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Some New Integrability Classes of Fourier Coefficients

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Using the fractional derivatives of Weyl type we generalize some known classes of Fourier coefficients and some inequalities of Sidon-Fomin type for trigonometric polynomials. New L^1-estimate of Telyakovskii type for fractional derivative of cosine series is also given. Finally, some embedding relations are presented.

1. Introduction and Preliminaries

The definition for fractional derivatives of certain trigonometric series was introduced firstly in [13] by Weyl (see also [3, p. 263]; [14, XII]).

For $\alpha \geq 0$, let us consider the series

$$\sum_{k=1}^{\infty} k^{\alpha} a_k \cos\left(kx + \frac{\alpha\pi}{2}\right). \tag{1}$$

If (1) is Fourier series of some function $f_\alpha(x)$, then its sum function is denoted by $\mathcal{D}_{\pm}^{(\alpha)} f(x)$ and is called the fractional derivative of order α of the series f, where

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\,. \tag{2}$$

The following inequality is known as Bernstein inequality for fractional derivatives of Weyl type for trigonometric polynomials in L^p space, $1 \leq p \leq \infty$.

Theorem A [3]. *If $T_n(x)$ is a trigonometric polynomial of order n, then for all $1 \leq p \leq \infty$ the following inequality holds*

$$\|\mathcal{D}_{\pm}^{(\alpha)} T_n\| \leq \begin{cases} \frac{2^{1-\alpha}}{\Gamma(2-\alpha)}\, n^{\alpha}\, \|T_n\|_p\,, & 0 \leq \alpha \leq 1 \\ n^{\alpha}\, \|T_n\|_p\,, & \alpha \geq 1\,. \end{cases}$$

In [9], we have proved the following theorem.

Theorem B. *Let $\{\lambda_k\}_{k=1}^n$ be a sequence of real numbers. Then for any $1 < p \le 2$, $n \ge 1$, $r = 0, 1, \dots$,*

$$\Big\| \sum_{k=1}^n \lambda_k D_k^{(r)}(x) \Big\|_1 \le M_p\, n^{r+1} \left(\frac{1}{n} \sum_{k=1}^n |\lambda_k|^p \right)^{1/p},$$

where M_p is a positive constant which depends only on p, $D_k^{(r)}$ is the r-th derivative of Dirichlet kernel D_k and $\|\cdot\|_1$ is the L^1-norm.

For $r = 0$ this inequality is known as Bojanić-Stanojević inequality, which is often used in the theory of trigonometric series, especially in L^1-integrability problems. Specially, if $|\lambda_k| \le 1$, $k = 1, 2, \dots, n$, we obtain the extension of the Sidon-Fomin type inequality ([6]), proved in [7].

Theorem C [7]. *Let $\{\lambda_k\}_{k=1}^n$ be a sequence of real numbers such that $|\lambda_k| \le 1$, for all k. Then there exists a constant $M > 0$ such that for any $n \ge 1$ and $r = 0, 1, \dots$,*

$$\Big\| \sum_{k=1}^n \lambda_k D_k^{(r)}(x) \Big\| \le M n^{r+1}\,.$$

A null-sequence of real numbers $\{a_k\}_{k=0}^\infty$ belongs to the class S_α, $\alpha \ge 0$, if there exists a monotone decreasing sequence $\{A_k\}_{k=0}^\infty$ such that $\sum\limits_{k=0}^\infty k^\alpha A_k < \infty$ and $|\Delta a_k| \le A_k$, for all k. In connection with the class S_α, $\alpha \ge 0$, in [12] we obtained new L^1-estimate for the series (1).

We note that for $\alpha = r = 1, 2, \dots$, this class was defined in [7] and for $\alpha = 0$ it is the Sidon-Telyakovskii class (see [6]).

Theorem D [12]. *Let the coefficients of the series* (1) *belong to the class S_α, $\alpha \ge 0$. Then the series* (1) *is a Fourier series of some $\mathcal{D}_\pm^{(\alpha)} f \in L^1(0,\pi)$ and the following inequality holds:*

$$\int_0^\pi |\mathcal{D}_\pm^{(\alpha)} f(x)|dx \le M \sum_{n=0}^\infty n^\alpha A_n\,, \qquad \text{where } 0 < M = M(\alpha) < \infty\,.$$

A null-sequence $\{a_k\}_{k=0}^\infty$ belongs to the class C_α, $\alpha \ge 0$, if for every $\varepsilon > 0$ there exists $\delta > 0$ independent of n, such that

$$\int_0^\delta \Big| \sum_{k=n}^\infty \Delta a_k \mathcal{D}_\pm^{(\alpha)} D_k(x) \Big| dx < \varepsilon\,.$$

For $\alpha=0$, we obtain the Garrett-Stanojević class C, and for $\alpha=r=1,2,\dots$ this class was defined by the author in [8].

Denote by I_m, the dyadic interval $[2^{m-1},2^m)$, $m\ge 1$.

A null-sequence $\{a_k\}_{k=1}^{\infty}$ belongs to the class $\mathcal{F}_{p\alpha}$, $p>1$, $\alpha\ge 0$, if

$$\sum_{m=1}^{\infty} 2^{m(1/q+\alpha)}\left(\sum_{k\in I_m}|\Delta a_k|^p\right)^{1/p}<\infty\,,\qquad \text{where } \frac{1}{p}+\frac{1}{q}=1\,.$$

This class for $\alpha=r=1,2,\dots$ was defined in [9] and for $\alpha=0$ it is the Fomin's class $\mathcal{F}_p$, $p>1$, defined in [1].

A null-sequence $\{a_k\}_{k=1}^{\infty}$ belongs to the class $S_{p\alpha}$, $p>0$, $\alpha\ge 0$, if there exists a monotone decreasing sequence $\{A_k\}_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} k^{\alpha}A_k<\infty \qquad \text{and} \qquad \frac{1}{n}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{A_k^p}=O(1)\,.$$

It is obvious that for $\alpha=0$ we obtain the class of Č. V. Stanojević and V. B. Stanojević (see [5]) and for $\alpha=r=1,2,\dots$, it is the class defined by the author in [8].

On the other hand, we say that the null-sequence $\{a_k\}_{k=1}^{\infty}$ belongs to the class $S_{p\alpha\beta}$, $p>1$, $0\le\beta\le\alpha$, if there exists a monotone decreasing sequence $\{A_k\}_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} k^{\alpha}A_k<\infty \qquad \text{and} \qquad \frac{1}{n^{p(\alpha-\beta)+1}}\sum_{k=1}^{n}\frac{|\Delta a_k|^p}{A_k^p}=O(1).$$

For $\beta=r\in\{0,1,\dots,[\alpha]\}$ this class was defined by Sheng in [4], but also it was refined by the author in [11].

Concerning the class $S_{p\alpha r}$, $p>1$, $\alpha\ge 0$, $r\in\{0,1,\dots,[\alpha]\}$ we proved in [10] the following theorem.

Theorem E [10]. *Let the coefficients of the series* (2) *belong to the class* $S_{p\alpha r}$, $1<p\le 2$, $\alpha\ge 0$, $r\in\{0,1,\dots,[\alpha]\}$. *Then the* r*-th derivative of the series* (2) *is a Fourier series of some* $f^{(r)}\in L^1(0,\pi)$ *and the following inequality holds:*

$$\int_0^{\pi}|f^{(r)}(x)|dx\le M_{p,\alpha}\sum_{n=1}^{\infty}n^{\alpha}A_n\,,$$

where $M_{p,\alpha}$ *is a positive constant which depends on* p *and* α.

2. Sidon-Fomin Type Inequalities and Applications

Theorem 1. *Let $\{\lambda_k\}_{k=1}^n$ be a sequence of real numbers. Then for any $1<p\le 2$ and $\alpha\ge 0$, $n\in\mathbb{N}$, the following inequality holds:*

$$\Big\|\sum_{k=1}^{n}\lambda_k\mathcal{D}_{\pm}^{(\alpha)}D_k(x)\Big\|\le\begin{cases}\frac{2^{1-\alpha}}{\Gamma(2-\alpha)}\,M_p n^{\alpha+1}\left(\frac{1}{n}\sum_{k=1}^{n}|\lambda_k|^p\right)^{1/p}, & 0\le\alpha\le 1\\ M_p n^{\alpha+1}\left(\frac{1}{n}\sum_{k=1}^{n}|\lambda_k|^p\right)^{1/p}, & \alpha\ge 1,\end{cases}$$

where the constant M_p depends only on p.

Proof. Applying firstly Theorem A, then Bojanić-Stanojević inequality, we obtain

$$\begin{aligned}\Big\|\sum_{k=1}^{n}\lambda_k\mathcal{D}_{\pm}^{(\alpha)}D_k(x)\Big\| &\le\begin{cases}\frac{2^{1-\alpha}}{\Gamma(2-\alpha)}\,n^{\alpha}\left\|\sum_{k=1}^{n}\lambda_kD_k(x)\right\|, & 0\le\alpha\le 1\\ n^{\alpha}\left\|\sum_{k=1}^{n}\lambda_kD_k(x)\right\|, & \alpha\ge 1\end{cases}\\ &\le\begin{cases}\frac{2^{1-\alpha}}{\Gamma(2-\alpha)}\,M_p n^{\alpha+1}\left(\frac{1}{n}\sum_{k=1}^{n}|\lambda_k|^p\right)^{1/p}, & 0\le\alpha\le 1\\ M_p n^{\alpha+1}\left(\frac{1}{n}\sum_{k=1}^{n}|\lambda_k|^p\right)^{1/p}, & \alpha\ge 1.\end{cases}\end{aligned}$$

Corollary 1. *Let $\{\lambda_k\}_{k=1}^n$ be a sequence of real numbers such that $|\lambda_k|\le 1$ for all k. Then, for all $\alpha\ge 0$, the following inequality holds*

$$\Big\|\sum_{k=1}^{n}\lambda_k\mathcal{D}_{\pm}^{(\alpha)}D_k(x)\Big\|\le\begin{cases}M\,\frac{2^{1-\alpha}}{\Gamma(2-\alpha)}\,n^{\alpha+1} & 0\le\alpha\le 1\\ M\,n^{\alpha+1} & \alpha\ge 1,\end{cases}$$

where $M>0$.

Theorem 2. *Let $\{a_k\}_{k=1}^n\in S_{p\alpha}$, $1<p\le 2$, $\alpha\ge 0$, $n\in\mathbb{N}$. Then*

$$\int_0^{\pi}\Big|\sum_{j=1}^{k}\frac{\Delta a_j}{A_j}\,\mathcal{D}_{\pm}^{(\alpha)}D_j(x)\Big|dx=\begin{cases}O_{p,\alpha}\left(k^{\alpha+1}\right), & 0\le\alpha\le 1\\ O_p\left(k^{\alpha+1}\right), & \alpha\ge 1.\end{cases}$$

Proof. Putting $\lambda_j=\dfrac{\Delta a_j}{A_j}$, $j=1,2,\dots,k$, in Theorem 1, we obtain

$$\begin{aligned}\int_0^{\pi}\Big|\sum_{j=1}^{k}\frac{\Delta a_j}{A_j}\,\mathcal{D}_{\pm}^{(\alpha)}D_j(x)\Big|dx &\le\begin{cases}\frac{2^{1-\alpha}}{\Gamma(2-\alpha)}\,M_p k^{\alpha+1}\left(\frac{1}{k}\sum_{j=1}^{k}\frac{|\Delta a_j|^p}{A_j^p}\right)^{1/p}, & 0\le\alpha\le 1\\ M_p k^{\alpha+1}\left(\frac{1}{k}\sum_{j=1}^{k}\frac{|\Delta a_j|^p}{A_j^p}\right)^{1/p}, & \alpha\ge 1\end{cases}\\ &=\begin{cases}O_{p,\alpha}\left(k^{\alpha+1}\right), & 0\le\alpha\le 1\\ O_p\left(k^{\alpha+1}\right), & \alpha\ge 1.\end{cases}\end{aligned}$$

Theorem 3. *Let* $\{a_k\}_{k=1}^n \in S_{p\alpha\beta}$, $1 \le p < 2$, $0 \le \beta \le \alpha$, $n \in \mathbb{N}$. *Then*

$$\int_0^\pi \Big|\sum_{j=1}^k \frac{\Delta a_j}{A_j}\,\mathcal{D}_\pm^{(\beta)} D_j(x)\Big|dx = \begin{cases} O_{p,\beta}\left(k^{\alpha+1}\right), & 0 \le \beta \le 1 \\ O_p\left(k^{\alpha+1}\right), & \beta \ge 1\,. \end{cases}$$

Proof. Putting $\lambda_j = \dfrac{\Delta a_j}{A_j}$, $j = 1, 2, \ldots, k$, in Theorem 1 and using the equality

$$n^{\beta+1}\left(\frac{1}{n}\sum_{k=1}^n \frac{|\Delta a_k|^p}{A_k^p}\right)^{1/p} = n^{\alpha+1}\left(\frac{1}{n^{p(\alpha-\beta)+1}}\sum_{k=1}^n \frac{|\Delta a_k|^p}{A_k^p}\right)^{1/p},$$

we prove the assertion.

Corollary 2. *Let* $\{a_k\}_{k=1}^n \in S_{p\alpha\beta}$, $1 < p \le 2$, $0 \le \beta \le \alpha$, $n \in \mathbb{N}$. *Then*

$$A_k \int_0^\pi \Big|\sum_{j=1}^k \frac{\Delta a_j}{A_j}\mathcal{D}_\pm^{(\beta)} D_j(x)\Big|dx = o(1), \qquad k \to \infty\,.$$

Proof. Applying Theorem 3 and using the known limit $k^{\alpha+1}A_k = o(1)$, $k \to \infty$, we obtain

$$A_k \int_0^\pi \Big|\sum_{j=1}^k \frac{\Delta a_j}{A_j}\mathcal{D}_\pm^{(\beta)} D_j(x)\Big|dx = \begin{cases} O_{p,\beta}\left(k^{\alpha+1}A_k\right), & 0 \le \beta \le 1 \\ O_p\left(k^{\alpha+1}A_k\right), & \beta \ge 1 \end{cases} = o(1), \ \ k \to \infty.$$

Using the same technique as that applied in the proof of Theorem E, by Theorem 3 and Corollary 2, we obtain the following theorem.

Theorem 4. *Let the coefficients of the series* (2) *belong to the class* $S_{p\alpha\beta}$, $1 < p \le 2$, $0 \le \beta \le \alpha$. *Then the fractional derivatives of order* β *of the series* (2) *is a Fourier series of some* $\mathcal{D}_\pm^{(\beta)} f \in L^1(0,\pi)$ *and*

$$\int_0^\pi \Big|\mathcal{D}_\pm^{(\beta)} f(x)\Big|dx \le \begin{cases} M_{p,\beta}\sum_{k=1}^\infty k^\alpha A_k\,, & 0 \le \beta \le 1 \\ B_p \sum_{k=1}^\infty k^\alpha A_k\,, & \beta \ge 1\,, \end{cases}$$

where $M_{p,\beta}$ *is a positive constant which depends on* p *and* β, *and* B_p *is a positive constant depending only on* p.

3. Embedding Relations

Using Corollary 1 and the technique of the proof of Theorem 4 in [7], we obtain the following theorem.

Theorem 5. *For all $\alpha \geq 0$ the following embedding relation holds: $S_\alpha \subset BV \cap C_\alpha$.*

Theorem 6. *For all $p > 1$, $0 \leq \beta \leq \alpha$ the following embedding relation holds: $S_{p\alpha\beta} \subset \mathcal{F}_{p\beta}$.*

Proof. By the condition $\dfrac{1}{n^{p(\alpha-\beta)+1}} \sum_{k=1}^{n} \dfrac{|\Delta a_k|^p}{A_k^p} = O(1)$ and the monotonicity of $\{A_k\}$ we obtain

$$\begin{aligned}\left(\sum_{k \in I_m} |\Delta a_k|^p\right)^{1/p} &\leq 2^{(m-1)\frac{1}{p}} \left(\frac{1}{2^{m-1}} \sum_{k \in I_m} \frac{|\Delta a_k|^p}{A_k^p}\right)^{1/p} A_{2^{m-1}} \\ &\leq K\, 2^{m/p}\, 2^{m(\alpha-\beta)}\, A_{2^{m-1}}\,,\end{aligned}$$

where K is an absolute constant. Hence,

$$\begin{aligned}\sum_{m=1}^{\infty} 2^{m(1/q+\beta)} \left(\sum_{k \in I_m} |\Delta a_k|^p\right)^{1/p} &\leq K \sum_{m=1}^{\infty} 2^{m(1/p+1/q)} 2^{m\alpha} A_{2^{m-1}} \\ &= K \sum_{m=1}^{\infty} 2^{m(1+\alpha)} A_{2^{m-1}} \\ &= K\, 2^{1+\alpha} \sum_{m=1}^{\infty} 2^{(m-1)(1+\alpha)} A_{2^{m-1}} < \infty\,.\end{aligned}$$

According to Theorem 2 (see also [8, Theorem]), we obtain the following result.

Theorem 7. *For all $1 < p \leq 2$ and $\alpha \geq 0$ the following embedding relation holds: $S_{p\alpha} \subset BV \cap C_\alpha$.*

Applying the same technique as in the proof of Theorem 3.2 in [9], by Theorem 1, we obtain the following theorem.

Theorem 8. *For all $1 < p \leq 2$ and $\alpha \geq 0$ the following embedding relation holds: $\mathcal{F}_{p\alpha} \subset BV \cap C_\alpha$.*

According to Theorem 6 and Theorem 8, we obtain the following result.

Theorem 9. *For all $1 < p \leq 2$, $0 \leq \beta \leq \alpha$, the following embedding relation holds: $S_{p\alpha\beta} \subset BV \cap C_\beta$.*

We note that a direct proof of this inclusion for $\beta = r \in \{0, 1, \ldots, [\alpha]\}$ was given also by the author in [11].

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