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Chebyshev Rational Functions

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Dedicated to Professor Bl. Sendov on the occasion of his 70th birthday

In this paper, Chebyshev polynomials of the first and second kind are introduced and studied for the system

$$\Big\{1, x, x^2, \dots, x^n, \frac{1}{x-a_1}, \frac{1}{x-a_2}, \dots, \frac{1}{x-a_m}\Big\}$$

where $a_k,\ |a_k| > 1$, are real. Bernstein and Markov type inequalities for the derivative of the Chebyshev polynomials of the first kind are proved (Theorem 3). Recursion formulas for these polynomials are established.

1. Introduction

Let us denote by $\{\phi_j\}_{j=0}^{n+m}$ the system of functions

$$1, x, x^2, \dots, x^n, \frac{1}{x-a_1}, \frac{1}{x-a_2}, \dots, \frac{1}{x-a_m} \tag{1}$$

where $\{a_j\}$ are preassigned distinct points such that $|a_j| > 1$. We shall assume that $\{a_j\}$ are real, althought our study can be easily modified to cover also the case of complex a_j. The functions (1) form a Chebyshev system on the interval $[-1, 1]$. We shall study some classical extremal problems in the class of generalized polynomials $p(x) = \sum_{k=0}^{n+m} c_k\phi_k(x)$.

Definition 1. *The Chebyshev polynomial of the first kind for the system* (1)*, or Chebyshev rational function, is defined as a solution of the extremal problem*

$$\min_{c_j}\Big\{\|p\|_{L^\infty_{[-1,1]}}\ :\ p(x) = \sum_{j=0}^{n+m} c_j\phi_j(x),\ c_n = 1\Big\}. \tag{2}$$

Note that $\phi_n(x) = x^n$, and thus, in the case $m = 0$ the solution coincides with the classical Chebyshev polynomial. The case $n = 0$ was considered in [2].

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2. Chebyshev Rational Functions

Throughout this paper we shall use the substitutions

$$x = \frac{1}{2}\Big(v + \frac{1}{v}\Big), \qquad a_k = \frac{1}{2}\Big(\alpha_k + \frac{1}{\alpha_k}\Big), \qquad |\alpha_k| < 1,$$

for complex numbers v, and for $|a_k| > 1$. Evidently, if $|v| = 1$, then $x \in [-1, 1]$, and if a_k are real and $|a_k| > 1$, then $-1 < \alpha_k < 1$.

Recall that the Blaschke product for $\alpha_1, \dots, \alpha_m$ is defined by

$$B_m(v) := \prod_{k=1}^{m} \frac{v - \alpha_k}{1 - \alpha_k v}$$

and $B_0(v) \equiv 1$. Let us denote

$$T_{n,m}(x) := \frac{M_{n,m}}{2}\left(f_{n,m}(v) + \frac{1}{f_{n,m}(v)}\right), \tag{3}$$

$$U_{n,m}(x) := \frac{M_{n,m}}{v - v^{-1}}\left(f_{n,m}(v) - \frac{1}{f_{n,m}(v)}\right), \tag{4}$$

where $f_{n,m}(v) = v^n B_{n,m}(v)$. The functions of type $T_{n,1}$ have been used by Achiezer in [1] to find the best uniform algebraic approximation of the function $(x - a)^{-1}$.

The rational function $T_{n,m}$ does not change, while $U_{n,m}$ changes its sign if we substitute v by v^{-1}, because $B_{n,m}(v^{-1}) = B_{n,m}^{-1}(v)$, for real α_k. So they are functions of x. This can be seen directly too.

It is easy to verify that the function $T_{n,m}$ can be represented in the form

$$T_{n,m}(x) = \sum_{k=0}^{n} A_k x^k + \sum_{k=1}^{m} \frac{B_k}{x - a_k}.$$

In order to make the coefficient A_n equal to 1 we choose the constant $M_{n,m}$ so that

$$\lim_{x\to\infty} \frac{T_{n,m}(x)}{x^n} = 1 \;\Leftrightarrow\; \lim_{v\to 0} \frac{T_{n,m}}{\big(\frac{1}{2}(v + v^{-1})\big)^n} = 1.$$

Consequently,

$$M_{0,m} = 2(-1)^m \frac{\prod_{k=1}^{m} \alpha_k}{\prod_{k=1}^{m} \alpha_k^2 + 1}, \qquad M_{n,m} = (-1)^m\, 2^{-n+1} \prod_{k=1}^{m} \alpha_k, \quad n > 0.$$

We call the rational functions $T_{n,m}$ and $U_{n,m}$, respectively, the Chebyshev rational function (or simply the Chebyshev polynomial) of the first kind and the Chebyshev polynomial of the second kind. We shall prove that $T_{n,m}$ is the extremal function in the problem (2).

Lemma 1. *For any x, $x \neq a_k$, we have*

$$T_{n,m}^2(x) + (1 - x^2)\, U_{n,m}^2(x) = M_{n,m}^2.$$

Proof. From equations (3) and (4) we find

$$\begin{aligned} f_{n,m} &= \frac{T_{n,m}}{M_{n,m}} + \frac{(v - v^{-1})\, U_{n,m}}{2M_{n,m}}, \\ f_{n,m}^{-1} &= \frac{T_{n,m}}{M_{n,m}} - \frac{(v - v^{-1})\, U_{n,m}}{2M_{n,m}}. \end{aligned}$$

If we multiply these two equations and note that $(v - v^{-1})^2 = 4(x^2 - 1)$, we shall arrive at the wanted relation.

Theorem 1. *The functions $T_{n,m}(x)$ and $\sqrt{1-x^2}\, U_{n,m}(x)$ oscillate between $M_{n,m}$ and $-M_{n,m}$ in the interval $[-1, 1]$, taking the values $\pm M_{n,m}$ alternatively in $(n + m + 1)$, and $(n + m)$ points, respectively.*

Proof. Lemma 1 implies that $|T_{n,m}(x)| \leq |M_{n,m}|$ and $|(1 - x^2)\, U_{n,m}(x)| \leq |M_{n,m}|$ for $x \in [-1, 1]$. The function $T_{n,m}(x)$ (respectively, $\sqrt{1-x^2}\, U_{n,m}(x)$) takes the values $\pm M_{n,m}$ if and only if $f_{n,m}(v) = \pm 1$ ($f_{n,m}(v) = \pm i$, respectively). Since $f_{n,m}(z)$ is analytic in the unit disk $\overline{D} := \{z : |z| \leq 1\}$ and has exactly $n + m$ zeros in D, then the theorem follows from the Argument Principle.

Theorem 2. *The Chebyshev polynomial of the first kind $T_{n,m}$ is the only solution of problem* (2)*.*

Proof. The function $T_{n,m}$ is of the type

$$T_{n,m} = \frac{P(x)}{Q(x)} = x^n + A_1 x^{n-1} + \cdots + A_n + \sum_{i=1}^{m} \frac{B_i}{x - a_i}.$$

If there exists another rational function of the same type, say,

$$\tilde{T}_{n,m} = \frac{\tilde{P}(x)}{Q(x)} = x^n + \tilde{A}_1 x^{n-1} + \cdots + \tilde{A}_n + \sum_{i=1}^{m} \frac{\tilde{B}_i}{x - a_i}$$

such that

$$\|\tilde{P}/Q\|_{L^\infty_{[-1,1]}} \leq \|P/Q\|_{L^\infty_{[-1,1]}},$$

then by Theorem 1, the difference $T - \tilde{T}$ will have at least $n + m$ zeros. But $P - \tilde{P}$ is a polynomial of degree $n + m - 1$, which implies $P \equiv \tilde{P}$, i.e., $T \equiv \tilde{T}$.

3. Differential Properties of the Chebyshev Polynomials

The next lemma is an analog of the classical result that the Chebyshev polynomial of the second kind is the derivative of that of the first kind.

Lemma 2. *For $x \neq a_k$, $k = 1, \dots, m$, we have*

$$T'_{n,m}(x) = U_{n,m}(x) R_{n,m}(x). \tag{5}$$

For $x \in (-1, 1)$ we have

$$\left(\sqrt{1-x^2}\, U_{n,m}(x)\right)' = -T_{n,m}(x) \frac{R_{n,m}(x)}{\sqrt{1-x^2}}, \tag{6}$$

and for $|x| > 1$, $x \neq a_k$, $k = 1, \dots, m$, we have

$$\left(\sqrt{x^2-1}\, U_{n,m}(x)\right)' = T_{n,m}(x) \frac{R_{n,m}(x)}{\sqrt{x^2-1}}, \tag{7}$$

where

$$R_{n,m}(x) = n + \sum_{k=1}^{m} \frac{\operatorname{sgn} a_k \cdot \sqrt{a_k^2 - 1}}{a_k - x}.$$

Proof. Evidently

$$\frac{dT_{n,m}(x)}{dx} = \frac{M}{2} \cdot \frac{d(f + f^{-1})}{dv} \cdot \frac{dv}{dx} = \frac{M}{2} \left(f - \frac{1}{f}\right) \frac{f'_v}{f}\, v'_x, \tag{8}$$

where

$$f = f_{n,m} = v^n \prod_{k=1}^{m} \frac{v - \alpha_k}{1 - \alpha_k v}.$$

On the other hand,

$$f'_v = n v^{n-1} \prod_{k=1}^{m} \frac{v - \alpha_k}{1 - \alpha_k v} + v^n \sum_{k=1}^{m} \frac{1 - \alpha_k^2}{(1 - \alpha_k v)^2} \prod_{i=1, i \neq k}^{m} \frac{v - \alpha_i}{1 - \alpha_i v}$$

and

$$\frac{f'_v}{f} = \frac{n}{v} + \sum_{k=1}^{m} \frac{1 - \alpha_k^2}{(1 - \alpha_k v)(v - \alpha_k)} = \frac{n}{v} + \frac{1}{v} \sum_{k=1}^{m} \frac{\operatorname{sgn} a_k \cdot \sqrt{a_k^2 - 1}}{a_k - x}, \tag{9}$$

because $(1 - \alpha_k^2)/2\alpha_k = \operatorname{sgn} a_k \cdot \sqrt{\alpha_k^2 - 1}$. Evidently, from $x = 2^{-1}(v + v^{-1})$ we get $v'_x/v = 2/(v - v^{-1})$. Then (9) yields

$$\frac{f'_v}{f}\, v'_x = \frac{2}{v - v^{-1}} \left(n + \sum_{k=1}^{m} \frac{\operatorname{sgn} a_k \cdot \sqrt{a_k^2 - 1}}{a_k - x}\right),$$

which together with (8) gives (5).

The proof of (6) and (7) is the same. We need just to note that

$$\sqrt{1-x^2}\,U_{n,m}(x) = \frac{v-v^{-1}}{2i}\cdot\frac{M_{n,m}}{2}\big(f_{n,m}(v) - f_{n,m}^{-1}(v)\big)$$

and

$$\sqrt{x^2-1}\,U_{n,m}(x) = \frac{v-v^{-1}}{2}\cdot\frac{M_{n,m}}{2}\big(f_{n,m}(v) - f_{n,m}^{-1}(v)\big).$$

If $x \in [-1,1]$, which is the most interesting case, we have

$$R_{n,m}(x) = n + \sum_{k=1}^{m} \frac{\sqrt{a_k^2-1}}{|a_k - x|}. \tag{10}$$

Lemma 3. *For $x \in [-1,1]$ we have*

$$0 < R_{n,m}(x) < \max\{R_{n,m}(1), R_{n,m}(-1)\}.$$

Proof. The first inequality follows from (10). Let us suppose that $\max\{R_{n,m}(1), R_{n,m}(-1)\} = R_{n,m}(1)$. Then, from

$$R_{n,m}(x) = n + \sum_{a_k>1} \frac{\sqrt{a_k^2-1}}{a_k - x} + \sum_{a_k<-1} \frac{\sqrt{a_k^2-1}}{|a_k| + x}$$

we find

$$\begin{aligned} &R_{n,m}(1) - R_{n,m}(x) \\ &\quad = (1-x)\left[\sum_{a_k>1}\Big(\frac{\sqrt{a_k^2-1}}{(a_k-1)(a_k-x)} - \sum_{a_k<-1}\frac{\sqrt{a_k^2-1}}{(|a_k|+1)(|a_k|+x)}\right] \\ &\quad \ge \frac{1-x}{2}\big(R_{n,m}(1) - R_{n,m}(-1)\big) \ge 0. \end{aligned}$$

The case $\max\{R_{n,m}(1), R_{n,m}(-1)\} = R_{n,m}(-1)$ can be proved in a similar manner.

The next theorem generalizes the known classical result.

Theorem 3. *The following inequalities are true:*

$$|T'_{n,m}(x)| \le |M_{n,m}|\,\frac{R_{n,m}(x)}{\sqrt{1-x^2}}, \qquad x \in (-1,1). \tag{11}$$

For $x \in [-1,1]$ we have

$$|T'_{n,m}(x)| \le |T'_{n,m}(1)| = |M_{n,m}|R_{n,m}^2(1), \text{ if } R_{n,m}(1) \ge R_{n,m}(-1), \tag{12}$$

$$|T'_{n,m}(x)| \le |T'_{n,m}(-1)| = |M_{n,m}|R_{n,m}^2(-1), \text{ if } R_{n,m}(-1) \ge R_{n,m}(1). \tag{13}$$

Proof. Let us observe that from Lemma 1 and equations (5) and (6) the following Bernstein-Szegö type equalities in the interval $[-1,1]$ follow:

$$(1-x^2)\left(T'_{n,m}(x)\right)^2 + T^2_{n,m}(x)R^2_{n,m}(x) = M^2_{n,m}R^2_{n,m}(x), \tag{14}$$

and for $x \in (-1,1)$

$$\left[\left(\sqrt{1-x^2}\,U_{n,m}(x)\right)'\right]^2 + U^2_{n,m}(x)R^2_{n,m}(x) = M^2_{n,m}\,\frac{R^2_{n,m}(x)}{1-x^2}. \tag{15}$$

Then the inequality (11) follows from (14).

Let us prove (12). Since

$$|T'_{n,m}(x)| = |U_{n,m}(x)|R_{n,m}(x)$$

and, in this case $R_{n,m}(x) \le R_{n,m}(1)$, it suffices to show that

$$|U_{n,m}(x)| \le |M_{n,m}|R_{n,m}(1), \qquad x \in [-1,1], \tag{16}$$

and

$$|U_{n,m}(1)| = |M_{n,m}|R_{n,m}(1). \tag{17}$$

From equation (6) it follows for $x \in (-1,1)$ that

$$-xU_{n,m}(x) + (1-x^2)U'_{n,m}(x) = -T_{n,m}(x)R_{n,m}(x).$$

Then, from $|U'_{n,m}(1)| < \infty$, if $x \to 1$ we obtain (17), such that $|T_{n,m}(1)| = |M_{n,m}|$. It remains to prove the inequality (16). From the last relations we see that $|U_{n,m}(-1)| < |U_{n,m}(1)|$. Let us suppose that the point b, $b \in (-1,1)$ is a point of local extremum of $U_{n,m}(x)$. Then $U'_{n,m}(b) = 0$, and (15) implies

$$U^2_{n,m}(b) = M^2_{n,m}\,\frac{R^2_{n,m}(b)}{b^2 + (1-b^2)R^2_{n,m}(b)}.$$

From this relation, if $R^2_{n,m}(b) \ge 1$, we have

$$b^2 + (1-b^2)R^2_{n,m}(b) \ge b^2 + (1-b^2) = 1$$

and

$$U^2_{n,m}(b) \le M^2_{n,m}R^2_{n,m}(b) \le M^2_{n,m}R^2_{n,m}(1).$$

If $R^2_{n,m}(b) < 1$, then $b^2 + (1-b^2)R^2_{n,m}(b) \ge R^2_{n,m}(b)$, and

$$U^2_{n,m}(b) \le M^2_{n,m} \le M^2_{n,m}R^2_{n,m}(1),$$

such that $R^2_{n,m}(1) > 1$. This completes the proof.

4. Recursion Formulas

The next Lemma is completely analogous to the classical case for $n \geq 2$.

Lemma 4. *For $n \geq 2$, the recursion formula*

$$T_{n,m}(x) = \frac{M_{n,m}}{M_{n-1,m}}\, 2xT_{n-1,m}(x) - \frac{M_{n,m}}{M_{n-2,m}}\, T_{n-2,m}(x) \tag{18}$$

is true. For $n = 1$ we have

$$T_{1,m}(x) = \frac{M_{1,m}}{M_{0,m}} \left[xT_{0,m}(x) + (x^2-1)U_{0,m}(x)\right], \tag{19}$$

$$U_{1,m}(x) = \frac{M_{1,m}}{M_{0,m}} \left[T_{0,m}(x) + xU_{0,m}(x)\right]. \tag{20}$$

If $n \geq 2$, since $M_{n,m}/M_{n-1,m} = 1/2$ and $M_{n,m}/M_{n-2,m} = 1/4$, we have

$$T_{n,m}(x) = xT_{n-1,m}(x) - \frac{1}{4}T_{n-2,m}(x).$$

Proof. The proof of (18) follows from the relations

$$\begin{aligned}
\frac{4x}{M_{n-1,m}}T_{n-1,m}(x) &= \left[v^{n-1}B_m(v) + v^{-n+1}B_m^{-1}(v)\right](v+v^{-1}) \\
&= \left[v^{n}B_m(v) + v^{-n}B_m^{-1}(v)\right] + \left[v^{n-2}B_m(v) + v^{-n+2}B_m^{-1}(v)\right] \\
&= \frac{2T_{n,m}}{M_{n,m}} + \frac{2T_{n-2,m}}{M_{n-2,m}}.
\end{aligned}$$

The proof of the equation (19) follows from the relations:

$$\frac{4xT_{0,m}(x)}{M_{0,m}} = vB_m(v) + v^{-1}B_m^{-1}(v) + vB_m^{-1}(v) + v^{-1}B_m(v),$$

and

$$\frac{4(x^2-1)U_{0,m}(x)}{M_{0,m}} = vB_m(v) + v^{-1}B_m^{-1}(v) - vB_m^{-1}(v) - v^{-1}B_m(v).$$

The proof of (20) follows from (19) after differentiation, via (5) and (6), and the relation $R_{1,m}(x) = 1 + R_{0,m}(x)$.

Lemma 5. *For $x \in [-1,1]$ and $m \geq 1$ the recursion formulas*

$$T_{n,m} = \frac{M_{n,m}}{M_{n,m-1}} \left(\frac{a_m x - 1}{a_m - x}\, T_{n,m-1} + \frac{\sqrt{a_m^2-1}\,(x^2-1)}{|a_m - x|}\, U_{n,m-1}\right) \tag{21}$$

and

$$U_{n,m} = \frac{M_{n,m}}{M_{n,m-1}} \left(\frac{a_m x - 1}{a_m - x}\, U_{n,m-1} + \frac{\sqrt{a_m^2-1}}{|a_m - x|}\, T_{n,m-1}\right) \tag{22}$$

are true.

Proof. Since $B_m(v) = B_{m-1}(v)\dfrac{v-\alpha_m}{1-\alpha_m v}$, we have

$$\begin{aligned}
&\frac{2T_{n,m-1}}{M_{n,m-1}}\left(\frac{v-\alpha_m}{1-\alpha_m v}+\frac{1-\alpha_m v}{v-\alpha_m}\right)\\
&= \left[v^n B_{m-1}(v)+\frac{1}{v^n B_{m-1}(v)}\right]\left[\frac{v-\alpha_m}{1-\alpha_m v}+\frac{1-\alpha_m v}{v-\alpha_m}\right]\\
&= v^n B_m(v) + v^{-n}B_m^{-1}(v) + v^{-n}B_{m-1}^{-1}(v)\frac{v-\alpha_m}{1-\alpha_m v} + v^n B_{m-1}(v)\frac{1-\alpha_m v}{v-\alpha_m}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\frac{(v-v^{-1})U_{n,m-1}}{M_{n,m-1}}\left(\frac{v-\alpha_m}{1-\alpha_m v}-\frac{1-\alpha_m v}{v-\alpha_m}\right)\\
&= v^n B_m(v) + v^{-n}B_m^{-1}(v) - v^{-n}B_{m-1}^{-1}(v)\frac{v-\alpha_m}{1-\alpha_m v} - v^n B_{m-1}(v)\frac{1-\alpha_m v}{v-\alpha_m}.
\end{aligned}$$

Now, if we add the last two equalities and observe that

$$\frac{v-\alpha_m}{1-\alpha_m v}+\frac{1-\alpha_m v}{v-\alpha_m} = 2\,\frac{a_m x-1}{a_m-x},$$

and

$$\left(v-\frac{1}{v}\right)\left(\frac{v-\alpha_m}{1-\alpha_m v}-\frac{1-\alpha_m v}{v-\alpha_m}\right) = 4\,\frac{\operatorname{sgn} a_m\cdot\sqrt{a_m^2-1}\,(x^2-1)}{a_m-x},$$

we obtain (21). The proof of (22) follows from (21) after differentiation, via (5) and (6) and the relation

$$R_{n,m}(x) = R_{n,m-1}(x) + \frac{\sqrt{a_m^2-1}}{|a_m-x|},\qquad x\in[-1,1].$$

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