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Entropic Schemes for Conservation Laws

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A new class of Godunov-type numerical methods (called here entropic) for solving nonlinear scalar conservation laws is introduced. This new class generalizes from the classical Godunov scheme. Convergence and error estimates for the entropic methods are proved. In the case of one space dimension the projection in an entropic scheme is characterized via approximations from above and below.

1. Introduction

We are interested in the scalar hyperbolic conservation law

$$\begin{cases} u_t + \operatorname{div}_x f(u) = 0, & (x,t) \in \mathbb{R}^d \times (0,\infty) \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where f is a given flux function. In recent years, there has been enormous activity in the development of the mathematical theory and in the construction of numerical methods for (1). Even though the existence-uniqueness theory of weak solutions is complete, there are many numerically efficient methods for which the questions of convergence and error estimates are still open. For example, the original MinMod, UNO, ENO, and WENO methods are known to be numerically robust, at least for piecewise smooth initial data u_0, but theoretical results about convergence are still missing [3, 7, 8, 19].

In this paper, we consider a class of the so-called Godunov-type schemes for solving (1), see [21]. There are two main steps in such schemes: evolution and projection. In the original Godunov scheme, the projection is onto piecewise constant functions – the cell averages. In a general Godunov-type method, the projection is onto piecewise polynomials. To determine the properties of a scheme it is necessary to study the properties of the projection operator. For example, it is important to know whether this operator reproduces polynomials of a given degree, whether it is total variation diminishing (*TVD*) or non-oscillatory. TVD and non-oscillation properties are invariants of the exact solution operator and many numerical methods are build to preserve one of these properties, see [7, 9, 10, 11, 12]. However, none of them is sufficient for convergence of such methods to the *entropy solution*, and more restrictions on the projection step are needed. For example, one can impose the so-called entropy inequalities [1, 18] or require that the projection step is entropy diminishing or entropic [2, 6, 14]. Alternatively, for a convex flux, one can impose one-sided stability on the projection and then prove convergence via Tadmor's Lip$'$ theory [17, 20].

In this paper, we follow the approach in [2, 6, 14]. We consider Godunov-type schemes with entropic projection. We restrict our attention to Godunov-type methods with exact evolution. A convergence result in this case is important since it is a key ingredient in the proof of convergence of the fully discrete schemes. Our main results are an error estimate for Godunov-type schemes with exact evolution and entropic projection step, and a characterization theorem of such schemes via one-sided approximations (from below and above) in the one-dimensional case.

2. Error Estimates for Entropic Schemes

Consider the initial value problem

$$\begin{cases} u_t + \operatorname{div}_x f(u) = 0, & (x,t) \in \mathbb{R}^d \times (0,T) \\ u(x,0) = u_0(x), & u_0 \in L^1(\mathbb{R}^d), \end{cases} \tag{2}$$

where $T > 0$ and f is Lipschitz continuous vector function, i.e., $f \in \operatorname{Lip}(1, L^\infty)$. A function

$$u \in C\big((0,T], L^1(\mathbb{R}^d)\big) := \Big\{ u : \mathbb{R}^d \to \mathbb{R} \mid u(t,\cdot) \in L^1,\ t \in (0,T], \\ \lim_{t' \to t} \|u(t,\cdot) - u(t',\cdot)\|_{L^1} = 0 \Big\}$$

is called the *entropy solution* of (2) if

$$-\int_0^T \int_{\mathbb{R}^d} \Big(|u-c|\varphi_t + \operatorname{sign}(u-c)\,(f(u)-f(c))\operatorname{div}_x\varphi \Big) dxdt \\ + \int_{\mathbb{R}^d} |u(x,T)-c|\varphi(x,T)\,dx - \int_{\mathbb{R}^d} |u_0(x)-c|\varphi(x,0)\,dx \le 0,$$

for all $c \in \mathbb{R}$ and all nonnegative continuously differentiable functions $\varphi = \varphi(x,t)$, compactly supported on $\mathbb{R}^d \times \mathbb{R}_+$. While there can be many weak solutions, it is well-known that the entropy solution of (2) is unique (see [15]). It is also known that if $f \in \operatorname{Lip}(1, L^\infty)$, then the entropy solution of (2) is *total variation diminishing* (TVD), i.e.,

$$|u(\cdot,t)|_{\mathrm{BV}(\mathbb{R}^d)} \le |u_0|_{\mathrm{BV}(\mathbb{R}^d)}, \qquad t > 0,$$

see [22] for a definition and properties of the space $\mathrm{BV}(\mathbb{R}^d)$.

Suppose that u is the entropy solution of (2) corresponding to the initial data $u_0 \in \mathrm{BV}(\mathbb{R}^d)$. Let $N \geq 1$ and $0 = t_0 < \cdots < t_N := T$. Let $v(x,t)$ be a right-continuous function in t such that, for each $n = 0, \ldots, N-1$, v is an entropy solution of

$$\begin{cases} u_t^n + \mathrm{div}_x f(u^n) = 0, & (x,t) \in \mathbb{R}^d \times (t_n, t_{n+1}) \\ u^n(\cdot, t_n) = v(\cdot, t_n), & v(\cdot, t_n) \in L^1(\mathbb{R}^d). \end{cases}$$

Note that v is uniquely determined by the functions $\{v(\cdot, t_n)\}_{n=0}^{N-1}$.

In the original Godunov method, $v(\cdot, t_n)$ is the average of $v(\cdot, t_n^-)$ on a cell I, where $v(\cdot, t_0^-) := u_0$. For a general Godunov-type method, $v(\cdot, t_n)$ is determined from $v(\cdot, t_n^-)$ by $v(\cdot, t_n) := P_h v(\cdot, t_n^-)$, where $P_h : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is a "projection" operator. For a function $g \in L^1(\mathbb{R}^d)$, $P_h g$ is usually a "simpler" function that makes it possible to solve (2) exactly with initial data $P_h g$ for small time. For the sake of simplicity, only regular grids are considered in this paper. That is, I is a generic d-dimensional cube with side $h > 0$, and $\mathcal{D}$ is a partition of $\mathbb{R}^d$. Hence, $|I| = h^d$ and $\cup_{\mathcal{D}} I = \mathbb{R}^d$.

The Godunov-type schemes considered in this paper are such that the projection operator P_h meets the following requirements:

(P1) P_h *is conservative*:

$$\int_I P_h g(x)\, dx = \int_I g(x)\, dx, \qquad g \in L^1(\mathbb{R}^d), \quad I \in \mathcal{D}.$$

(P2) P_h *has the approximation property*: For any $g \in \mathrm{BV}(\mathbb{R}^d)$,

$$\|P_h g - g\|_{L^1(\mathbb{R}^d)} \leq C_0 h |g|_{\mathrm{BV}(\mathbb{R}^d)},$$

where C_0 is a non-negative constant.

(P3) P_h *is entropic*:

$$\int_I \left(|P_h g(x) - l| - |g(x) - l|\right) dx \leq 0, \qquad g \in \mathrm{BV}(\mathbb{R}^d), \tag{3}$$

for all $I \in \mathcal{D}$ and all $l \in \mathbb{R}$.

An example of a projection operator satisfying (P1)-(P3) is the *averaging operator* A_h: for $g \in L^1$, we define $A_h g$ to be the piecewise constant function such that

$$A_h g|_I := \frac{1}{h} \int_I g\, dx, \qquad I \in \mathcal{D}.$$

Definition 1. *A Godunov-type scheme is called* total variation bounded *(TVB) if there exists a constant C_2 such that*

$$|v(\cdot, t_n)|_{\mathrm{BV}(\mathbb{R}^d)} \leq C_1 |u_0|_{\mathrm{BV}(\mathbb{R}^d)}$$

for all $n = 0, \ldots, N$.

With this notation, we have the following result.

Theorem 1. *Let u be the entropy solution of* (2) *with initial condition* $u_0 \in \mathrm{BV}$. *Also, let v be the numerical solution obtained by a TVB Godunov-type method satisfying* (P1)–(P3) *and $hN \le C_2 T$, for an absolute constant C_2. Then*

$$\|v(\cdot,T) - u(\cdot,T)\|_{L^1(\mathbb{R}^d)} \le C\sqrt{h}\,|u_0|_{\mathrm{BV}(\mathbb{R}^d)}$$

where C depends on T and C_i, $0 \le i \le 2$.

Proof. This result in the one-dimensional case follows from the more general result given in [14]. The proof of the theorem in the general case is based on a version of Kuznetsov's error estimates [4, 14, 16, 13]. Using the approach in [4] (see also [14]), we arrive at the following estimate for the Godunov-type method described above:

$$\begin{aligned}&\|v(\cdot,T) - u(\cdot,T)\|_{L^1(\mathbb{R}^d)} \le C\sqrt{h}\,|u_0|_{\mathrm{BV}(\mathbb{R}^d)}\\ &+\sum_{n=1}^{N}\sum_{I\in\mathcal{D}}\int_{\mathbb{R}^d}\int_I c_I(y)\{|P_h v(x,t_n^-) - u(y,t_n)| - |v(x,t_n^-) - u(y,t_n)|\}dxdy,\end{aligned}$$

where C depends on T and C_i, $0 \le i \le 2$, and $c_I(y)$ is a function of I, h and y but independent of x. Using that and (P3), we conclude

$$\begin{aligned}\|v(\cdot,T) \;&-\; u(\cdot,T)\|_{L^1(\mathbb{R}^d)} \le C\sqrt{h}\,|u_0|_{\mathrm{BV}(\mathbb{R}^d)}\\ &+\; \sum_{n=1}^{N}\sum_{I\in\mathcal{D}}\int_{\mathbb{R}^d} c_I(y)\,\sup_{l\in\mathbb{R}}\Big(\int_I\{|P_h v(x,t_n^-) - l| - |v(x,t_n^-) - l|\}dx\Big)dy,\\ &\le\; C\sqrt{h}\,|u_0|_{\mathrm{BV}(\mathbb{R}^d)}.\end{aligned}$$

3. Characterization of Entropic Schemes

In this section, we consider the one-dimensional case ($d = 1$). Let us consider a partition $\mathcal{D} = \cup_j I_j$, where $I_j := [jh, (j+1)h)$, $j \in \mathbb{Z}$ and $h > 0$. In the context of conservation laws, it is reasonable to assume that the projection P_h is a *co-monotone* operator. That is, if g is non-increasing (non-decreasing) on I_j, for some $j \in \mathbb{Z}$, then $P_h g$ is also non-increasing (non-decreasing) on I_j. This property is satisfied for the so-called *non-oscillatory* methods [5, 7, 10, 11, 12] in all intervals but the ones near an extremum. We call P_h entropic on I if (3) holds for that interval and all $l \in \mathbb{R}$. Let G and G_h be the primitive functions of g and $P_h g$ respectively, i.e.,

$$G(x) = \int_{\infty}^{x} g(y)\,dy, \qquad G_h(x) = \int_{\infty}^{x} P_h g(y)\,dy.$$

We have the following characterization of co-monotone entropic projections.

Theorem 2. *Let* $g, P_h g \in \mathrm{BV}(\mathbb{R})$*, and* P_h *be co-monotone and conservative.*

(i) *If* g *is non-decreasing on* I_j*, then* P_h *is entropic on* I_j *if and only if*

$$G(x) \leq G_h(x), \qquad \text{for any } \ x \in I_j.$$

(ii) *If* g *is non-increasing on* I_j*, then* P_h *is entropic on* I_j *if and only if*

$$G(x) \geq G_h(x), \qquad \text{for any } \ x \in I_j.$$

Proof. The *if* direction was proved for linear $P_h g$ in [1], the general case is similar and it also follows from Corollary 1.3 in [4]. The *only if* direction can be proved using standard real analysis arguments using appropriate choices for l in (3).

Remark 1. It can be shown that Theorem 2 holds for any partition $\mathcal{D}$. It will be interesting to see what is the analog of this characterization in the multidimensional case.

Remark 2. In the case when g is not monotone on I_j, the canonical choice for $P_h g$ is the average of g on I_j. That choice was used in [1] to construct an entropic method with linear $P_h g$ on each I_j, $j \in \mathbb{Z}$. Using Theorem 2, it is easy to show that their method is the best possible entropic, co-monotone and conservative method with piecewise linear approximation in the projection step.

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