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Cubature Formulae for the Sphere and the Ball in $\mathbb{R}^n$

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We construct explicitly the unique cubature for the unit ball in $\mathbb{R}^n$ based on integrals over spheres (balls), centered at the origin, that integrates exactly all m-harmonic functions. We show that there are no cubatures of this type with higher degree of precision. In particular, this gives integration rule for all polynomials in n variables of degree $2m-1$.

1. Introduction

Recent problems from practice require the use of generalizations of the classical quadratures for an interval, based on point evaluations. The interest is focused on explicit construction of multivariate cubatures, based on different type of data available for the recovery (see, for example [4, 5]). Except for a few cases, there are no known explicit formulae that integrate exactly all polynomials in n variables of degree as high as possible. However, in the last few years, an approach that utilizes the theory of polyharmonic functions was used to obtain cubatures that are exact for classes of multivariate algebraic polynomials (see [2, 3, 6]).

A function u, defined on a simply connected domain $D \subset \mathbb{R}^n$, is called a polyharmonic function of order m (or m-harmonic function) (see [1, 7]) if $u \in C^{2m-1}(\bar{D}) \cap C^{2m}(D)$ and it satisfies the equation

$$\Delta^m u(\mathbf{x}) = 0, \quad \mathbf{x} \in D, \quad \text{where} \quad \Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad \Delta^m := \Delta\Delta^{m-1}.$$

In particular, when $m=1$ ($m=2$), u is called harmonic (biharmonic). We denote the set of all m-harmonic functions on the ball with radius r, $B(r) := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < r\}$, by $H^m(B(r))$. Here $|\mathbf{x}| = (\sum_{i=1}^n x_i^2)^{1/2}$, and when $r=1$ we will write B instead of $B(1)$.

Note that every polynomial in n variables of degree $2m-1$ is a polyharmonic function of order m. Thus, any approximation rule that applies to polyharmonic functions would apply to the corresponding set of algebraic polynomials.

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This gives the possibility to exploit the properties of polyharmonic functions for the purpose of studying polynomial approximation problems, especially when the domain of interest is the sphere or the ball.

We investigate cubature of the form

$$\int_{\mathcal{D}(r)} \mu u \approx \sum_{j=0}^{m-1} C_j(r) \int_{\mathcal{D}(r_j)} \mu u, \qquad r \neq r_j,\ j = 0, \dots, m-1, \tag{1}$$

where μ is a weight function and $\mathcal{D}(r)$ is either the sphere or the ball in $\mathbb{R}^n$, centered at the origin with radius $r \in (0,1)$. We say that p is the polyharmonic degree of precision (PDP) of (1) if this cubature is exact for all $u \in H^p(\mathcal{D}(r))$, and p is the biggest number with this property. This notion is a generalization of the notion of algebraic degree of precision (ADP) for classical quadratures.

We construct explicitly the unique formula, that integrates exactly all polyharmonic functions of order m, and prove that there is no formula of this type with precision higher than m.

In the case of integration over a ball, we show a direct relation between (1) (with $r = 1$) and the one-dimensional interval quadrature

$$\int_{-1}^{1} \nu f \approx \sum_{j=0}^{m-1} A_j \int_{-r_j}^{r_j} \nu f \tag{2}$$

with even weight ν. We also prove that there is a unique formula of type (2) that is exact for all polynomials of degree $2m-1$, and show that this is the highest possible precision.

The proofs are based on basic properties of harmonic functions and on the following representation of m-harmonic functions (see [3, Lemma 2]).

Lemma 1. *Let $\phi_0, \dots, \phi_{m-1}$, be a basis in the space of univariate algebraic polynomials of degree $m-1$. For each $u \in H^m(B)$ there exist unique functions $b_0, \dots, b_{m-1}$, each harmonic in B, such that*

$$u(\mathbf{x}) = \sum_{j=0}^{m-1} \phi_j(|\mathbf{x}|^2) b_j(\mathbf{x}), \qquad \mathbf{x} \in B.$$

Aside of the theory of cubatures, formulae (1) are interesting on their own since they can be viewed as extensions of the Pizzetti formula for polyharmonic functions (see [9, 2]),

$$\int_{B(r)} u(\mathbf{x})\, d\mathbf{x} = \pi^{n/2} r^n \sum_{k=0}^{m-1} \frac{r^{2k}}{2^{2k}\Gamma(n/2+k+1)} \cdot \frac{\Delta^k u(0)}{k!},$$

and its analogue on the sphere [7] (here Γ is the Gamma function).

2. Formulae for the Sphere and the Ball

First, we investigate cubature of the form

$$\int_{S(r)} u(\xi)\, d\sigma(\xi) \approx \sum_{j=0}^{m-1} C_j(r) \int_{S(r_j)} u(\xi)\, d\sigma(\xi), \qquad r \in (0,1),\ r \neq r_j, \tag{3}$$

with $0 < r_0 < \cdots < r_{m-1} < 1$ fixed, where $S(r) := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = r\}$ and $d\sigma$ is the $(n-1)$-dimensional surface measure. Observe that (3) is not exact for

$$L(\mathbf{x}) := \prod_{j=0}^{m-1} (|\mathbf{x}|^2 - r_j^2) \in \pi_{2m}(\mathbb{R}^n),$$

and hence the polyharmonic degree of precision of (3) can be at most m.

Next, for every r, we find the unique formula of type (3), that is exact for all $u \in H^m(B(r))$. In particular, this cubature will have ADP $= 2m-1$ (PDP $= m$), and can be viewed as a multidimensional analogue of the Gaussian quadrature in the one-dimensional case. More precisely, the following theorem holds.

Theorem 1. *Given any $0 < r_0 < \cdots < r_{m-1} < 1$ and information $\{\int_{S(r_j)} u(\xi)\, d\sigma(\xi)\}_{j=0}^{m-1}$, for every $0 < r < 1$, $r \neq r_j$, there is a unique cubature formula*

$$\int_{S(r)} u(\xi)\, d\sigma(\xi) \approx \sum_{j=0}^{m-1} A_j(r) \int_{S(r_j)} u(\xi)\, d\sigma(\xi),$$

exact for all $u \in H^m(B(r))$. Its weights are

$$A_j(r) = \frac{r^{n-1}}{r_j^{n-1}} \cdot \frac{\omega(r^2)}{(r^2 - r_j^2)\omega'(r_j^2)}, \qquad \omega(t) := (t - r_0^2) \ldots (t - r_{m-1}^2).$$

An approach similar to the technique in [3] can be applied to construct cubature with the data given being integrals of u and its consecutive normal derivatives $\{\frac{\partial^k u}{\partial \nu^k}\}$, namely

$$\int_{S(r)} u(\xi)\, d\sigma(\xi) \approx \sum_{j=0}^{m-1} \sum_{k=0}^{\nu_j - 1} C_{jk}(r) \int_{S(r_j)} \frac{\partial^k u}{\partial \nu^k}(\xi)\, d\sigma(\xi), \qquad r \in (0,1),$$

for any given multiplicities $\nu_0, \ldots, \nu_{m-1}$.

Next, we consider cubature for the integral over the ball $B(r)$,

$$\int_{B(r)} \mu(|\mathbf{x}|) u(\mathbf{x})\, d\mathbf{x} \approx \sum_{j=0}^{m-1} C_j(r) \int_{B(r_j)} \mu(|\mathbf{x}|) u(\mathbf{x})\, d\mathbf{x}, \tag{4}$$

where the weight $\mu : [0,1] \to \mathbb{R}$ has the property

$$\mu(qt) = q^a \mu(t), \tag{5}$$

for some constant a.

Theorem 2. *Let $0 < r_0 < \cdots < r_{m-1} < 1$ be given radii. For every $0 < r < 1$, $r \neq r_j$, and every weight μ, satisfying* (5)*, there is a unique cubature*

$$\int_{B(r)} \mu(|\mathbf{x}|)u(\mathbf{x})\,d\mathbf{x} \approx \sum_{j=0}^{m-1} A_j(r) \int_{B(r_j)} \mu(|\mathbf{x}|)u(\mathbf{x})\,d\mathbf{x}, \tag{6}$$

exact for all $u \in H^m(B(r))$. Its weights are

$$A_j(r) = (-1)^j \cdot \frac{r^{n+a}}{r_j^{n+a}} \cdot \frac{W_j(r^2, r_0^2, \dots, r_{m-1}^2)}{V(r_0^2, \dots, r_{m-1}^2)},$$

where

$$W_j(r^2, r_0^2, \dots, r_{m-1}^2) = \begin{vmatrix} r_0^2 - r^2 & r_0^4 - r^4 & \dots & r_0^{2(m-1)} - r^{2(m-1)} \\ \dots & \dots & \dots & \dots \\ r_{j-1}^2 - r^2 & r_{j-1}^4 - r^4 & \dots & r_{j-1}^{2(m-1)} - r^{2(m-1)} \\ & & & \\ r_{j+1}^2 - r^2 & r_{j+1}^4 - r^4 & \dots & r_{j+1}^{2(m-1)} - r^{2(m-1)} \\ \dots & \dots & \dots & \dots \\ r_{m-1}^2 - r^2 & r_{m-1}^4 - r^4 & \dots & r_{m-1}^{2(m-1)} - r^{2(m-1)} \end{vmatrix}$$

and V is the Vandermond determinant.

Further, we show that m is the highest possible precision for cubatures (4). To do that, we investigate the univariate interval quadrature (2) and its connection to (4).

3. Quadrature Based on Intervals

Here, we consider quadrature of type (2) with even weight ν (similar formulae were investigated in [4, 8]). The following theorem holds.

Theorem 3. *For every $0 < r_0 < \cdots < r_{m-1} < 1$ and even continuous positive weight ν there is a unique quadrature of type* (2) *with ADP$= 2m - 1$. There is no quadrature of this type with ADP$\geq 2m$.*

A simple relation between the polyharmonic degree of precision of cubature (4) (with $r = 1$) and the algebraic degree of precision of the univariate interval quadrature (2) is given by the next lemma.

Lemma 2. *Cubature* (4) *(with* $r = 1$*) has polyharmonic degree of precision* p *if and only if the quadrature rule*

$$\int_{-1}^{1} \widetilde{\mu}(t) f(t)\,dt \approx \sum_{j=0}^{m-1} C_j(1) \int_{-r_j}^{r_j} \widetilde{\mu}(t) f(t)\,dt,$$

with

$$\widetilde{\mu}(t) := \begin{cases} \mu(t) t^{n-1}, & 0 < t < 1 \\ \mu(-t)|t|^{n-1}, & -1 < t < 0, \end{cases}$$

has algebraic degree of precision $2p - 1$.

Lemma 2 and Theorem 3 show that there is no cubature of type (4) with PDP $> m$, and therefore the explicit formula (6) is the only one with the highest possible degree of precision.

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