CONSTRUCTIVE THEORY OF FUNCTIONS, Varna 2002
(B. Bojanov, Ed.), DARBA, Sofia, 2003, pp. 397-403.

Majorization of Zeros of Polynomials

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Let f be a monic polynomial of degree n with zeros $z_1, \dots, z_n$. We show that

$$\begin{aligned} \sum_{\nu=1}^{k} \phi\big(|z_\nu|\big) &\leq \phi\big(M_0(f)\big) + (k-1)\phi(1) \qquad (k = 1, \dots, n-1), \\ \sum_{\nu=1}^{n} \phi\big(|z_\nu|\big) &\leq \phi\big(M_0(f)\big) + (n-2)\phi(1) + \phi\big(|f(0)|/M_0(f)\big), \end{aligned}$$

where ϕ is any non-decreasing function such that $\phi \circ \exp$ is convex on $\mathbb{R}$ and $M_0(f)$ is the so-called Mahler measure of f. These inequalities describe a weak majorization. Certain upper bounds for the Mahler measure allow us to establish more explicit results, which are still sharp.

1. Introduction and Statement of Results

First, we introduce the following notation (see [3, p.10]). For any vector $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote by $(x_{[1]}, \dots, x_{[n]})$ a rearrangement of the components of $\boldsymbol{x}$ such that $x_{[1]} \geq \cdots \geq x_{[n]}$. Now we can define the term *majorization* as follows.

Definition 1. *For two vectors $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{b} = (b_1, \dots, b_n)$, we say that $\boldsymbol{a}$ is* weakly majorized *by $\boldsymbol{b}$, and write this as $\boldsymbol{a} \prec_{\mathrm{w}} \boldsymbol{b}$, if*

$$\sum_{j=1}^{k} a_{[j]} \leq \sum_{j=1}^{k} b_{[j]} \qquad (k = 1, \dots, n). \tag{1}$$

Furthermore, we say that $\boldsymbol{a}$ is (strongly) majorized *by $\boldsymbol{b}$, and write this as $\boldsymbol{a} \prec \boldsymbol{b}$, if in* (1) *equality occurs for $k = n$.*

Clearly, the inequalities (1) remain true if, on the left-hand side, we use any other arrangement of the components of $\boldsymbol{a}$.

A result of Weyl (see [8] or [3, p. 116, A.2]) states that, if $\boldsymbol{a} \prec_{\mathrm{w}} \boldsymbol{b}$ and ψ is any non-decreasing convex function on $\mathbb{R}$, then

$$\big(\psi(a_1), \ldots, \psi(a_n)\big) \prec_{\mathrm{w}} \big(\psi(b_1), \ldots, \psi(b_n)\big) .$$

Majorizations have been studied not only in various branches of mathematics, but also in other subjects such as economics (see [3]). They are of interest since they provide some information about the distribution of the components of $\boldsymbol{a}$ as compared to those of $\boldsymbol{b}$. They also imply individual bounds for the components of $\boldsymbol{a}$.

In this paper, we are interested in majorizations for the moduli of the zeros of a polynomial f. Our main result refines and extends Theorem 1 in [5]. We shall use the quantity

$$M_0(f) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\,\mathrm{d}\theta\right),$$

which has sometimes been called the *Mahler measure* of f.

Furthermore, we shall denote by $\mathcal{F}$ the class of all non-decreasing functions $\phi : (0,\infty) \to \mathbb{R}$ for which $\phi \circ \exp$ is convex on $\mathbb{R}$. Examples of functions belonging to $\mathcal{F}$ are $\phi(x) = \log x$, $\phi(x) = \max\{a, \log x\}$ for any $a \in \mathbb{R}$, and $\phi(x) = x^p$ for any $p > 0$.

Theorem 1. *Let f be a monic polynomial of degree n with zeros $z_1, \ldots, z_n$. Then, for any $\phi \in \mathcal{F}$,*

$$\sum_{\nu=1}^{k} \phi\big(|z_\nu|\big) \le \phi\big(M_0(f)\big) + (k-1)\phi(1) \qquad (k = 1, \ldots, n-1),$$

$$\sum_{\nu=1}^{n} \phi\big(|z_\nu|\big) \le \phi\big(M_0(f)\big) + (n-2)\phi(1) + \phi\big(|f(0)|/M_0(f)\big) .$$

Equality is attained throughout when

$$f(z) = (z - z_1)(z - z_n)\prod_{\nu=2}^{n-1}\left(z - \mathrm{e}^{\mathrm{i}\theta_\nu}\right),$$

where $|z_1| \ge 1$, $|z_n| \le 1$, and $\theta_2, \ldots, \theta_{n-1} \in [0, 2\pi)$.

Note that the class of polynomials for which equality is attained is relatively large since it can be described by $n+2$ independent real parameters. Moreover, the proof will show that, for any monic polynomial, there is equality in at least two of the relations when ϕ is the logarithm.

However, from a practical point of view, Theorem 1 may not be easily applicable since, in general, the determination of the Mahler measure $M_0(f)$ can be as difficult as the calculation of the zeros. This is shown by the following observation.

Proposition 1. *For a polynomial f of degree $n \geq 5$, it is in general not possible to express $M_0(f)$ in terms of the coefficients of f by means of a finite number of rational operations and radicals.*

For this reason, it is desirable to have results in terms of the L^2 norm

$$\|f\| := \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(\mathrm{e}^{\mathrm{i}t})\right|^2\,\mathrm{d}t\right)^{1/2} = \left(\sum_{\nu=0}^{n}|a_\nu|^2\right)^{1/2}$$

of a polynomial $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$. From Theorem 1, we can deduce the following statement.

Corollary 1. *Let f be a monic polynomial of degree n with zeros $z_1, \dots, z_n$. Define*

$$N_\pm(f) := \sqrt{\frac{1}{2}\left(\|f\|^2 \pm \sqrt{\|f\|^4 - 4|f(0)|^2}\right)}.$$

Then, for any $\phi \in \mathcal{F}$,

$$\sum_{\nu=1}^{k}\phi\big(|z_\nu|\big) \leq \phi\big(N_+(f)\big) + (k-1)\phi(1) \qquad (k = 1, \dots, n-1),$$

$$\sum_{\nu=1}^{n}\phi\big(|z_\nu|\big) \leq \phi\big(N_+(f)\big) + (n-2)\phi(1) + \phi\big(N_-(f)\big).$$

Equality is attained throughout when $f(z) = z^n + \mathrm{e}^{\mathrm{i}\theta}$, where $\theta \in [0, 2\pi)$.

Finally, we state another consequence of Theorem 1. It is slightly weaker than Corollary 1, but it allows us to interpret the polynomial f as a perturbation of a binomial. As such, the result is a counterpart to Theorem 4 in [5].

Corollary 2. *Let f be a monic polynomial of degree n with zeros $z_1, \dots, z_n$ and let $f_*(z) := z^n + \mathrm{e}^{\mathrm{i}\theta}$, where $\theta \in \mathbb{R}$. Then, for any $\phi \in \mathcal{F}$,*

$$\sum_{\nu=1}^{k}\phi\big(|z_\nu|\big) \leq \phi\big(1+\|f-f_*\|\big) + (k-1)\phi(1) \qquad (k = 1, \dots, n-1),$$

$$\sum_{\nu=1}^{n}\phi\big(|z_\nu|\big) \leq \phi\big(1+\|f-f_*\|\big) + (n-2)\phi(1) + \phi\big(|f(0)|/(1+\|f-f_*\|)\big).$$

Equality is attained throughout when $f = f_*$.

2. Auxiliary Results

It is well known (see, e.g., [1, Theorem 184]) that, for any monic polynomial f, we have $M_0(f) \leq \|f\|$. The following lemma gives a refinement of this inequality.

Lemma 1. *For any monic polynomial f, we have, in the notation of Corollary 1,*

$$M_0(f) \leq N_+(f). \tag{2}$$

Proof. Let $f(z) = \prod_{\nu=1}^{n}(z - z_\nu)$, and let

$$\{1, \dots, n\} = I_1 \cup I_2 \qquad (I_1 \cap I_2 = \emptyset)$$

be any decomposition of the set of indices into two disjoint subsets. Then, by an inequality of Vicente Gonçalves [7],

$$\prod_{\mu \in I_1} |z_\mu|^2 + \prod_{\nu \in I_2} |z_\nu|^2 \leq \|f\|^2, \tag{3}$$

where a product has to be replaced by 1 if the corresponding subset of indices is the empty set. Since

$$M_0(f) = \prod_{\nu=1}^{n} \max\{1, |z_\nu|\} \tag{4}$$

(see [2, p.98, formula(2)] or [4, p.105]), there exists a decomposition such that

$$M_0(f) = \prod_{\mu \in I_1} |z_\mu|.$$

Moreover, $z_1 \cdots z_n = (-1)^n f(0)$. Hence, it follows from (3) that

$$M_0(f)^2 + |f(0)|^2 M_0(f)^{-2} \leq \|f\|^2,$$

that is,

$$M_0(f)^4 - \|f\|^2 M_0(f)^2 + |f(0)|^{-2} \leq 0.$$

This is a quadratic inequality for $M_0(f)^2$, which implies (2). $\square$

Lemma 2. *For $a \in \mathbb{C}$, define*

$$\Phi(x) := \left[x + (1 - |a|)^2\right]\left[x + (1 + |a|)^2\right]$$

and

$$\Psi(x) := \left[2 - 2\Re a + |1 - a|^2 + x + 4\sqrt{x + |1 - a|^2}\right]^2.$$

Then

$$\Phi(x) \leq \Psi(x) \qquad (x \geq 0). \tag{5}$$

Proof. At $x = 0$, we have

$$\begin{aligned}\Psi(0) &= \left[2 - 2\Re a + |1-a|^2 + 4|1-a|\right]^2 = \left[1 - |a|^2 + 2|1-a|^2 + 4|1-a|\right]^2 \\ &= \left(1 - |a|^2\right)^2 + 4\left(|1-a|^2 + 2|1-a|\right)\left(|1-a|^2 + 2|1-a| + 1 - |a|^2\right) \\ &= \Phi(0) + 4\left(|1-a|^2 + 2|1-a|\right)\left[(1 + |1-a|)^2 - |a|^2\right].\end{aligned}$$

Since, by the triangular inequality, $|a| \le 1 + |1-a|$, we see that the term in square brackets is non-negative, and so $\Phi(0) \le \Psi(0)$.

Next, we compare the derivatives of Φ and Ψ. Obviously,

$$\Psi'(x) = 2\left[2 + |1-a|^2 - 2\Re a + x + 4\sqrt{x + |1-a|^2}\right]\left(1 + \frac{2}{\sqrt{x + |1-a|^2}}\right).$$

Since

$$2 - 2\Re a + 2\sqrt{x + |1-a|^2} \ \ge\ 2(1 - \Re a) + 2|1-a| \ \ge\ 0,$$

we conclude that

$$\begin{aligned}\Psi'(x) &\ge 2\left[|1-a|^2 + x + 2\sqrt{x + |1-a|^2}\right]\left(1 + \frac{2}{\sqrt{x + |1-a|^2}}\right) \\ &\ge 2[|1-a|^2 + 2|1-a| + x] + 8 \\ &= 2(|1-a| + 1)^2 + 2x + 6 \ge 2|a|^2 + 2x + 6 = \Phi'(x) + 4.\end{aligned}$$

Altogether, we have shown that $\Phi(0) \le \Psi(0)$ and $\Phi'(x) \le \Psi'(x)$ for $x \ge 0$. This implies that (5) holds. □

Lemma 3. *Let f be a monic polynomial of positive degree and let $f_*(z) := z^n + e^{i\theta}$ for any $\theta \in \mathbb{R}$. Then, in the notation of Corollary 1,*

$$N_+(f) \ \le\ 1 + \|f - f_*\|.$$

Proof. First, we note that

$$\|f - f_*\| = \sqrt{\|f\|^2 - 2\Re a}\,,$$

where $a = f(0)e^{-i\theta}$. Hence we have to show that

$$\left[\frac{1}{2}\left(\|f\|^2 + \sqrt{\|f\|^4 - 4|a|^2}\right)\right]^{1/2} - 1 \ \le\ \sqrt{\|f\|^2 - 2\Re a}\,.$$

By a simple calculation, we find that this inequality is equivalent to

$$\left(\|f\|^2 - 2|a|\right)\left(\|f\|^2 + 2|a|\right) \ \le\ \left[2 + \|f\|^2 - 4\Re a + 4\sqrt{\|f\|^2 - 2\Re a}\right]^2. \qquad (6)$$

Finally, introducing $S := \sum_{\nu=1}^{n-1} |a_\nu|^2$, we have

$$\|f\|^2 = 1 + |a|^2 + S, \qquad \|f\|^2 - 2\Re a = S + |1-a|^2,$$

and so (6) may be rewritten as

$$\left[S+(1-|a|)^2\right]\left[S+(1+|a|)^2\right] \le \left[2-2\Re a+|1-a|^2+S+4\sqrt{S+|1-a|^2}\right]^2,$$

which is true, as a consequence of Lemma 2 and the fact that $S \ge 0$. This completes the proof. □

3. Proofs of the Results

Proof of Theorem 1. Clearly, (4) implies that, for any $k \in \{1,\ldots,n\}$, we have

$$\sum_{j=1}^{k} \log|z_j| \le \sum_{\nu=1}^{n} \log\bigl(\max\{1,|z_\nu|\}\bigr) = \log M_0(f).$$

Moreover,

$$\sum_{j=1}^{n} \log|z_j| = \log|f(0)|.$$

Hence we have the (strong) majorization

$$\bigl(\log|z_1|,\ldots,\log|z_n|\bigr) \prec \bigl(\log M_0(f),0,\ldots,0,\log(|f(0)|/M_0(f)\bigr), \qquad (7)$$

and so the result follows from Weyl's theorem. □

Proof of Proposition 1. With the help of a computer algebra program (such as *Maple*), it is readily verified that the polynomial

$$h(z) := z^5 - 10z^4 + 11z^3 - 10z^2 + 6z - 5$$

has a zero at $z_1 := 8.880\,35\,275\ldots$ and two pairs of conjugate zeros inside the unit circle. Hence, in view of (4),

$$f(z) := (z-1)^{n-5}\,h(z) \qquad (n \ge 5)$$

is a monic polynomial such that $M_0(f) = z_1$. It is shown in [6, pp. 393–394] that the Galois group of $h(-z)$ is the symmetric group. This implies that none of the zeros of h is contained in an extension of the field of rational numbers by radicals. Thus $M_0(f)$ cannot be expressed in terms of the coefficients of f by means of rational operations and radicals. □

Proof of Corollaries 1 and 2. The relation (7) is equivalent to

$$\sum_{j=1}^{k} \log|z_j| \le \log M_0(f) \qquad (k = 1,\ldots,n-1),$$

$$\sum_{j=1}^{n} \log|z_j| = \log|f(0)|.$$

This implies that, if $K \geq M_0(f)$, then also

$$\big(\log|z_1|, \ldots, \log|z_n|\big) \prec \big(\log K, 0, \ldots, 0, \log(|f(0)|/K)\big).$$

According to Lemmas 1 and 3, the numbers $N_+(f)$ and $1 + \|f - f_*\|$ are upper bounds for $M_0(f)$. Using these bounds as K, and applying again Weyl's result, we obtain the conclusions of the corollaries. □

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