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Orthogonal Product Systems of Rational Functions

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In a Hilbert space the Gram-Schmidt method can be used to construct orthogonal systems. Orthogonal polynomials, the Franklin system and its generalizations, the Malmquist–Takenaka systems are examples that can be derived this way [1], [3], [11].

Another class of orthogonal systems can be constructed from conditionally orthogonal systems by multiplications [1], [4], [5]. Several classical systems, including the trigonometric system, the Walsh system or the Vilenkin system, character systems of additive and multiplicative groups of local fields [11], [12], Walsh–similar systems recently introduced by Sendov [14], [15], [16] belong to this class. These systems have important theoretical properties that are useful in numerical computations, too. For instance Fourier coefficients and partial sums can be computed by applying fast algorithms similar to FFT [6].

In this paper we investigate product systems generated by Blaschke functions. Discrete rational orthogonal functions of this type are useful in control theory [2].

1. Introduction

In this section we recall some notions and results on *unitary dyadic martingale differences* (shortly: UDMD) systems introduced in [12]. Let as fix a probability space $(X, \mathcal{A}, \mu)$. The *conditional expectation* (CE) of the function f with respect to the sub-σ-algebra $\mathcal{B} \subseteq \mathcal{A}$ is denoted by $E^{\mathcal{B}}f$. The L^q-space of $\mathcal{B}$-measurable functions will be denoted by $L^q(\mathcal{B}) := L^q(X, \mathcal{B}, \mu)$. Instead of $L^q(X, \mathcal{A}, \mu)$ we write L^q. It is well-known that for $1 \le q \le \infty$ the map $L^q \ni f \to E^{\mathcal{B}}f$ is a bounded linear projection onto $L^q(\mathcal{B})$, and $\|E^{\mathcal{B}}f\|_q \le \|f\|_q$. We note that if $\mathcal{B} := \{X, \emptyset\}$ is the trivial σ-algebra, then $E^{\mathcal{B}}f = \int_X f\, d\mu$, i.e., CE is a generalization of the integral (see [17]).

The conditional expectation operator has a simple form if $\mathcal{B}$ is an *atomic σ-algebra*, i.e., if $\mathcal{B}$ is generated by the collection of pairwise disjoint sets:

$$\mathcal{B} := \sigma\{I_j : j = 1, 2, \dots, m\}, \qquad I_i \cap I_j = \emptyset \ (1 \le i < j \le m),\ \cup_{j=1}^m I_j = X.$$

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The sets I_j $(j = 1, \dots, m)$ are called the *atoms* of $\mathcal{B}$ and the $\mathcal{B}$-measurable functions are exactly the step functions, constant on the I_j's. This m-dimensional space coincides with $L^1(\mathcal{B})$. Denote the collection of atoms in $\mathcal{B}$ by $\widehat{\mathcal{B}}$. Then the conditional expectation is of the form

$$(E^{\mathcal{B}} f)(x) = \frac{1}{\mu(I)} \int_I f \, d\mu \qquad (x \in I \in \hat{\mathcal{B}}).$$

In order to get orthonormal product systems we fix a stochastic basis, i.e., an increasing sequence of sub-σ-algebras of $\mathcal{A}$:

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n \subset \cdots \subset \mathcal{A},$$

and a sequence $\Phi = (\phi_k, k \in \mathbb{N}^*)$ $(\mathbb{N}^* := \{1, 2, \dots\})$ of *adapted standardized functions.* This means that the functions ϕ_k are $\mathcal{A}_k$-measurable and

$$E_{k-1}(\phi_k) = 0, \quad E_{k-1}(|\phi_k|^2) = 1 \qquad (k \in \mathbb{N}^*), \tag{1}$$

where E_k denotes the conditional expectation with respect to $\mathcal{A}_k$. In other words (1) means that Φ is a *normalized martingale difference sequence* with respect to the stochastic basis $(\mathcal{A}_k,\ k = 0, 1, \dots)$.

For the definition of *product systems* of Φ we shall use the expansion of natural numbers with respect to the base 2. It is well-known that every number $m \in \mathbb{N} := \{0, 1, \dots\}$ can uniquely be written in the form

$$m = \sum_{k=1}^{\infty} m_k 2^{k-1},$$

where $m_k \in \{0, 1\}$. Then for each $m \in \mathbb{N}$ we define the product

$$\psi_m := \prod_{k=1}^{\infty} \phi_k^{m_k}.$$

The system $\Psi = (\psi_m, m \in \mathbb{N})$ is called the *product system* of the system Φ.

It is known (see [4], [5], [12]) that conditions (1) imply that the product system Ψ is an orthonormal system (ONS) with respect to the scalar product

$$\langle f, g \rangle = \int_X f \overline{g} \, d\mu \qquad (f, g \in L^2).$$

Theorem 1. *Let Ψ be the product system of Φ satisfying* (1). *Then Ψ is an orthonormal system.*

The stochastic basis $(\mathcal{A}_k, k \in \mathbb{N})$ is called *dyadic* if for every k the σ-algebra $\mathcal{A}_k$ is atomic and every atom $I \in \widehat{\mathcal{A}}_k$ can be split into two atoms $I', I'' \in \widehat{\mathcal{A}}_{k+1}$, such that $\mu(I') = \mu(I'')$. If $|\phi_k| = 1$, then the system Φ satisfying (1) is called a system of *unitary dyadic martingale differences*, or UDMD-system. In the dyadic case the atoms I_k^n $(k = 0, 1, \dots, 2^n - 1)$ of $\widehat{\mathcal{A}}_n$ are ordered so that

$$I_k^n = I_{2k}^{n+1} \cup I_{2k+1}^{n+1} \qquad (k = 0, 1, \dots, 2^n - 1,\ n \in \mathbb{N}).$$

The definition of product systems implies that the Dirichlet kernels

$$D_{2^n}(x,t) := \sum_{k=0}^{2^n-1} \psi_k(x)\overline{\psi_k(t)} \qquad (x, t \in X,\ n \in \mathbb{N})$$

of the system Ψ can be written in the product form

$$D_{2^n}(x,t) := \prod_{k=0}^{n-1} (1 + \phi_k(x)\overline{\phi_k(t)}) \qquad (x, t \in X,\ n \in \mathbb{N}).$$

The functions $D_{2^n}(x,t)$ $(x \in X)$ are constant on the atoms of $\widehat{\mathcal{A}}_n$. Furthermore, it turns out (see [12, Theorem 4, pp. 99) that in the case of dyadic stochastic basis we have

$$D_{2^n}(x,t) = 2^n \delta_{\hat{x}\hat{t}},$$

where $\hat{x}$ denotes the atom in $\widehat{\mathcal{A}}_n$ containing the point $x \in X$ and δ_{uv} is the Kronecker symbol.

Introduce the following system of functions:

$$h_0(x) := 1, \qquad h_{2^n+k}(x) := 2^{-n/2}\phi_n(x) D_{2^n}(x, I_k^n),$$

where $0 \le k < 2^n, n \in \mathbb{N}$ and $x \in X$.

The system $H = (h_n, n \in \mathbb{N})$ is called the *Haar-system* generated by the system Φ (see [7]).

It is easy to show that *the Haar-system generated by the UDMD system is an orthonormal system.*

In this paper we investigate dyadic stochastic base and *finite product systems.* We suppose that $\mathcal{A} = \mathcal{A}_{\mathcal{N}}$ is the collection of subsets of X. In this case X has 2^N elements and the Fourier-coefficients with respect to the system Ψ can be written in the form

$$c_k^{\Psi} = \langle f, \psi_k \rangle = 2^{-N} \sum_{x \in X} f(x)\overline{\psi}_k(x) \qquad (k = 0, 1, \dots, 2^N - 1). \tag{2}$$

Furthermore, each function $f : X \to \mathbb{C}$ can be reconstructed from the coefficients $(c_k^{\Psi}, 0 \le k < 2^N)$ by

$$f(x) = \sum_{k=0}^{2^N-1} c_k^{\Psi} \psi_k(x) \qquad (x \in X). \tag{3}$$

In order to compute the Ψ-Fourier coefficients of a function f or to reconstruct f from the c_k^{Ψ}'s by formula (2) and (3) one needs $2^N \cdot 2^N$ multiplications and $2^N(2^N-1)$ additions. In the trigonometric case, there is an algorithm which computes the discrete Fourier coefficients using $O(N2^N)$ algebraic operations (additions or multiplications). This algorithm is called the *Fast Fourier Transform* or, briefly, FFT. It was shown (see [6], [8], [9], [10], [13]) that such an algorithm exists for any Ψ-transform provided Ψ is a product system of systems satisfying (1).

In the case of the Haar-Fourier coefficients

$$c_k^H = \langle f, h_k\rangle = 2^{-N}\sum_{x\in X} f(x)\overline{h}_k(x) \qquad (k=0,1,\dots,2^N-1)$$

can be computed by using $O(2^N)$ operation and the same holds for the reconstruction of function from Haar-Fourier coefficients [10].

2. Rational UDMD Systems

In this section we shall use dyadic stochastic base generated by function systems. Let $S : X \to \mathbb{X}$ be an $\mathcal{A}$-measurable function and let us denote by $\mathcal{B} := \sigma(S)$ the σ-algebra generated by S. Then $\sigma(S)$ is of the form

$$\sigma(S) = \{S^{-1}(H) : H \in \mathcal{A}\},$$

where $S^{-1}(H)$ is the pre image of $H \in \mathcal{A}$. It is known, that the $\mathcal{B}$-measurable functions $f : X \to \mathbb{C}$ are of the form $f = g \circ S$, where $g : X \to \mathbb{C}$ is a Borel-measurable function. If the image of S is finite, then $\mathcal{B}$ is atomic and $\mathcal{B} = \{S^{-1}(y) : y \in Im\, S\}$. Here $S^{-1}(y)$ is the pre image of the singleton $\{y\}$. Especially, if $S_1 = A \circ S$ with the function $A : X \to X$, then $\mathcal{C} := \sigma(S_1) \subseteq \mathcal{B} := \sigma(S)$ and the atoms of $\mathcal{C}$ can be written as union of atoms belonging to $\mathcal{B}$:

$$S_1^{-1}(c) = S^{-1}(A^{-1}(c)) = \bigcup_{b\in Im\, S,\ A(b)=c} S^{-1}(b).$$

Hence, it follows that the conditional expectation $E^{\mathcal{C}}$ of the $\mathcal{B}$ measurable function $f = g \circ S$ at the atom $C = S_1^{-1}(c)$ can be expressed in the form

$$(E^{\mathcal{C}} f)(C) = \frac{1}{\mu(C)} \sum_{b\in Im\, S,\ A(b)=c} g(b)\mu(S^{-1}(b)). \tag{4}$$

In order to define the dyadic stochastic bases we fix a sequence $A_j : X \to X$ $(j = 1, 2, \dots)$ of twofold maps. This means that we suppose that for every $x \in X$ and $j \in \mathbb{N}$ the pre image $A_j^{-1}(x)$ is a set with two elements. Introduce the maps

$$S_k^n := A_{k+1} \circ \cdots \circ A_n \qquad (0 \le k < n, n = 1, 2, \dots), \tag{5}$$

$$S_n^n(x) := x \qquad (x \in X, n \in \mathbb{N}).$$

For a fixed number $y \in X$ set

$$X_n := X_n^y := \{x \in X : S_0^n(x) = y\} \qquad (n \in \mathbb{N}).$$

Then X_n has 2^n elements.

Fix $N \in \mathbb{N}^*$ and for $0 \leq n \leq N$ denote by $\mathcal{A}_n^N$ the σ-algebra generated by the restriction of S_n^N to X_N. On X_N let us introduce the discrete measure defined by $\mu_N(\{x\}) := 2^{-N}$ $(x \in X_N)$. Then $(X, \mathcal{A}_N^N, \mu_N)$ is a probability space. It is easy to check that

$$\mathcal{A}_0^N = \{X_N, \emptyset\} \subset \mathcal{A}_1^N \subset \cdots \subset \mathcal{A}_N^N \tag{6}$$

is a dyadic stochastic basis and $\mu_N(H) = 2^{-n}$ if $H \in \hat{\mathcal{A}}_n^N$. We shall say that the function $g_n : X \to \mathbb{C}$ *is odd with respect to the map* A_n if the condition $A_n(x_1) = A_n(x_2), x_1 \neq x_2$ $(x_1, x_2 \in X)$ implies $g_n(x_1) = -g_n(x_2)$. Moreover, if $|g_n(x)| = 1$ $(x \in X, n \in \mathbb{N}^*)$, *then the functions* $\phi_n^N := g_n \circ S_n^N$ $(n = 1, \ldots, N)$ *form a UDMD system with respect to the stochastic basis* (6). By (5), $S_{n-1}^N = A_n \circ S_n^N$. Consequently, by using (4) for $\phi_n^N = g_n \circ S_n^N$ we get

$$E_{n-1}\phi_n^N = \frac{1}{2} \sum_{A_n(b)=c} g_n(b) = 0.$$

In this section we investigate discrete martingale differences constructed by rational functions. To this end, let us denote by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ the open unit disc. In our construction the Blaschke functions

$$B_b(z) := \frac{z-b}{1-\bar{b}z} \qquad (z \in \mathbb{C})$$

play a basic role. If the parameter b belongs to $\mathbb{D}$, then the restriction of B_b to $\mathbb{D}$ is a bijection of $\mathbb{D}$. Furthermore B_b is a $1-1$ map on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. For every complex number $a \in \mathbb{D}$ we introduce the Blaschke products of order two:

$$A_a(z) := B_a(z)B_{-a}(z) = B_{a^2}(z^2) \qquad (z \in \mathbb{C},\ a \in \mathbb{D}).$$

The map $A_a : \mathbb{T} \to \mathbb{T}$ is twofold and $A_a(z_1) = A_a(z_2)$, $z_1 \neq z_2$ $(z_1, z_2 \in \mathbb{T})$ implies $z_1 = -z_2$. Consequently the identity map $g(z) = z$ is an odd map with respect to A_a. It can be proved that for every sequence $a_n \in \mathbb{D}$ $(n \in \mathbb{N}^*)$ the functions $\phi_n^N := A_{n+1} \circ \cdots \circ A_N$ $(n < N)$ are rational functions with 2^{N-n} poles outside the closed unit disc. Moreover, this is a UDMD system with respect to the stochastic basis (6).

Theorem 2. *The system* $(\phi_n^N, 1 \leq n \leq N)$ *generated by the sequence* $a_n \in \mathbb{D}$ $(n \in \mathbb{N}^*)$ *is a UDMD system of rational functions and consequently the product system* Ψ *is a discrete rational orthonormal system. The Fourier coefficients with respect to the system* Ψ *can be computed by a fast algorithm using* $O(N2^N)$ *operations.*

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