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On a Conjecture of Polynomial Zeros

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Sendov's conjecture on the critical points was stated more than 45 years ago and is still open in the general case. Recently, new conjectures concerning zeros of the s-th derivative of a polynomial were formulated by Bl. Sendov. We try to investigate some of them by computer. Here we give a summary of the numerical experiments we have done.

1. Introduction

The following famous conjecture of Bl. Sendov on the critical points of a polynomial was formulated in 1957.

Conjecture 1. *If all the zeros of a polynomial $p(z) = (z - z_1) \dots (z - z_n)$, $n \geq 2$, lie in the unit disk $D := \{z \in \mathbb{C} : |z| \leq 1\}$ in the complex plane, then every disk with center z_i and radius 1, $i = 1, \dots, n$, contains at least one zero of the derivative $p'(z)$.*

For a detailed account on the results related to Conjecture 1 the reader is referred to the recent survey by Bl. Sendov [3]. Moreover, a variety of new conjectures concerning zeros of the s-th derivative of a polynomial are formulated therein. We are going to discuss some of these conjectures.

First we recall the basic notation.

$\mathcal{P}_n$ – the set of all polynomials $p(z) = (z - z_1) \dots (z - z_n)$ of degree n with zeros $z_i \in D$, $i = 1, \dots, n$;

$A(p)$ – the set of all zeros of the polynomial $p(z)$;

$A(p^{(s)})$ – the set of all zeros of the s-th derivative of the polynomial $p(z)$;

$H(p)$ – the convex hull of the set $A(p)$;

$D(c; r)$ – the disk in the complex plane $\mathbb{C}$ with center c and radius r;

$\rho(a; B) = \inf_{b \in B} |a - b|$, where $a \in \mathbb{C}$ and $B \subset \mathbb{C}$;

$\rho(A; B) = \sup_{a \in A} \rho(a; B)$, where $A, B \subset \mathbb{C}$.

The following is a generalization of Conjecture 1.

Conjecture 2 (Sendov [3]). *For every polynomial $p \in \mathcal{P}_n$ we have*

$$\rho(A(p); A(p^{(s)})) \le \frac{2s}{s+1}, \qquad s = 1, \dots, n-1.$$

For $s = 1$ this is exactly Conjecture 1 and for $s = n-1$ it is trivially true. Here are some other particular cases in which Conjecture 2 is shown to be true (see [4]):

- $s = 2$ and $n = 3, 4, 5, 6$;
- $s = 3$ and $n = 4, 5, 6, 7, 8, 9$;
- $2s \ge n$ and $n \ge 5$;
- the corners of $H(p)$ lie on the the unit circle.

In the case $s = n-2$ the following estimates were given in [4].

(i) For $n = 2m+1$ (n – odd),

$$\rho(A(p); A(p^{(n-2)})) \ge \frac{2(n-1)}{n+1},$$

and the extremal polynomial is expected to be

$$p(z) = (z-1)\left(z^2 + \frac{2m}{m+1}\, z + 1\right)^m. \tag{1}$$

(ii) For $n = 2m+2$ (n – even),

$$\rho(A(p); A(p^{(n-2)})) \ge 1 + \sqrt{\frac{(n-2)(n-4)}{n(n+2)}},$$

and

$$p(z) = (z-1)(z+1)\left(z^2 + 2\sqrt{\frac{m^2-1}{m(m+2)}}\, z + 1\right)^m \tag{2}$$

is a candidate for an extremal polynomial.

However, the estimate $\frac{2s}{s+1}$ for $\rho(A(p); A(p^{(s)}))$ in Conjecture 2 is not exact for all n and s. In our discussions with Prof. Bl. Sendov he stated a possible improvement of this estimate.

Conjecture 3 (Sendov [5]). *For every polynomial $p \in \mathcal{P}_n$ we have*

$$\rho(A(p); A(p^{(s)})) \le \left(\frac{2s}{s+1}\right)^{1/(n-s)}, \qquad s = 1, \dots, n-1.$$

We try to verify Conjecture 2 and Conjecture 3 by computer programs. Below we present an algorithm and give a summary of numerical experiments we have done. Note that Peterson [2] have also used a computer code for *Maple* to produce plots and illustrate the distribution of the critical points of a polynomial.

2. The Algorithm

Let us consider the set of polynomials $\mathcal{P}_n$ and fix the derivative order to be s. Here we describe an algorithm for numerical evaluation of the deviation $\rho(A(p); A(p^{(s)}))$. By N we denote the number of random samples for each step, and by M the number of steps we do. The algorithm we have used for our investigations is based on randomly varying the zeros of the polynomials in small domains. Because of the continuous dependence of the critical points of a polynomial on its zeros, it is natural to expect that for large N the numerical results will be sufficiently close to the exact values of the estimated deviation.

Step 1. At the first step we make N random choices of the zeros $z_1, \dots, z_n \in D$ of a polynomial $p(z)$.

For each such a choice we compute:

- approximately the zeros $\xi_1, \dots, \xi_{n-s}$ of the s-th derivative $p^{(s)}(z)$ using the Weierstrass-Dochev method;
- the deviation ρ of the set $\{\xi_1, \dots, \xi_{n-s}\}$ from the set $\{z_1, \dots, z_n\}$, i.e.,

$$\rho_\nu^{(1)} = \rho(A(p); A(p^{(s)})) = \sup_{1\le i\le n} \inf_{1\le j\le n-s} |z_i - \xi_j|, \qquad \nu = 1, \dots, N.$$

At the end we have $\rho^{(1)} := \sup\limits_{1\le\nu\le N} \rho_\nu^{(1)}$ and the location of the zeros $z_1^{(1)}, \dots, z_n^{(1)}$ of the extremal polynomial for this series of random choices.

Step μ, $\mu = 2, \dots, M$. At each next step we try to minimize the deviation $\rho(A(p); A(p^{(s)}))$, varying the zeros of the polynomial in small neighbourhoods of the zeros of the extremal polynomial from the previous step. More precisely, we make N random choices of the zeros

$$z_1 \in G_1,\ \dots,\ z_i \in G_i,\ \dots,\ z_n \in G_n,$$

of the polynomial $p(z)$. The domains G_i could be disks

$$G_i = D(z_i^{(\mu-1)}; \varepsilon) \qquad (\text{resp. } G_i = D(z_i^{(\mu-1)}; \varepsilon) \cap D),$$

or annular sectors

$$G_i = \big\{ z \in D : \arg z \in [\arg z_i^{(\mu-1)} - \varepsilon, \arg z_i^{(\mu-1)} + \varepsilon], \\ |z| \in [|z_i^{(\mu-1)}| - \delta, |z_i^{(\mu-1)}| + \delta] \big\},$$

for appropriately chosen $\varepsilon > 0$, $\delta > 0$.

Then, for each such a choice we compute as above:

- approximately the zeros $\xi_1, \dots, \xi_{n-s}$ of the s-th derivative $p^{(s)}(z)$ using the Weierstrass-Dochev method;
- the deviation ρ of the set $\{\xi_1, \dots, \xi_{n-s}\}$ from the set $\{z_1, \dots, z_n\}$, i.e.,

$$\rho_\nu^{(\mu)} = \rho(A(p); A(p^{(s)})) = \sup_{1 \le i \le n} \inf_{1 \le j \le n-s} |z_i - \xi_j|, \qquad \nu = 1, \dots, N.$$

At the end we have $\rho^{(\mu)} := \sup_{1 \le \nu \le N} \rho_\nu^{(\mu)}$ and the location of the zeros $z_1^{(\mu)}, \dots, z_n^{(\mu)}$ of the extremal polynomial for this series of random choices.

Finally, we find an approximation

$$\rho^* := \rho^{(M)} \approx \sup_{p \in \mathcal{P}_n} \rho(A(p); A(p^{(s)})).$$

3. Numerical Results

We have done numerical experiments varying the following parameters: the number of iterations $N = 1000, 2000, \dots, 20000$; the number of steps $M = 10, 15, 20, \dots, 50$; the required accuracy for the Weierstrass-Dochev method 0.001, 0.0001, 0.00001, 0.000001; the size of the small domains in which we vary the zeros of the polynomial $\varepsilon, \delta = 0.005, 0.01, \dots, 0.1$.

n	s	$\frac{2s}{s+1}$	$\left(\frac{2s}{s+1}\right)^{1/(n-s)}$	ρ^*
4	2	1.333333	1.154701	1.154558
5	2	1.333333	1.100642	1.065
5	3	1.5	1.224745	1.3326
6	2	1.333333	1.074570	0.985179
6	3	1.5	1.144714	1.211527
6	4	1.6	1.264911	1.333333
7	2	1.333333	1.059224	0.88565
7	3	1.5	1.106682	1.02345
7	4	1.6	1.169607	1.267805
7	5	1.666666	1.290994	1.484047

The results from the table suggest that Conjecture 2 is true.

Conjecture 3 is not true in general.

There exist polynomials for which the deviation $\rho(A(p); A(p^{(s)}))$ is greater than $(\frac{2s}{s+1})^{1/(n-s)}$. For example, for $n = 5$ and $s = 3$ (also for $n = 7$ and $s = 5$) the zeros of the polynomial produced by our program are very close to the zeros of the polynomial (1) supposed to be maximal for $s = n - 2$ in [4].

An interesting observation from our numerical experiments is that there are locally maximal polynomials. Depending on the initial positions of the polynomial zeros and the size (parameters ε, δ) of the small domains G_i in which we vary the zeros, the procedure may converge to different polynomials. In that sense, it is important to choose properly the parameters ε, δ.

4. The Case When the Vertices of $H(p)$ Lie on the Unit Circle

Suppose that the vertices of $H(p)$ lie on the unit circle $C = \{z : |z| = 1\}$. As we mentioned above, Sendov [4] gave a proof of Conjecture 2 in this case, using a slightly modified version of the following theorem.

Theorem (Meir and Sharma [1]). *Let $p \in \mathcal{P}_n$ and $z = 1$ be a zero of p of multiplicity ν. Then at least one zero of $p^{(s)}$, $1 \le s \le n-1$, lies in the disk $D(\frac{\nu}{s+1}; 1 - \frac{\nu}{s+1})$.*

It is no accident that many of the books and articles concerning zeros of polynomials are referred to be devoted "on the geometry of polynomials". Using the Meir and Sharma theorem we may restate Conjecture 2 in the case when the vertices of $H(p)$ lie on the unit circle as a problem from elementary geometry as follows.

Let F be a convex polygon in the plane such that all its vertices $z_1, \dots, z_k$ lie on the unit circle. To each vertex z_j we prescribe a positive integer ν_j, $j = 1, \dots, k$. If s is a positive integer with $s < \nu_1 + \cdots + \nu_k$, show that the set of disks $\left\{D(\frac{\nu_j z_j}{s+1}; 1 - \frac{\nu_j}{s+1})\right\}_{j=1}^{k}$ covers F.

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