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On Uniform Summability of Discrete (Interpolatory) Processes

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The aim of this paper is to construct a wide class of discrete trigonometric or algebraic polynomial processes which are uniformly convergent in a suitable Banach space of continuous functions.

1. Introduction

The Lagrange interpolation is one of the most natural discrete approximating tools on an interval $I \subset \mathbb{R}$. However, as it was proved by G. Faber in 1914, there is no point system for which the corresponding sequence of Lagrange interpolatory polynomials converges uniformly for all continuous functions. It is natural to ask how to construct such processes which are uniformly convergent in suitable spaces of continuous functions.

One way of achieving this aim is to loosen the strict condition on the degree of interpolating polynomials, thus introducing free parameters to be suitably determined for the uniform convergence (see [8, Chapter II], [2], [16], [14]). The success of a construction like this strongly depends on the matrix of nodes.

Another way to obtain uniformly convergent discrete processes is to consider suitable sums of the Lagrange interpolatory polynomials (see [1], [6], [3], [15], [10], [11]).

In Section 2 we shall define a wide class of discrete processes using the so called Θ-summation and we shall formulate a very general problem with respect to the uniform convergence in a suitable Banach space of continuous functions. The aim of this paper is to solve this problem choosing the parameters and the Banach spaces in various ways. The uniform convergence of the corresponding processes would follow immediately from their explicit form.

Several interpolatory properties of the corresponding polynomials will be also given from which many earlier results of the interpolation theory can be obtained as corollaries of our general theorems.

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2. A General Construction of Discrete Processes

Let $I \subset \mathbb{R}$ be an interval and let us fix the natural numbers m and N. Consider a point system $X_N := \{x_{N,N} < x_{N-1,N} < \cdots < x_{1,N}\}$, a discrete measure (or nonnegative weights) $\mu_N := \{\mu_{N,N}, \mu_{N-1,N}, \dots, \mu_{1,N}\}$ and a basis $P_m := \{p_0, p_1, \dots, p_m\}$ in the linear space of trigonometric or algebraic polynomials with real coefficients of degree not greater than m.

We investigate summation processes generated by a function Θ as defined below. Let us denote by Φ the set of *summation functions* $\Theta : [0, +\infty) \to \mathbb{R}$ satisfying the following requirements:

(i) $\operatorname{supp} \Theta \subset [0, 1]$,
(ii) $\lim_{t\to 0+} \Theta(t) = \Theta(0) := 1$,
(iii) the limits $\Theta(t_0 \pm 0) := \lim_{t\to t_0 \pm 0} \Theta(t)$ exist and they are finite at every point $t_0 \in [0, +\infty)$,
(iv) for all $t \geq 0$ the function value $\Theta(t)$ lies in the closed interval determined by $\Theta(t - 0)$ and $\Theta(t + 0)$.

It may be shown that (see [10, p. 161]) Θ is continuous except at most at a countable set of points in $[0, 1]$.

For an arbitrary function $f : I \to \mathbb{R}$, we define

$$\big(S^{\Theta}_{m,N} f\big)(x) := S^{\Theta}_{m,N}\big(f, X_N, \mu_N, P_m, x\big) := \sum_{l=0}^{m} \Theta\big(\frac{l}{m}\big) c_{l,N}(f) p_l(x), \tag{1}$$

where

$$c_{l,N}(f) := c_{l,N}(f, X_N, \mu_N, P_m) := \sum_{k=1}^{N} f(x_{k,N}) p_l(x_{k,N}) \mu_{k,N}.$$

With any two index sequences $(m_n, n \in \mathbb{N} := \{1, 2, \dots\})$ and $(N_n, n \in \mathbb{N})$ we associate the sequence of polynomials

$$\big(S^{\Theta}_{m_n,N_n} f, n \in \mathbb{N}\big) \tag{2}$$

for all $f : I \to \mathbb{R}$.

The following problem will be investigated:

Problem 1. *Choose the parameters $X_{N_n}, \mu_{N_n}, P_{m_n}$, so that the corresponding sequence* (2) *tends uniformly to f in a suitable subspace of continuous functions for a fairly wide class of summation functions Θ.*

3. The Trigonometric Case

Let us fix a natural number $N \in \mathbb{N}$ and consider the equidistant point system $X_N := \{x_{k,N} := k\frac{2\pi}{N} \mid k = 0, 1, \dots, N-1\}$, the discrete measure $\mu_{k,N} := \frac{1}{N}$ $(k = 0, 1, \dots, N-1)$, and the complex trigonometric system $p_j(x) := e^{ijx}$ $(x \in \mathbb{R},\ j \in \mathbb{Z})$.

In this case we assume that the summation function $\Theta : \mathbb{R} \to \mathbb{R}$ is even and on the interval $[0, +\infty)$ it satisfies the conditions of Section 2. The convergence will be considered in the Banach space $(C_{2\pi}, \|\cdot\|_\infty)$, where $C_{2\pi}$ denotes the linear space of complex valued 2π-periodic continuous functions defined on $\mathbb{R}$ and $\|\cdot\|_\infty$ is the supremum norm.

A *necessary and sufficient* condition may be given for the summation function Θ which guarantees the uniform convergence of (2).

Theorem 1 ([10, Theorem 1]). *Suppose that one of the following two conditions holds:*

(a) $\Theta \in \Phi$ *and for the index sequences* $(m_n, n \in \mathbb{N})$, $(N_n, n \in \mathbb{N})$ *we have* $\lim_{n\to+\infty} m_n = +\infty$ *and* $\lim_{n\to+\infty}(N_n - m_n) = +\infty$,

(b) $\Theta \in \Phi$, Θ *is continuous at the point* 1 *and the index sequences* $(m_n, n \in \mathbb{N})$ *and* $(N_n, n \in \mathbb{N})$ *satisfy the relations* $\lim_{n\to+\infty} m_n = +\infty$ *and* $N_n \geq m_n(1 + o(1))$ $(n \to +\infty)$.

Then the sequence $(S^{\Theta}_{m_n,N_n} f, n \in \mathbb{N})$ *converges uniformly on* $\mathbb{R}$ *to* f *for every* $f \in C_{2\pi}$ *if and only if the Fourier transform of* Θ, *i.e., the function*

$$\hat{\Theta}(x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Theta(t)e^{-ixt}dt \qquad (x \in \mathbb{R}),$$

is Lebesgue integrable on $\mathbb{R}$.

This statement is a discrete version of a well-known fundamental result in the theory of Fourier series (see [7, p. 168]). We also remark that by choosing different parameters of these operators, different orders of the uniform convergence can be attained (see [10, Section 5]).

Several interpolatory properties of $S^{\Theta}_{m,N} f$ can be seen immediately using the following result:

Theorem 2 ([11, Lemma 3]). *Suppose that the function* $\Theta : \mathbb{R} \to \mathbb{R}$ *is an even function supported in* $[-1, 1]$, $\Theta(1) = 0$ *and* $N \geq m$.

(a) The polynomial $S^{\Theta}_{m,N} f$ *interpolates the function* f *at the points of* X_N *if and only if*

$$\Theta\Big(\frac{j}{m}\Big) + \Theta\Big(\frac{N-j}{m}\Big) = 1 \qquad (j = 1, 2, \ldots, N-1).$$

(b) If r *is a positive integer, then* $\big(S^{\varphi}_{m,N} f\big)^{(r)}(x_{k,N}) = 0$ $(k = 0, 1, \ldots, N-1)$ *if and only if*

$$j^r\Theta\Big(\frac{j}{m}\Big) + (-1)^r(N-j)^r\Theta\Big(\frac{N-j}{m}\Big) = 0, \qquad (j = 1, 2, \ldots, N-1).$$

Theorems 1 and 2 generalize a lot of earlier results in interpolation theory, giving also the order of convergence (see [11]).

Here we emphasize on the following: The $(0, r)$ lacunary interpolatory polynomials $(r \in \mathbb{N})$ can be obtained by a suitable Θ-summation and the convergence behaviour of these processes can be easily described immediately from their explicit forms (compare [11] with [8, Section 4 of Chapter VIII]).

4. The Algebraic Cases

The algebraic cases are more complicated. Our far-reaching program is to find the analogue of Theorem 1. Next we formulate some results in this direction.

4.1. Processes Using the Roots of Chebyshev Polynomials

Let us fix $N \in \mathbb{N}$ and consider a given point system $X_N \subset [-1,1]$. The *index of the point* $x \in X_N$ is defined to be 1 if $x \in (-1,1)$ and is $1/2$ if $x \in \{-1,1\}$. The *index of the point system* X_N is the sum of the indices of its points. It will be denoted by $I_{X_N} =: I_N$. It is clear that $I_N = N, N-1/2$, or $N-1$, for any X_N.

Let us define the measure μ_N by

$$\mu_{k,N} := \begin{cases} \dfrac{1}{2I_N}, & \text{if } x_{k,N} \in \{-1,1\} \\ \dfrac{1}{I_N}, & \text{if } x_{k,N} \in (-1,1) \end{cases} \qquad (k = 1,2,\dots,N,\ N \in \mathbb{N}). \tag{3}$$

We shall choose the basis in the following way

$$P_m := \{T_0, \sqrt{2}T_1, \sqrt{2}T_2, \dots, \sqrt{2}T_m\} \subset \mathcal{P}_m,$$

where $T_l(x) := \cos(l \arccos x)$ ($l \in \mathbb{N}_0 := \{0,1,2,\dots\}$) is the Chebyshev polynomial of the first kind.

For every $m, N \in \mathbb{N}$ the point system X_N determines uniquely the parameters (X_N, μ_N, P_m) of sequence (2). Therefore, in the sequel, speaking about an X_N*-system*, we mean (X_N, μ_N, P_m).

We shall consider the following four X_N-systems.

$$\mathbf{T}_N := \Big\{x_{k,N} := \cos\frac{2k-1}{2N}\pi \mid k = 1,2,\dots,N\Big\},$$

$\mathbf{T}_N$ are the roots of T_N (the Chebyshev polynomial of the first kind);

$$\mathbf{U}_N^{\pm} := \Big\{x_{k,N} := \cos\frac{k-1}{N-1}\pi \mid k = 1,2,\dots,N\Big\},$$

$\mathbf{U}_N^{\pm}$ are the roots of U_{N-2} (the Chebyshev polynomial of the second kind) supplemented with the endpoints -1 and 1;

$$\mathbf{V}_N^{-} := \Big\{x_{k,N} := \cos\frac{2k-1}{2N-1}\pi \mid k = 1,2,\dots,N\Big\},$$

$\mathbf{V}_N^{-}$ are the roots of V_{N-1} (the Chebyshev polynomial of the third kind) supplemented with -1;

$$\mathbf{W}_N^{+} := \Big\{x_{k,N} := \cos\frac{2(k-1)}{2N-1}\pi \mid k = 1,2,\dots,N\Big\},$$

$\mathbf{W}_N^+$ are the roots of W_{N-1} (the Chebyshev polynomial of the fourth kind) supplemented with 1.

In the following statement we give a *sufficient* condition on the summation function Θ which guarantees the uniform convergence of (2) for the above four point systems.

Theorem 3 ([13, Theorem 6.2]). *Let X_N be one of the point systems $\mathbf{T}_N$, $\mathbf{U}_N^{\pm}$, $\mathbf{V}_N^-$, $\mathbf{W}_N^+$. Suppose that $m_n \to +\infty$ if $n \to +\infty$ and $m_n \le 2I_{N_n}$ $(n \in \mathbb{N})$. If for a given summation function $\Theta \in \Phi$, the function*

$$\hat{\Theta}(x) := \frac{1}{2\pi}\int_0^{+\infty} \Theta(t)\cos(tx)\,dt \qquad \big(x \in [0,+\infty)\big)$$

is Lebesgue integrable on $[0,+\infty)$, then the sequence $S^{\Theta}_{m_n,N_n} f$ $(n \in \mathbb{N})$ converges uniformly on $[-1,1]$ to f for all $f \in C[-1,1]$.

The polynomials $S^{\Theta}_{2I_N,N} f$ are of degree $< 2I_N$. It is clear that among them there are a lot which interpolate the function f at the points of X_N. In [13, Theorem 5.1] we give a *necessary and sufficient* condition on the summation function Θ to satisfy this requirement.

4.2. Processes Based on the Roots of Orthogonal Polynomials

Let us choose the parameters X_N, μ_N, P_m, defined in Section 2 in the following way. Suppose that $w : I \to \mathbb{R}$ is a weight, X_N consists of the roots of $p_N(w)$ (the orthonormal polynomial for the weight w), $\mu_{k,N}$'s are the corresponding Cotes numbers and $P_m := \{p_0(w), \dots, p_m(w)\}$.

The starting result is due to Grünwald [3]: The Rogosinski type summation, based on the roots of $\mathbf{T}_N$, is uniformly convergent. On the Jacobi roots, the uniform convergence takes place only on $[a,b] \subset (-1,1)$ (see [18] and [17]).

A significant observation shows that if we consider the *weighted approximation* with another suitable weight ϱ, then uniform convergence may be obtained on the *whole* interval I. Therefore, in these cases new difficulties arise: the suitable choice of the weight ϱ. In [12] we investigated this problem for the roots of Jacobi polynomials. A special case of the results in [12] is formulated below.

Theorem 4 ([12, Corollary 4.2]). *Let $w(x) := (1-x)^{\alpha}(1+x)^{\beta}$ $(x \in (-1,1)$, $\alpha, \beta \ge -1/2$. Choose X_N, μ_N and P_m as above. Assume that $\Theta : [0,1] \to \mathbb{R}$, is convex, $\Theta(0) = 1$, $\Theta(1) = 0$, $\Theta \in \mathrm{Lip}_1[0,1]$. Then*

$$\lim_{N\to+\infty} \|(f - S^{\Theta}_{N,N} f)\varrho\|_{\infty} = 0 \qquad (\forall f \in C_{\varrho}),$$

where $\varrho(x) = (1-x)^{\frac{\alpha}{2}+\frac{1}{4}}(1+x)^{\frac{\beta}{2}+\frac{1}{4}}$ $(x \in (-1,1))$ and $C_{\varrho} := \{f \in C(-1,1) \mid \lim_{\pm 1} f\varrho = 0\}$.

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