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Solving Equations by *q*-iterative Methods and *q*-Sendov Conjecture

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We develop methods which include q-derivatives for solving equations. They are very useful when the continuous function does not have fine smooth properties. We will discuss the convergence and accuracy of those methods and compare them with well-known methods.

1. Introduction

At the last quarter of XX century, q-calculus appears like a connection between mathematics and physics (see [5], [6]). It has a lot of applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hyper geometric functions and in other sciences like quantum theory, mechanics and theory of relativity.

Let $q \in (0,\ 1)$. A q-natural number $[n]_q$ is defined by $[n]_q := 1+q+\cdots+q^{n-1}$, $n \in \mathbb{N}$. Generally, a q-complex number $[a]_q$ is $[a]_q := (1-q^a)/(1-q),\ a \in \mathbb{C}$. The factorial of a number $[n]_q$ is $[0]_q! := 1,\ [n]_q! := [n]_q[n-1]_q \cdots [1]_q,\ n \in \mathbb{N}$.

The q-derivative of a function $f(z)$ is

$$(D_q f)(z) := \frac{f(z)-f(qz)}{z-qz} \quad (z \neq 0), \qquad (D_q f)(0) := \lim_{z\to 0}(D_q f)(z),$$

and high q-derivatives are $D_q^0 f := f,\ D_q^n f := D_q(D_q^{n-1} f),\ n = 1,2,3,\ldots$ Notice that a continuous function on an interval, which does not include 0, is continuously q-differentiable.

Jackson's q-Taylor formula (see [3], [4] and [2]) is given by

$$f(z) = \sum_{k=0}^{\infty} \frac{\big(D_q^k f\big)(a)}{[k]_q!}(z-a)^{(k)},$$

where $(z-a)^{(0)} = 1,\ (z-a)^{(k)} = \prod_{i=0}^{k-1}(z-aq^i),\ k \in \mathbb{N}$.

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In q-analysis, we define q-integral by

$$I_q(f) = \int_0^a f(t)d_q(t) := a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n, \quad I_1(f) = I(f) = \int_0^a f(t)\ dt.$$

Notice that $I(f) = \lim_{q\uparrow 1} I_q(f)$. Also,

$$\int_a^b f(t)d_q(t) := \int_0^b f(t)d_q(t) - \int_0^a f(t)d_q(t).$$

The next q-Taylor formula with a remainder term

$$f(z) = \sum_{k=0}^{n-1} \frac{\big(D_q^k f\big)(a)}{[k]_q!}(z-a)^{(k)} + R_n(f,z,a,q),$$

where

$$R_n(f,z,a,q) = \int_{t=a}^{t=z} (z-qt)^{(n-1)} \frac{\big(D_q^n f\big)(t)}{[n-1]_q!} d_q(t).$$

is given in the paper of Ernst [2] (also see Jing and Fan [4]).

2. Analysis of the Convergence of an Iterative Process by q-derivative

Our purpose is to formulate and prove a theorem for scanning the convergence of an iterative process

$$x_{k+1} = \Phi(x_k), \qquad k = 0,1,2,\ldots,$$

by q-analysis. We have studied it for the first time in our paper [7].

The next lemma is needed.

Lemma 1. *Let $\Phi(x)$ be a continuous function on $[a,b]$ $(0 \notin [a,b])$. Then, for all x and y such that $a < x < y < b$, it is valid*

$$\Phi(y) - \Phi(x) = \big(D_{x/y}\Phi\big)(y)(y-x), \qquad \Phi(y) - \Phi(x) = \big(D_{y/x}\Phi\big)(x)(y-x).$$

Proof. Taking $q' = x/y$ for the first, and $q'' = y/x$, for the second case, we derive the statement.

Now, we can prove the main result of this section.

Theorem 1. *Suppose that* $\Phi(x)$ *is a continuous function on* $[a,b]$ $(0 \notin [a,b])$, *which satisfies the next conditions:*

(i) $\Phi : [a,b] \mapsto [a,b]$;

(ii) $\big(\forall q \in (\min\{a,b\}/\max\{a,b\},1)\big)\big(\forall x \in (a,b)\big) : \ \big|(D_q f)(x)\big| \le d < 1.$

Then the iterative process $x_{k+1} = \Phi(x_k)$, $k \in \mathbb{N}_0$, *with initial value* $x_0 \in [a,b]$, *is converging to the fixed point of* $\Phi(x)$, *i.e.,* $\lim\limits_{k\to\infty} x_k = \xi$, $\Phi(\xi) = \xi$.

Proof. Let us consider the series $\xi = x_0 + \sum\limits_{k=0}^{\infty}(x_{k+1} - x_k)$. Let $x_k^{(M)} = \max\{x_k, x_{k-1}\}$, $x_k^{(m)} = \min\{x_k, x_{k-1}\}$ and $q = x_k^{(m)}/x_k^{(M)}$. According to Lemma 1, we have

$$\Phi(x_k) - \Phi(x_{k-1}) = (D_q\Phi)(x_k^{(M)})(x_k - x_{k-1}).$$

So,

$$\big|x_{k+1} - x_k\big| = \big|\big(D_q\Phi\big)(x_k^{(M)})\big||x_k - x_{k-1}| \le d|x_k - x_{k-1}|.$$

A repeated use of the last estimate yields $\big|x_{k+1} - x_k\big| \le d^k|x_1 - x_0|$, and therefore

$$\sum_{k=0}^{\infty}|x_{k+1} - x_k| \le |x_1 - x_0|\sum_{k=0}^{\infty} d^k = \frac{|x_1 - x_0|}{1-d}.$$

Hence, the series S converges and $\xi = \lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} x_{n+1}$. Since $\Phi(x)$ is a continuous function, we have

$$\xi = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\Phi(x_n) = \Phi(\lim_{n\to\infty} x_n) = \Phi(\xi).$$

Definition 1. *An iterative method* $x_{k+1} = \Phi(x_k)$ *with a fixed point* ξ *has* $(r;q)$*-order of convergence if there exists a constant* $C_r \in \mathbb{R}^+$ *such that*

$$|\xi - x_{n+1}| \le C_r|(\xi - x_n)^{(r)}|$$

for large enough n.

3. On q-Newton Method

Suppose that the equation $f(x) = 0$ has a unique isolated solution $x = \xi$. If x_n is an approximation for the exact solution ξ, by using Jackson's q-Taylor formula, we have

$$0 = f(\xi) \approx f(x_n) + (D_q f)(x_n)(\xi - x_n) \quad \Rightarrow \quad \xi \approx x_n - \frac{f(x_n)}{(D_q f)(x_n)}.$$

So, we can construct *q-Newton method*

$$x_{n+1} = x_n - \frac{f(x_n)}{(D_q f)(x_n)} \quad \text{or} \quad x_{n+1} = x_n \left(1 - \frac{1-q}{1 - f(qx_n)/f(x_n)}\right).$$

This method, written in the form

$$x_{n+1} = x_n - \frac{x_n - qx_n}{f(x_n) - f(qx_n)} f(x_n),$$

reminds of the method of chords (secants).

The next theorem is a q-analogue of a well-known result (see Bakhvalov [1]).

Theorem 2. *Let the equation $f(x) = 0$ has a unique isolated root $x = \xi$ and $a > 0$, $1 \le p \le 2$. If the function $f(x)$ satisfies*

(i) $|(D_q f)(x)| \ge M_1^{p-1} > 0$,

(ii) $|f(x) - f(y) - (D_q f)(y)(x-y)| < L^{p-1}|x-y|^p$,

then, for all initial values $x_0 \in (\xi - b, \xi + b)$, where $b = \min\{a, M_1/L\}$, the q-Newton method converges to the exact solution of the equation $f(x) = 0$ and

$$|\xi - x_n| \le \frac{M_1}{L}\left(\frac{L}{M_1}|\xi - x_0|\right)^{p^n}.$$

Proof. We can write the q-Newton method in the form

$$(D_q f)(x_n)(x_{n+1} - x_n) = -f(x_n).$$

From the condition (ii), we have

$$|f(\xi) - f(x_n) - (D_q f)(x_n)(\xi - x_n)| < L^{p-1}|\xi - x_n|^p.$$

Hence, using the fact that $f(\xi) = 0$, we obtain

$$|(D_q f)(x_n)(\xi - x_{n+1})| < L^{p-1}|\xi - x_n|^p.$$

By condition (i),

$$|\xi - x_{n+1}| < \frac{L^{p-1}}{|(D_q f)(x_n)|}|\xi - x_n|^p < \left(\frac{L}{M_1}\right)^{p-1}|\xi - x_n|^p.$$

Now, if $x_n \in (\xi - b, \xi + b)$, then

$$|\xi - x_{n+1}| < \left(\frac{L}{M_1}\right)^{p-1} b^p = \left(\frac{L}{M_1}\right)^{p-1} b^{p-1} b \le b.$$

Let us denote $c = L/M_1$. Now

$$|\xi - x_{n+1}| < c^{p-1}|\xi - x_n|^p \quad \Rightarrow \quad c\,|\xi - x_{n+1}| < c^p|\xi - x_n|^p,$$

which yields the final conclusion.

4. An Error Estimation

First of all, we shall give a q-analogue of the well-known mean value theorem for integrals which we have proved in our paper [8].

Theorem 3. *Let $f(x)$ and $g(x)$ be some continuous functions on $[a,b]$. Then there exists $\hat{q} \in (0,1)$ such that*

$$\big(\forall q \in (\hat{q},1)\big)\big(\exists \tau \in (a,b)\big): \quad I_q(fg) = g(\tau) I_q(f).$$

Let us return to the q-Taylor formula with a remainder term.

Theorem 4. *Let $f(x)$ be a continuous function on $[a,b]$ and $R_n(f,z,c,q)$, $(z, c \in (a,b))$ be the remainder term in the q-Taylor formula. Then there exists $\hat{q} \in (0,1)$ such that for all $q \in (\hat{q},1)$, one can find a point $\tau \in (a,b)$ between c and z which satisfies*

$$R_n(f,z,c,q) = \frac{(D_q^n f)(\tau)}{[n-1]_q!} \int_{t=c}^{t=z} (z-qt)^{(n-1)} d_q(t).$$

Proof. Since $f(x)$ is a continuous function on $[a,b]$, it can be expanded by the q-Taylor formula of order n at the point c with the remainder term

$$R_n(f,z,c,q) = \int_{t=c}^{t=z} (z-qt)^{(n-1)} \, \frac{(D_q^n f)(t)}{[n-1]_q!} d_q(t).$$

Notice that the functions $(z-t)^{(n-1)}$ and $(D_q^n f)(t)/[n-1]_q!$ are continuous on the segment between c and z which is contained in (a,b). According to Theorem 3, there exists $\hat{q} \in (0,1)$ such that for all $q \in (\hat{q},1)$ one can find a point τ between c and z such that the statement of theorem is valid.

Now, we are ready to prove the main theorem of this section.

Theorem 5. *Suppose that the function $f(x)$ is continuous on $[a,b]$ and the equation $f(x) = 0$ has a unique isolated solution $\xi \in (a,b)$. If the conditions*

$$|(D_q f)(x)| \geq M_1 > 0, \qquad |(D_q^2 f)(x)| \leq M_2$$

are satisfied for all $x \in (a,b)$, then there exists $\hat{q} \in (0,1)$ such that for all $q \in (\hat{q},1)$, the iterations obtained by the q-Newton method satisfy

$$|\xi - x_{k+1}| \leq \frac{M_2}{(1+q)M_1} |(\xi - x_k)^{(2)}|,$$

i.e., the q-Newton method has $(2;q)$-order of convergence.

Proof. From the formulation of the q-Newton method we have

$$x_{k+1} - \xi = x_k - \xi - \frac{f(x_k)}{(D_q f)(x_k)}.$$

Hence

$$f(x_k) + (D_q f)(x_k)(\xi - x_k) = (D_q f)(x_k)(\xi - x_{k+1}).$$

By using q-Taylor's formula of order $n = 2$ at the point x_k for $f(\xi)$, we have

$$f(\xi) = f(x_k) + (D_q f)(x_k)(\xi - x_k) + R_2(f, \xi, x_k, q).$$

Since $f(\xi) = 0$, we obtain $(D_q f)(x_k)(\xi - x_{k+1}) = -R_2(f, \xi, x_k, q)$, i.e.,

$$|\xi - x_{k+1}| = \frac{|R_2(f, \xi, x_k, q)|}{|(D_q f)(x_k)|}.$$

According to Theorem 4, there exists $\hat{q} \in (0, 1)$ such that for all $q \in (\hat{q}, 1)$ a point $\tau \in (a, b)$ can be found such that

$$R_2(f, \xi, x_k, q) = (D_q^2 f)(\tau) \int_{t=x_k}^{t=\xi} (\xi - qt) d_q(t).$$

Evaluating the last integral, by definition, we obtain

$$\int_{t=x_k}^{t=\xi} (\xi - qt) d_q(t) = \frac{(\xi - x_k)(\xi - q x_k)}{1 + q}.$$

Thus

$$|\xi - x_{k+1}| = \frac{|(D_q^2 f)(\tau))|}{|(D_q f)(x_k)|} \; \frac{(\xi - x_k)^{(2)}}{1 + q}.$$

Using now the conditions the function $f(x)$ and its q-derivatives have been supposed to satisfy we finish the proof of the theorem.

5. On *q*-Sendov Conjectures

These methods can be successfully used for finding all zeros of complex polynomials. Our numerical investigations persuade us that the next conjectures might be true.

Conjecture 1 (***q***-Sendov). *If all zeros of a polynomial lie in the unit circle, then the circle of radius* 1 *centered at each of them contains a zero of the q-derivative of the polynomial.*

Conjecture 2 (Strong ***q***-Sendov). *If all zeros of a polynomial p lie in the unit circle, then there exists $\hat{q} \in (0, 1)$ such that for all $q \in (\hat{q}, 1)$ the circle of radius q centered at each zero of p contains a zero of the q-derivative of the polynomial.*

Remark. The bound $\hat{q}$ can not be excluded from Conjecture 2. Here is a simple counterexample for it. Let us consider the polynomial $p(z) = 4z^2 - 1$ whose zeros are $z_1 = -1/2$ and $z_2 = 1/2$. Now, its q-derivative is $(D_q p)(z) = (1+q)z$. Obviously, for $q < 1/2$, the zero $w = 0$ of $(D_q p)(z)$ does not lie in any of the disks $\{z : |z - z_k| < q\}$ $(k = 1, 2)$.

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