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Approximate Recovery of Functions and Besov Spaces of Dominating Mixed Smoothness

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We derive an estimate from above for the approximate optimal recovery of bivariate periodic functions taken from a Besov space of dominating mixed smoothness.

1. Approximate Optimal Recovery

We study the effectiveness of the approximation by generalized sampling operators. Let F be a class of continuous, periodic functions defined on $\mathbb{T}^2 = [0, 2\pi)^2$. Then, following [15, Chapter 4, Section 5], we consider for fixed m, $\xi = (\xi^1, \xi^2, \ldots, \xi^m)$, $\xi^j \in \mathbb{T}^2$, $j = 1, \ldots, m$, and $\psi_1(x_1, x_2), \ldots, \psi_m(x_1, x_2)$ the linear operator

$$\Psi_m(f, \xi)(x_1, x_2) := \sum_{j=1}^{m} f(\xi^j)\, \psi_j(x_1, x_2)$$

and define the quantities

$$\Psi_m(F, \xi, L_p(\mathbb{T}^2)) := \sup_{f \in F} \| \Psi_m(f, \xi) - f \,| L_p(\mathbb{T}^2)\|$$

and

$$\varrho_m(F, L_p(\mathbb{T}^2)) := \inf_{\psi_1, \ldots, \psi_m} \inf_{\xi} \Psi_m(F, \xi, L_p(\mathbb{T}^2)) \,.$$

Hence $\varrho_m(F, L_p(\mathbb{T}^2))$ measures the optimal approximate recovery of the functions from F. Here we are interested in the case when F is the unit ball in a Besov space $S^r_{p,q}B(\mathbb{T}^2)$ of dominating mixed smoothness (a definition will be given below). Our main result reads as follows.

Theorem 1. *Let $1 < p < \infty$, $1 \le q \le \infty$, and $r > 1/p$. Let F be the unit ball in $S^r_{p,q}B(\mathbb{T}^2)$. For any natural number m there exists a system of points $\xi^1, \ldots, \xi^m \in \mathbb{T}^2$, a collection of trigonometric polynomials $\psi_1(x_1, x_2), \ldots, \psi_m(x_1, x_2)$, and a constant C (independent of m) such that*

$$\sup_{f \in F} \| \Psi_m(f, \xi) - f \,| L_p(\mathbb{T}^2)\| \le C\, m^{-r} (\log m)^{r+1-1/q} \,. \qquad (1)$$

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Remark 1. Our proof will be constructive. The functions $\psi_j(x_1, x_2)$, $j = 1, \ldots, m$, are always certain tensor products of shifts of the (one-dimensional) Dirichlet kernel. Also the points ξ are given explicitly, cf. Section 2.

Remark 2. In case $q = \infty$ the estimate (1) has been proved earlier by Temlyakov, cf. [15, Chapter 4, Theorem 5.1].

2. The Sampling Operator

As usual, $\mathbb{N}$ stands for the natural numbers, by $\mathbb{N}_0$ we denote the natural numbers including 0 and by $\mathbb{Z}^d$ the d-tupels of integers. Let

$$\mathcal{D}_m(t) := \sum_{|k| \le m} e^{ikt}, \qquad t \in \mathbb{T}, \quad m \in \mathbb{N}_0,$$

be the Dirichlet kernel and let

$$I_m f(t) := \frac{1}{2m+1} \sum_{\ell=0}^{2m} f(t_\ell)\, \mathcal{D}_m(t - t_\ell), \qquad t_\ell = \frac{2\pi\ell}{2m+1},$$

be the unique trigonometric polynomial of degree less than or equal to m which interpolates f at the nodes t_ℓ. We do not need the complete sequence of interpolatory polynomials of a given function. We concentrate on a dyadic subsequence. To have a convenient notation we put $L_j := I_{2^j}$, $j = 0, 1, \ldots$ By $L_{j,k} := L_j \otimes L_k$ we denote the tensor product of L_j and L_k. The sampling operators B_m we are going to study are defined as

$$B_m := \sum_{j=0}^{m} L_{j,m-j} - \sum_{j=0}^{m-1} L_{j,m-j-1}, \qquad m = 1, 2, \ldots$$

This is Smolyak's construction (sometimes called Smolyak algorithm or blending operators) with respect to the L_j, cf. e.g. [2, 11, 12, 15, 17]. We collect a few properties of B_m. Therefore we need some further notations. As usual, let

$$c_k(f) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(t)\, e^{-ikt}\, dt, \qquad k \in \mathbb{Z}^d,$$

be the Fourier coefficient of $f \in L_1(\mathbb{T}^d)$. We put

$$\mathcal{T}_m := \Big\{ \Big(\frac{2\pi\ell_1}{2^{j+1}+1}, \frac{2\pi\ell_2}{2^{m-j+1}+1} \Big) \,:\, 0 \le \ell_1 \le 2^{j+1},\ 0 \le \ell_2 \le 2^{m-j+1}, \quad j = 0, \ldots, m \Big\}.$$

Lemma 1. *Let $m \in \mathbb{N}$.*

(i) *B_m uses samples of f from the sparse grid $\mathcal{T}_m \cup \mathcal{T}_{m-1}$.*

(ii) *It holds $c_k(B_m f) = 0$ if*

$$k \notin H_m := \left\{ (\ell_1, \ell_2) \,:\, \exists r \in (\mathbb{N}_0 \cap [0,m]) \ \text{s.t.}\ |\ell_1| \le 2^r \ \text{and}\ |\ell_2| \le 2^{m-r} \right\}.$$

(iii) *Suppose that f is a trigonometric polynomial with harmonics from H_m. Then $B_m f = f$.*

Proof. Using the projection property of L_j the proof is elementary, but see also [14].

3. Besov Spaces of Dominating Mixed Smoothness

For us it is convenient to introduce the Besov spaces by making use of a Littlewood-Paley decomposition, cf. [6, 9]. Let

$$P_0 = (-1,1)\,, \qquad P_j = \{x \,:\, 2^{j-1} \le |x| < 2^j\}\,, \qquad j \in \mathbb{N}\,,$$
$$P_{j,k} = P_j \times P_k\,, \qquad j,\, k \in \mathbb{N}_0\,.$$

As an abbreviation we shall use

$$f_{j,k}(x) = \sum_{\ell \in P_{j,k}} c_\ell(f)\, e^{i\ell x}\,, \qquad x \in \mathbb{T}^2, \quad j,k \in \mathbb{N}_0\,,$$

which results in

$$f = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} f_{j,k}\,.$$

Let $1 < p < \infty$, $1 \le q \le \infty$, and $r > 0$. Then the Besov space $S^r_{p,q}B(\mathbb{T}^2)$ of dominating mixed smoothness is the collection of all functions $f \in L_p(\mathbb{T}^2)$ such that

$$\|\, f \,|S^r_{p,q}B(\mathbb{T}^2)\| := \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{r(j+k)q}\, \|f_{j,k}\,|L_p(\mathbb{T}^2)\|^q \right)^{1/q} < \infty\,.$$

For $r > 1/p$ one knows that $S^r_{p,q}B(\mathbb{T}^2)$ contains continuous functions only, cf. [9, 2.4.1].

4. The Approximation Power of B_m

Let I be the identity operator (we do not indicate the space where I is considered, hoping this will be clear from the context). We recall the identity

$$\begin{aligned} I \otimes I - B_m = (I - L_m) \otimes L_0 + I \otimes (I - L_m) \\ + \sum_{j=0}^{m-1} (I - L_j) \otimes (L_{m-j} - L_{m-j-1}) \,, \end{aligned} \tag{2}$$

valid for each $m \in \mathbb{N}$, cf. [2, Prop. 1.4/2] or [17], the following assertion concerning tensor products of Sobolev spaces

$$W^r_p(\mathbb{T}) \otimes_{\alpha_p} W^r_p(\mathbb{T}) = S^r_p W(\mathbb{T}^2) \,, \qquad 1 < p < \infty \,,\ r \geq 0 \,, \tag{3}$$

(here α_p denotes the p-nuclear norm and $S^r_p W(\mathbb{T}^2)$ denotes a Sobolev space of dominating mixed smoothness), cf. [13], and

$$\| \, f - L_j f \, | L_p(\mathbb{T}) \| \leq c \, 2^{-jr} \, \| \, f \, | W^r_p(\mathbb{T}) \| \tag{4}$$

with some constant c independent of f and j ($1 < p < \infty$, $r > 1/p$, cf. [3, 4, 15, 10]). Since α_p is an uniform norm it follows from (2)–(4) that

$$\| \, f - B_m f \, | L_p(\mathbb{T}^2) \| \leq C \, m \, 2^{-mr} \, \| \, f \, | S^r_p W(\mathbb{T}^2) \| \tag{5}$$

($1 < p < \infty$, $r > 1/p$) holds with some constant C independent of f and m. In what follows we shall show that one can replace the factor m on the right-hand side of (5) by m^γ with $\gamma < 1$. We denote $a \sim b$ if there exists a constant $c > 0$ (independent of the context dependent relevant parameters) such that

$$c^{-1} \, a \leq b \leq c \, a \, .$$

Proposition 1. *Suppose* $1 < p < \infty$, $1 \leq q \leq \infty$, *and* $r > 1/p$. *Then*

$$\| \, I - B_m \, : \, S^r_{p,q} B(\mathbb{T}^2) \mapsto L_p(\mathbb{T}^2) \| \sim \, m^{1-\frac{1}{q}} \, 2^{-mr} \, . \tag{6}$$

Proof. Step 1. Using the projection property of L_j we derive

$$\Big((I - L_j) \otimes (L_{m-j} - L_{m-j-1}) \Big) \, f_{u,v} = 0 \tag{7}$$

if either $j \geq u$ or if $m - j - 1 \geq v$. Next we recall the Littlewood-Paley characterization of $S^r_p W(\mathbb{T}^2)$. If $1 < p < \infty$ and $r \geq 0$, then

$$\bigg\| \bigg(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{r(j+k)2} \, |f_{j,k}|^2 \bigg)^{1/2} \bigg| L_p(\mathbb{T}^2) \bigg\| \tag{8}$$

generates an equivalent norm on $S^r_p W(\mathbb{T}^2)$, cf. [6]. Let r_0 be a real number such that $1/p < r_0 < r$. Further, we shall use the abbreviation $a_+ = \max\{a, 0\}$ for real numbers a. We derive from (4), (7), and (8)

$$
\begin{aligned}
&\Big\| \sum_{j=0}^{m-1} \Big((I - L_j) \otimes (L_{m-j} - L_{m-j-1}) \Big) f_{u,v} \,|L_p(\mathbb{T}^2) \Big\| \\
&\le \sum_{j=\max\{0,m-v\}}^{\min\{u-1,m-1\}} \big\| (I - L_j) \otimes (L_{m-j} - L_{m-j-1})\, f_{u,v} \,|L_p(\mathbb{T}^2) \big\| \\
&\le c_1\, 2^{-mr_0} \big(\min\{u-1, m-1\} - \max\{0, m-v\} \big)_+ \, \| f_{u,v} \,|S^{r_0}_p W(\mathbb{T}^2) \| \\
&\le c_2\, 2^{-mr_0} \big(\min\{u-1, m-1\} - \max\{0, m-v\} \big)_+ 2^{(u+v)r_0} \| f_{u,v} \,|L_p(\mathbb{T}^2) \|
\end{aligned}
$$

for some constant c_2 independent of f. Because of (3), (2) and (4) this implies

$$
\begin{aligned}
&\| f_{u,v} - B_m f_{u,v} \,|L_p(\mathbb{T}^2) \| \\
&\le c\, 2^{-mr_0} \big(\min\{u, m\} - \max\{0, m-v\} \big)_+ \, 2^{(u+v)r_0} \, \| f_{u,v} \,|L_p(\mathbb{T}^2) \| \\
&\quad + \big\| \big((I - L_m) \otimes L_0 \big) f_{u,v} \,|L_p(\mathbb{T}^2) \big\| + \big\| \big(I \otimes (I - L_m) \big) f_{u,v} \,|L_p(\mathbb{T}^2) \big\| \qquad (9) \\
&\le c \Big(1 + \big(\min\{u, m\} - \max\{0, m-v\} \big)_+ \Big) \, 2^{(u+v-m)r_0} \, \| f_{u,v} \,|L_p(\mathbb{T}^2) \| ,
\end{aligned}
$$

for some constant c independent of f, m, u and v and for arbitrary $r_0 > 1/p$. Here we used also the boundedness of L_0 considered as a mapping of $W^{r_0}_p(\mathbb{T})$ into $L_p(\mathbb{T})$, cf. (3) and (4).

Step 2. Let $1/q + 1/q' = 1$. For given m we shall use the splitting $f = f_1 + f_2 + f_3 + f_4 + f_5$, where

$$
f_1 = \sum_{u+v \le m} f_{u,v} \, , \qquad f_2 = \sum_{u=1}^{m} \sum_{v=m-u+1}^{m} f_{u,v} \, , \qquad f_3 = \sum_{u=0}^{m} \sum_{v=m+1}^{\infty} f_{u,v} \, ,
$$

$$
f_4 = \sum_{u=m+1}^{\infty} \sum_{v=0}^{m} f_{u,v} \, , \quad \text{and} \quad f_5 = \sum_{u=m+1}^{\infty} \sum_{v=m+1}^{\infty} f_{u,v} \, .
$$

Lemma 1(iii) yields $B_m f_1 = f_1$. Furthermore, with $r_0 < r$ we derive from (9) and Hölder's inequality

$$
\begin{aligned}
&\| f_2 - B_m f_2 | L_p(\mathbb{T}^2) \| \le c_1 \sum_{u=1}^{m} \sum_{v=m-u+1}^{m} 2^{(u+v-m)r_0} (u+v-m) \| f_{u,v} | L_p(\mathbb{T}^2) \| \\
&\le c_1\, 2^{-mr_0} \Bigg(\sum_{u=1}^{m} \sum_{v=m-u+1}^{m} 2^{(u+v)(r_0-r)q'} \, (u+v-m)^{q'} \Bigg)^{1/q'} \| f_2 \,|S^r_{p,q} B(\mathbb{T}^2) \| \\
&= c_1\, 2^{-mr} \Bigg(\sum_{u=1}^{m} \sum_{\ell=1}^{u} 2^{\ell (r_0-r) q'} \, \ell^{q'} \Bigg)^{1/q'} \| f_2 \,|S^r_{p,q} B(\mathbb{T}^2) \| \\
&\le c_2\, (m+1)^{1/q'} \, 2^{-mr} \, \| f_2 \,|S^r_{p,q} B(\mathbb{T}^2) \| \, .
\end{aligned}
$$

Similarly we proceed in estimating $f_i - B_m f_i$, $i = 3, 4, 5$. We find

$$\begin{aligned} \| f_3 - B_m f_3 \,|L_p(\mathbb{T}^2)\| &\le c_1 \sum_{u=0}^{m} \sum_{v=m+1}^{\infty} 2^{(u+v-m)r_0}\,(1+u)\,\| f_{u,v}\,|L_p(\mathbb{T}^2)\| \\ &\le c_1 \,\| f_3 \,|S^r_{p,\infty}B(\mathbb{T}^2)\| \sum_{u=0}^{m} \sum_{v=m+1}^{\infty} 2^{(u+v)(r_0-r)} 2^{-mr_0}(1+u) \\ &\le c_2\, 2^{-mr}\,\| f_3 \,|S^r_{p,\infty}B(\mathbb{T}^2)\|\,, \end{aligned}$$

and analogously

$$\| f_i - B_m f_i \,|L_p(\mathbb{T}^2)\| \le c\, 2^{-mr}\,\| f_i \,|S^r_{p,\infty}B(\mathbb{T}^2)\|\,, \qquad i = 4, 5\,.$$

This proves the estimate from above.

Step 3. Estimate from below. We employ lacunary series as test functions. Let

$$f_m(x_1, x_2) := \sum_{u=2}^{m-1} e^{i2^u x_1 + i2^{m-u+1} x_2}\,, \qquad m = 3, 4, \ldots \tag{10}$$

Then

$$B_m f_m(x_1, x_2) = -(m-2)\, e^{-i(x_1+x_2)} + \sum_{u=2}^{m-1} e^{i2^u x_1 - ix_2} + \sum_{u=2}^{m-1} e^{-ix_1 + i2^{m-u+1} x_2}\,.$$

Obviously

$$\| f_m \,|S^r_{p,q}B(\mathbb{T}^2)\| \quad \sim \quad m^{1/q}\, 2^{mr}\,. \tag{11}$$

To calculate the L_p-norm of f_m and B_m we shall use the following Littlewood-Paley assertion, cf. [6]. There exist positive constants A_p and B_p such that

$$A_p \,\| f \,|L_p(\mathbb{T}^2)\| \le \left\| \Big(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |f_{j,k}(x)|^2 \Big)^{1/2} \Big| L_p(\mathbb{T}^2) \right\| \le B_p \,\| f \,|L_p(\mathbb{T}^2)\|$$

holds for all $f \in L_p(\mathbb{T}^2)$ $(1 < p < \infty)$. This yields

$$\| f_m \,|L_p(\mathbb{T}^2)\| \qquad \sim \qquad m^{1/2}\,, \tag{12}$$

$$\| B_m f_m \,|L_p(\mathbb{T}^2)\| \qquad \sim \qquad m\,, \tag{13}$$

if $1 < p < \infty$. Combining (11) with (12) and (13) the estimate from below follows. The proof is complete.

Corollary 1. (i) *Suppose* $1 < p \le 2$ *and* $r > 1/p$. *Then*

$$\| I - B_m \,:\, S^r_p W(\mathbb{T}^2) \mapsto L_p(\mathbb{T}^2)\| \quad \sim \quad m^{1/2}\, 2^{-mr}\,.$$

(ii) *Suppose* $2 < p < \infty$ *and* $r > 1/p$. *Then there exists a constant* c *such that*

$$\| I - B_m \,:\, S^r_p W(\mathbb{T}^2) \mapsto L_p(\mathbb{T}^2)\| \le c\, m^{1-1/p}\, 2^{-mr}\,.$$

Proof. The estimate from above becomes a consequence of the continuous embedding $S^r_pW(\mathbb{T}^2) \hookrightarrow S^r_{p,\max\{p,2\}}B(\mathbb{T}^2)$, cf. [8]. In (i) the estimate from below follows from

$$\| f_m \,|S^r_pW(\mathbb{T}^2)\| \quad \sim \quad m^{1/2}\, 2^{mr}\,,$$

where f_m are the functions defined in (10).

Proof of the Theorem. Because of $|\mathcal{T}_m| \le (m+1)\, 2^{m+2}$ the sampling operator B_m can be used to estimate ϱ_{M_m}, where $M_m = (3m+1)\, 2^{m+1}$. An application of Proposition 1 yields an upper bound for this particular sequence ϱ_{M_m}, $m = 1, 2, \ldots$. Using the monotonicity of the numbers ϱ_M we arrive at (1). This completes the proof.

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