Approximate Recovery of Functions and Besov Spaces of Dominating Mixed Smoothness

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We derive an estimate from above for the approximate optimal recovery of bivariate periodic functions taken from a Besov space of dominating mixed smoothness.

1. Approximate Optimal Recovery

We study the effectiveness of the approximation by generalized sampling operators. Let \( F \) be a class of continuous, periodic functions defined on \( T^2 = [0, 2\pi)^2 \). Then, following [15, Chapter 4, Section 5], we consider for fixed \( m, \xi = (\xi^1, \xi^2, \ldots, \xi^m), \xi^j \in T^2, j = 1, \ldots, m \), and \( \psi_1(x_1, x_2), \ldots, \psi_m(x_1, x_2) \) the linear operator

\[
\Psi_m(f, \xi)(x_1, x_2) := \sum_{j=1}^{m} f(\xi^j) \psi_j(x_1, x_2)
\]

and define the quantities

\[
\Psi_m(F, \xi, L^p(T^2)) := \sup_{f \in F} \left\| \Psi_m(f, \xi) - f \right\|_{L^p(T^2)}
\]

and

\[
g_m(F, L^p(T^2)) := \inf_{\psi_1, \ldots, \psi_m} \inf_{\xi} \Psi_m(F, \xi, L^p(T^2)).
\]

Hence \( g_m(F, L^p(T^2)) \) measures the optimal approximate recovery of the functions from \( F \). Here we are interested in the case when \( F \) is the unit ball in a Besov space \( S^{r}_{p,q}B(T^2) \) of dominating mixed smoothness (a definition will be given below). Our main result reads as follows.

**Theorem 1.** Let \( 1 < p < \infty, 1 \leq q \leq \infty, \) and \( r > 1/p \). Let \( F \) be the unit ball in \( S^{r}_{p,q}B(T^2) \). For any natural number \( m \) there exists a system of points \( \xi^1, \ldots, \xi^m \in T^2 \), a collection of trigonometric polynomials \( \psi_1(x_1, x_2), \ldots, \psi_m(x_1, x_2) \), and a constant \( C \) (independent of \( m \)) such that

\[
\sup_{f \in F} \left\| \Psi_m(f, \xi) - f \right\|_{L^p(T^2)} \leq C m^{-r} (\log m)^{r+1-1/q}.
\]

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Remark 1. Our proof will be constructive. The functions $\psi_j(x_1, x_2)$, $j = 1, \ldots, m$, are always certain tensor products of shifts of the (one-dimensional) Dirichlet kernel. Also the points $\xi$ are given explicitly, cf. Section 2.

Remark 2. In case $q = \infty$ the estimate (1) has been proved earlier by Temlyakov, cf. [15, Chapter 4, Theorem 5.1].

2. The Sampling Operator

As usual, $\mathbb{N}$ stands for the natural numbers, by $\mathbb{N}_0$ we denote the natural numbers including 0 and by $\mathbb{Z}^d$ the $d$-tupels of integers. Let

$$D_m(t) := \sum_{|k| \leq m} e^{ikt}, \quad t \in \mathbb{T}, \quad m \in \mathbb{N}_0,$$

be the Dirichlet kernel and let

$$I_m f(t) := \frac{1}{2m+1} \sum_{\ell=0}^{2m} f(t\ell) D_m(t-t\ell), \quad t\ell = \frac{2\pi\ell}{2m+1},$$

be the unique trigonometric polynomial of degree less than or equal to $m$ which interpolates $f$ at the nodes $t\ell$. We do not need the complete sequence of interpolatory polynomials of a given function. We concentrate on a dyadic subsequence. To have a convenient notation we put $L_j := I_{2^j}$, $j = 0, 1, \ldots$ By $L_{j,k} := L_j \otimes L_k$ we denote the tensor product of $L_j$ and $L_k$. The sampling operators $B_m$ we are going to study are defined as

$$B_m := \sum_{j=0}^{m} L_{j,m-j} - \sum_{j=0}^{m-1} L_{j,m-j-1}, \quad m = 1, 2, \ldots$$

This is Smolyak’s construction (sometimes called Smolyak algorithm or blending operators) with respect to the $L_j$, cf. e.g. [2, 11, 12, 15, 17]. We collect a few properties of $B_m$. Therefore we need some further notations. As usual, let

$$c_k(f) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(t) e^{-ikt} dt, \quad k \in \mathbb{Z}^d,$$

be the Fourier coefficient of $f \in L_1(\mathbb{T}^d)$. We put

$$T_m := \left\{ \left( \frac{2\pi\ell_1}{2^{j+1}+1}, \frac{2\pi\ell_2}{2^{m-j+1}+1} \right) : 0 \leq \ell_1 \leq 2^{j+1}, 0 \leq \ell_2 \leq 2^{m-j+1}, \right. \right. \left. \left. \quad j = 0, \ldots, m \right\}. \]
Lemma 1. Let $m \in \mathbb{N}$.

(i) $B_m$ uses samples of $f$ from the sparse grid $\mathcal{T}_m \cup \mathcal{T}_{m-1}$.

(ii) It holds $c_k(B_m f) = 0$ if

\[ k \not\in H_m := \{ (\ell_1, \ell_2) : \exists r \in (\mathbb{N}_0 \cap [0, m]) \text{ s.t. } |\ell_1| \leq 2^r \text{ and } |\ell_2| \leq 2^{m-r} \} . \]

(iii) Suppose that $f$ is a trigonometric polynomial with harmonics from $H_m$. Then $B_m f = f$.

Proof. Using the projection property of $L_j$ the proof is elementary, but see also [14].

3. Besov Spaces of Dominating Mixed Smoothness

For us it is convenient to introduce the Besov spaces by making use of a Littlewood-Paley decomposition, cf. [6, 9]. Let

\[ P_0 = (-1, 1), \quad P_j = \{ x : 2^{j-1} \leq |x| < 2^j \}, \quad j \in \mathbb{N}, \]

\[ P_{j,k} = P_j \times P_k, \quad j, k \in \mathbb{N}_0. \]

As an abbreviation we shall use

\[ f_{j,k}(x) = \sum_{\ell \in P_{j,k}} c_\ell(f) e^{i\ell x}, \quad x \in \mathbb{T}^2, \quad j, k \in \mathbb{N}_0, \]

which results in

\[ f = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} f_{j,k}. \]

Let $1 < p < \infty$, $1 \leq q \leq \infty$, and $r > 0$. Then the Besov space $S_{p,q}^r B(\mathbb{T}^2)$ of dominating mixed smoothness is the collection of all functions $f \in L_p(\mathbb{T}^2)$ such that

\[ \| f \|_{S_{p,q}^r B(\mathbb{T}^2)} := \left( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{(j+k)q} \| f_{j,k} \|_{L_p(\mathbb{T}^2)}^q \right)^{1/q} < \infty. \]

For $r > 1/p$ one knows that $S_{p,q}^r B(\mathbb{T}^2)$ contains continuous functions only, cf. [9, 2.4.1].
4. The Approximation Power of $B_m$

Let $I$ be the identity operator (we do not indicate the space where $I$ is considered, hoping this will be clear from the context). We recall the identity

$$I \otimes I - B_m = (I - L_m) \otimes L_0 + I \otimes (I - L_m) + \sum_{j=0}^{m-1} (I - L_j) \otimes (L_{m-j} - L_{m-j-1}),$$

valid for each $m \in \mathbb{N}$, cf. [2, Prop. 1.4/2] or [17], the following assertion concerning tensor products of Sobolev spaces

$$W^r_p(T) \otimes_{\alpha_p} W^r_p(T) = S^r_p(W(T^2)), \quad 1 < p < \infty, \quad r \geq 0,$$

(3)

(here $\alpha_p$ denotes the $p$-nuclear norm and $S^r_p(W(T^2))$ denotes a Sobolev space of dominating mixed smoothness), cf. [13], and

$$\|f - L_j f|L_p(T)\| \leq c 2^{-j \gamma} \|f|W^r_p(T)\|$$

(4)

with some constant $c$ independent of $f$ and $j$ ($1 < p < \infty, \ r > 1/p$, cf. [3, 4, 15, 10]). Since $\alpha_p$ is an uniform norm it follows from (2)–(4) that

$$\|f - B_m f|L_p(T^2)\| \leq C m 2^{-m r} \|f|S^r_p(W(T^2))\|$$

(5)

($1 < p < \infty, \ r > 1/p$) holds with some constant $C$ independent of $f$ and $m$. In what follows we shall show that one can replace the factor $m$ on the right-hand side of (5) by $m^\gamma$ with $\gamma < 1$. We denote $a \sim b$ if there exists a constant $c > 0$ (independent of the context dependent relevant parameters) such that

$$c^{-1} a \leq b \leq c a.$$

**Proposition 1.** Suppose $1 < p < \infty, \ 1 \leq q \leq \infty$, and $r > 1/p$. Then

$$\|I - B_m : S^r_{p,q}(T^2) \mapsto L_p(T^2)\| \sim m^{1 - \frac{1}{q}} 2^{-m r}.$$  

(6)

**Proof. Step 1.** Using the projection property of $L_j$ we derive

$$\left((I - L_j) \otimes (L_{m-j} - L_{m-j-1})\right) f_{u,v} = 0$$

(7)

if either $j \geq u$ or if $m - j - 1 \geq v$. Next we recall the Littlewood-Paley characterization of $S^r_p(W(T^2))$. If $1 < p < \infty$ and $r \geq 0$, then

$$\left\|\left(\sum_{j=0}^{\infty} 2^{r(j+k)/2} |f_{j,k}|^2\right)^{1/2} \left|L_p(T^2)\right|\right\|$$

(8)
generates an equivalent norm on $S_p^c W(T^2)$, cf. [6]. Let $r_0$ be a real number such that $1/p < r_0 < r$. Further, we shall use the abbreviation $a_+ = \max\{a,0\}$ for real numbers $a$. We derive from (4), (7), and (8)

\[
\left\| \sum_{j=0}^{m-1} (I - L_j) \otimes (L_{m-j} - L_{m-j-1}) f_{u,v} |L_p(T^2)| \right\| 
\leq \sum_{j=\text{max}(0,m-v)}^{\min\{u-1,m-1\}} \left\| (I - L_j) \otimes (L_{m-j} - L_{m-j-1}) f_{u,v} |L_p(T^2)| \right\| 
\leq c_1 2^{-mr_0} \left( \min\{u-1,m-1\} - \max\{0,m-v\} \right) \| f_{u,v} |S_p^c W(T^2)| \|
\leq c_2 2^{-mr_0} \left( \min\{u-1,m-1\} - \max\{0,m-v\} \right) + 2^{(u+v)r_0} \| f_{u,v} |L_p(T^2)| \|
\]

for some constant $c_2$ independent of $f$. Because of (3), (2) and (4) this implies

\[
\| f_{u,v} - B_m f_{u,v} |L_p(T^2)| \|
\leq c 2^{-mr_0} \left( \min\{u,m\} - \max\{0,m-v\} \right) + 2^{(u+v)r_0} \| f_{u,v} |L_p(T^2)| \|
\]

(9)

for some constant $c$ independent of $f, m, u$ and $v$ and for arbitrary $r_0 > 1/p$. Here we used also the boundedness of $L_0$ considered as a mapping of $W_p^p(T)$ into $L_p(T)$, cf. (3) and (4).

Step 2. Let $1/q + 1/q' = 1$. For given $m$ we shall use the splitting $f = f_1 + f_2 + f_3 + f_4 + f_5$, where

\[
f_1 = \sum_{u+v \leq m} f_{u,v}, \quad f_2 = \sum_{m} f_{u,v}, \quad f_3 = \sum_{0}^{\infty} f_{u,v}, \quad f_4 = \sum_{v}^{\infty} f_{u,v}, \quad f_5 = \sum_{u}^{\infty} f_{u,v}.
\]

Lemma 1(iii) yields $B_m f_1 = f_1$. Furthermore, with $r_0 < r$ we derive from (9) and Hölder’s inequality

\[
\| f_2 - B_m f_2 |L_p(T^2)| \| \leq c_1 \sum_{u}^{m} \sum_{v}^{m} 2^{(u+v-m)r_0} |u + v - m| \| f_{u,v} |L_p(T^2)| \|
\leq c_1 2^{-mr_0} \left( \sum_{u=1}^{m} \sum_{v=1}^{m} 2^{(u+v)(m-u)} (u + v - m)^{1/q'} \right) \| f_2 |S_p^r B(T^2)| \|
\]

(10)

\[
= c_1 2^{-mr} \left( \sum_{u=1}^{m} \sum_{\ell=1}^{m} q^{(m-u)q'} \right) \frac{1}{q'} \| f_2 |S_p^r B(T^2)| \|
\leq c_2 (m+1)^{1/q'} 2^{-mr} \| f_2 |S_p^r B(T^2)| .
\]
Similarly we proceed in estimating \( f_i - B_m f_i, \ i = 3, 4, 5 \). We find

\[
\| f_3 - B_m f_3 \|_{L_p(T^2)} \leq c_1 \sum_{u=0}^{\infty} \sum_{v=m+1}^{\infty} 2^{(u+v-m)r_0} (1 + u) \| f_{u,v} \|_{L_p(T^2)}
\]

\[
\leq c_1 \| f_3 \|_{S_{p,\infty}^r(B(T^2))} \sum_{u=0}^{\infty} \sum_{v=m+1}^{\infty} 2^{(u+v)(r_0-r)} 2^{-mr_0} (1 + u)
\]

\[
\leq c_2 2^{-mr} \| f_3 \|_{S_{p,\infty}^r(B(T^2))},
\]

and analogously

\[
\| f_i - B_m f_i \|_{L_p(T^2)} \leq c 2^{-mr} \| f_i \|_{S_{p,\infty}^r(B(T^2))}, \quad i = 4, 5.
\]

This proves the estimate from above.

**Step 3.** Estimate from below. We employ lacunary series as test functions. Let

\[
f_m(x_1, x_2) := \sum_{u=2}^{m-1} e^{2^{u}x_1 + 2^{m-u+1}x_2}, \quad m = 3, 4, \ldots
\]

Then

\[
B_m f_m(x_1, x_2) = -(m - 2) e^{-i(x_1 + x_2)} + \sum_{u=2}^{m-1} e^{2^{u}x_1 - ix_2} + \sum_{u=2}^{m-1} e^{-ix_1 + 2^{m-u+1}x_2}.
\]

Obviously

\[
\| f_m \|_{S_{p,q}^r B(T^2)} \sim m^{1/q} 2^{mr}.
\]

To calculate the \( L_p \)-norm of \( f_m \) and \( B_m \) we shall use the following Littlewood-Paley assertion, cf. [6]. There exist positive constants \( A_p \) and \( B_p \) such that

\[
A_p \| f \|_{L_p(T^2)} \leq \left( \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |f_{j,k}(x)|^2 \right)^{1/2} \| L_p(T^2) \| \leq B_p \| f \|_{L_p(T^2)}
\]

holds for all \( f \in L_p(T^2) \) (\( 1 < p < \infty \)). This yields

\[
\| f_m \|_{L_p(T^2)} \sim m^{1/2},
\]

\[
\| B_m f_m \|_{L_p(T^2)} \sim m,
\]

if \( 1 < p < \infty \). Combining (11) with (12) and (13) the estimate from below follows. The proof is complete.

**Corollary 1.** (i) Suppose \( 1 < p \leq 2 \) and \( r > 1/p \). Then

\[
\| I - B_m : S_{p}^r W(T^2) \to L_p(T^2) \| \sim m^{1/2} 2^{-mr}.
\]

(ii) Suppose \( 2 < p < \infty \) and \( r > 1/p \). Then there exists a constant \( c \) such that

\[
\| I - B_m : S_{p}^r W(T^2) \to L_p(T^2) \| \leq c m^{1-1/p} 2^{-mr}.
\]
Approximate Recovery

Proof. The estimate from above becomes a consequence of the continuous embedding $S^r_p W(T^2) \hookrightarrow S^r_{p, \max(p,2)} B(T^2)$, cf. [8]. In (i) the estimate from below follows from

$$\|f_m|S^r_p W(T^2)\| \sim m^{1/2} 2^{mr},$$

where $f_m$ are the functions defined in (10).

Proof of the Theorem. Because of $|T_m| \leq (m + 1) 2^{m+2}$ the sampling operator $B_m$ can be used to estimate $\varrho_{M_m}$, where $M_m = (3m + 1) 2^{m+1}$. An application of Proposition 1 yields an upper bound for this particular sequence $\varrho_{M_m}$, $m = 1, 2, \ldots$. Using the monotonicity of the numbers $\varrho_M$ we arrive at (1). This completes the proof.

References


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