}_k z),$$

$$\omega(z) := \omega_1'(z)\omega_2(z) - \omega_2'(z)\omega_1(z) \qquad (z \in \mathbb{C}).$$

It is clear that ω is a polynomial of degree $2N-2$. It is easy to show (see [6]) that if $c \in \mathbb{C}$ is a root of ω, then c^* is also a root of ω with the same multiplicity. Let us denote by $c_1, c_1^*, \dots, c_s, c_s^*$ the pairwise distinct roots of ω of multiplicity $\nu_1, \nu_1, \dots, \nu_s, \nu_s$, respectively.

Theorem 2. *The numbers $z_n := w_n = e^{i\tau_n} \in \mathbb{T}_N^a$, $\tau_n := \theta_N^{-1}(2\pi(n-1)/N)$ $(n = 1, \dots, N)$ are the solutions of the equilibrium equations*

$$\sum_{k=1, k\neq n}^{N} \frac{1}{z_n - z_k} = \sum_{j=1}^{s}\Big(\frac{\nu_j}{2}\cdot\frac{1}{z_n - c_j} + \frac{\nu_j}{2}\cdot\frac{1}{z_n - c_j^*}\Big) \quad (n = 1, \dots, N). \tag{1}$$

In the special case $a_0 = a_1 = \ldots = 0$ we reobtain the trigonometric system, the set of nodal points $\mathbb{T}_N^a$ is the set of the roots of unity of order N. In this case the equilibrium condition becomes

$$\sum_{k=1, k\neq n}^{N} \frac{1}{x_n - x_k} = \frac{N-1}{2}\cdot\frac{1}{x_n} \qquad (n = 1, \dots, N).$$

This special case can be found in [7, p. 425]. Moreover, for this case the following minimum property is proved: The potential energy

$$W = -\log \prod_{1\le j<k\le N} |e^{i\theta_k} - e^{i\theta_j}|$$

attains its minimum when $e^{i\theta_k}$ $(k = 1, \dots, N)$ are the roots of the equilibrium equation $x^N - 1 = 0$.

These motivate the interest in the examination of the same minimum property of $w_n = e^{i\tau_n} \in \mathbb{T}_N^a,\ \tau_n := \theta_N^{-1}(2\pi(n-1)/N)\ \ (n = 1, \dots, N)$.

In this paper we prove that the point $(e^{i\tau_1}, \dots, e^{i\tau_N})$ is a stationary point for the potential function

$$W(z_1, \dots, z_N) := -\log \frac{\prod_{1\le j<k\le N} |z_j - z_k|}{\prod_{j=1}^{s}\prod_{k=1}^{N} |z_k - c_j|^{\nu_j/2}\, |z_k - \overline{c_j^{-1}}|^{\nu_j/2}}$$

defined on the unit circle.

2. The Main Result

Theorem 3. *The point $(\tau_1, \dots, \tau_N)$ is a stationary point of the potential*

$$W(e^{it_1}, \dots, e^{it_N}) = -\log \frac{\prod_{1\le j<k\le N} |e^{it_j} - e^{it_k}|}{\prod_{j=1}^{s}\prod_{k=1}^{N} |e^{it_k} - c_j|^{\nu_j/2}\, |e^{it_k} - \overline{c_j^{-1}}|^{\nu_j/2}} \tag{2}$$

$((t_1, \dots, t_N) \in \mathbb{R}^N)$, *i.e,*

$$\frac{\partial W(e^{i\tau_1}, \dots, e^{i\tau_N})}{\partial t_n} = 0 \qquad (n = 1, \dots, N).$$

For $a_1 = \dots = a_N = a$ (when we reobtain the discrete Laguerre functions) similar property was proved in [8].

3. Proof

Let us denote $c_j = r_j e^{i\alpha_j}$ and $\rho_j = r_j + 1/r_j$. The potential function defined by (2) can be written in the following form

$$\begin{aligned} W(e^{it_1}, \dots, e^{it_N}) &= -\log \frac{\prod_{1\le j<k\le N} |e^{it_j} - e^{it_k}|}{\prod_{j=1}^{s}\prod_{k=1}^{N} |e^{it_k} - c_j|^{\nu_j/2}\, |e^{it_k} - \overline{c_j^{-1}}|^{\nu_j/2}} \\ &= -\log \frac{\prod_{1\le j<k\le N} |2\sin\frac{t_j - t_k}{2}|}{\prod_{j=1}^{s}\prod_{k=1}^{N} |\rho_j - 2\cos(t_k - \alpha_j)|^{\nu_j/2}}. \end{aligned}$$

$$\frac{\partial W}{\partial t_n} = \sum_{k=1, k\ne n}^{N} \frac{\sin(t_n - t_k)}{4\sin^2\frac{t_n - t_k}{2}} + \sum_{j=1}^{s} \frac{-\nu_j}{2} \cdot \frac{2\sin(t_n - \alpha_j)}{\rho_j - 2\cos(t_n - \alpha_j)}.$$

From the equilibrium conditions (1) we obtain that

$$\sum_{k=1,k\neq n}^{N} \frac{\overline{w_n - w_k}}{|w_n - w_k|^2} = \sum_{j=1}^{s} \frac{\nu_j}{2}\Big(\frac{\overline{w_n - c_j}}{|w_n - c_j|^2} + \frac{\overline{w_n - c_j^*}}{|w_n - c_j^*|^2}\Big).$$

This means that

$$\sum_{k=1,k\neq n}^{N} \frac{\cos\tau_n - \cos\tau_k}{|w_n - w_k|^2} = \sum_{j=1}^{s} \frac{\nu_j}{2}\Big(\frac{\cos\tau_n - r_j\cos\alpha_j}{|w_n - c_j|^2} + \frac{\cos\tau_n - (\cos\alpha_j)/r_j}{|w_n - c_j^*|^2}\Big)$$

and

$$\sum_{k=1,k\neq n}^{N} \frac{\sin\tau_n - \sin\tau_k}{|w_n - w_k|^2} = \sum_{j=1}^{s} \frac{\nu_j}{2}\Big(\frac{\sin\tau_n - r_j\sin\alpha_j}{|w_n - c_j|^2} + \frac{\sin\tau_n - (\sin\alpha_j)/r_j}{|w_n - c_j^*|^2}\Big).$$

Multiplying the first equality by $\sin\tau_n$, the second one by $\cos\tau_n$ and taking the difference we get

$$\sum_{k=1,k\neq n}^{N} \frac{\sin(\tau_n - \tau_k)}{|w_n - w_k|^2} = \sum_{j=1}^{s} \frac{\nu_j}{2}\Big(\frac{r_j\sin(\tau_n - \alpha_j)}{|w_n - c_j|^2} + \frac{\sin(\tau_n - \alpha_j)}{r_j|w_n - c_j^*|^2}\Big). \tag{3}$$

The right-hand side of this equality can be written in the following form

$$\begin{aligned}\sum_{j=1}^{s} \frac{\nu_j}{2}&\Big(\frac{r_j\sin(\tau_n - \alpha_j)}{|w_n - c_j|^2} + \frac{\sin(\tau_n - \alpha_j)}{r_j|w_n - c_j^*|^2}\Big)\\ &= \sum_{j=1}^{s} \frac{\nu_j}{2}\sin(\tau_n - \alpha_j)\,\frac{r_j|w_n - c_j^*|^2 + |w_n - c_j|^2/r_j}{\big(\rho_j - 2\cos(\tau_n - \alpha_j)\big)^2}\\ &= \sum_{j=1}^{s} \frac{\nu_j}{2}\sin(\tau_n - \alpha_j)\,\frac{2\big(\rho_j - 2\cos(\tau_n - \alpha_j)\big)}{\big(\rho_j - 2\cos(\tau_n - \alpha_j)\big)^2}\\ &= \sum_{j=1}^{s} \frac{\nu_j}{2}\,\sin(\tau_n - \alpha_j)\,\frac{2}{\rho_j - 2\cos(\tau_n - \alpha_j)}.\end{aligned} \tag{4}$$

From (3) and (4) we obtain that

$$\sum_{k=1,k\neq n}^{N} \frac{\sin(\tau_n - \tau_k)}{4\sin^2\frac{\tau_n - \tau_k}{2}} = \sum_{j=1}^{s} \frac{\nu_j}{2}\cdot\frac{2\sin(\tau_n - \alpha_j)}{\big(\rho_j - 2\cos(\tau_n - \alpha_j)\big)},$$

and consequently

$$\frac{\partial W(\tau_1,\ldots,\tau_N)}{\partial t_n} = \sum_{k=1,k\neq n}^{N} \frac{\sin(\tau_n - \tau_k)}{4\sin^2\frac{\tau_n - \tau_k}{2}} + \sum_{j=1}^{s} \frac{-\nu_j}{2}\cdot\frac{2\sin(\tau_n - \alpha_j)}{\rho_j - 2\cos(\tau_n - \alpha_j)} = 0.$$

In a natural way arises the question whether this stationary point is a minimum point for the potential function?

Problem. *Let* $c_j \in \mathbb{D}$*,* $j = 1, \dots, s$*,* $N \in \mathbb{N}$*,* $N \geq 2$*, be given numbers. If* N *unit masses at the variable point* $z_1, \dots, z_N \in \mathbb{T}$ *and* $2s$ *fixed masses* $q_j = -\nu_j/2$ *at* c_j *and* $1/\overline{c_j^{-1}}$ *are considered, for what position of the points* $z_1, \dots, z_N$ *does the potential function*

$$W(z_1, \dots, z_N) = -\log \frac{\prod\limits_{1 \leq j < k \leq N} |z_j - z_k|}{\prod\limits_{j=1}^{s} \prod\limits_{k=1}^{N} |z_k - c_j|^{\nu_j/2} \, |z_k - \overline{c_j^{-1}}|^{\nu_j/2}}$$

become minimum?

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