CONSTRUCTIVE THEORY OF FUNCTIONS, Varna 2005
(B. Bojanov, Ed.), pp. 1-10
Marin Drinov Academic Publishing House, Sofia, 2006

# Enhanced Asymptotic Approximation and Approximation of Truncated Functions by Linear Operators

Ulrich Abel and Mircea Ivan

Our concern is the investigation of local convergence properties of linear approximation operators. In the first part we continue the work by Kirov and Popova. Given a sequence of linear operators $L_n$ new operators $L_{n,r}$ can be constructed by application of $L_n$ to the $r$-th partial sum of the Taylor series of the approximated function. We derive the complete asymptotic expansion for the operators $L_{n,r}$ as $n$ tends to infinity, provided that the underlying operators $L_n$ possess such a property. In the second part we study approximation of truncated functions by Szász-Mirakjan-Durrmeyer operators.

## 1. Enhanced Asymptotic Approximation by Linear Operators

### 1.1. Introduction

Let $I \subseteq \mathbb{R}$ be an interval. Given a sequence of linear operators $L_n : C(I) \to C(I)$ $(n = 0, 1, 2, \dots)$, Kirov and Popova [10] associated to each $L_n$ a new operator $L_{n,r}$ $(r = 0, 1, 2, \dots)$ defined by

$$(L_{n,r} f)(x) = (L_n P_{x,r} f)(x), \tag{1}$$

where $P_{x,r} f$ is the $r$-th Taylor polynomial

$$P_{x,r}(f; t) = \sum_{j=0}^{r} \frac{f^{(j)}(t)}{j!} (x - t)^j \tag{2}$$

of the function $f \in C^r(I)$ in a neighbourhood of the point $t \in I$. For $r = 0$, we have $L_{n,0} \equiv L_n$. Kirov and Popova studied the properties of the operators (1) and proved a Korovkin-type theorem. In the previous paper [9] Kirov studied the special case, where $L_n \equiv B_n$ is the Bernstein polynomial. He gave an asymptotic relation for $(L_{n,r}f)(x)$ as $n \to \infty$ [9, Th. 2], which, however, does not give much insight in the asymptotic behaviour.

The purpose of this paper is the investigation of the asymptotic behaviour of sequences $L_{n,r}$ of operators (1) originating from approximation properties of the operators $L_n$ as $n$ tends to infinity.

Throughout the paper let $(\varphi_k)_{k=1}^{\infty}$ be a sequence of functions defined on $\mathbb{N}$, such that for each $k \in \mathbb{N}$

$$\lim_{n\to\infty} \varphi_k(n) = 0 \quad \text{and} \quad \varphi_{k+1}(n) = o(\varphi_k(n)) \quad (n \to \infty).$$

We consider operators $L_n$ satisfying an asymptotic relation and derive for the corresponding operators $L_{n,r}$ a complete asymptotic expansion of the form

$$L_{n,r}(f;x) \sim f(x) + \sum_{k=1}^{\infty} \varphi_k(n) c_k^{[r]}(f;x) \qquad (n \to \infty)$$

with certain coefficients $c_k^{[r]}(f;x) \quad (k = 1, 2, \dots)$ independent of $n$.

### 1.2. Asymptotic Approximation by Operators $L_{n,r}$

As first main result we obtain the complete asymptotic expansion of the operators $L_{n,r}$, provided the operators $L_n$ possess a complete asymptotic expansion. Throughout the paper we assume that the functions $f$ under consideration admit derivatives of sufficiently high orders.

**Theorem 1** ([1]). *Let $q, r \in \mathbb{N}$. Suppose that the linear operators $L_n : C(I) \to C(I)$ satisfy, for $x \in I$, an asymptotic expansion*

$$(L_n f)(x) = f(x) + \sum_{k=1}^{q} \varphi_k(n) \sum_{\ell=\ell_k}^{L_k} g_{k,\ell}(x) f^{(\ell)}(x) + o(\varphi_q(n)) \qquad (n \to \infty) \quad (3)$$

*with integers $L_k \geq \ell_k \geq 1$ and certain values $g_{k,\ell}(x)$ independent of $n$. Then, the operators $L_{n,r}$, as defined in* (1)*), possess the asymptotic expansion*

$$\begin{aligned}(L_{n,r}f)(x) =& f(x) + (-1)^r \sum_{k=1}^{q} \varphi_k(n) \sum_{\ell=\max\{\ell_k, r+1\}}^{L_k} \binom{\ell-1}{r} g_{k,\ell}(x) f^{(\ell)}(x) \\ &+ o(\varphi_q(n)) \qquad (n \to \infty).\end{aligned}$$

**Remark 1.** Several known linear approximation operators satisfy an asymptotic expansion of the form (3)) with the special sequence $L_k = 2k$ $(k \in \mathbb{N})$. Under this additional condition all terms of the first sum in (3) with $1 \leq k < \lfloor r/2 \rfloor + 1$ vanish. Note that in the case $r \leq 2q - 1$, such operators satisfy

$$(L_{n,r}f)(x) = f(x) + O\big(\varphi_{\lfloor r/2 \rfloor + 1}(n)\big) \qquad (n \to \infty).$$

Applications to special operators can be found in [1].

### 1.3. Asymptotic Approximation by Operators $L^{\Delta}_{n,r,\alpha}$

For practical use the operators $L_{n,r}$ are not easy to handle, since they require the existence of all derivatives $f', f'', \ldots, f^{(r)}$. For certain functions $f$, their computation may demand great effort. Moreover, the operators $L_{n,r}$ have the lack, that the derivatives must exist on the whole interval $I$.

It would be desirable to establish operators similar to $L_{n,r}$ which improve the order of convergence (locally) even if $f$ possesses only local smoothness properties, but work without any derivative of $f$. To overcome this difficulty we construct such operators by replacing the derivatives $f^{(j)}$ in the Taylor polynomial $P_{x,r}$ by suitable differences of the function $f$. It turns out that these new operators improve the degree of approximation in the same way as the operators $L_{n,r}$. In this section we consider only the most common case $\varphi_k(n) = n^{-k}$ $(k = 1, 2, \ldots)$.

Analogously to the Taylor polynomial $P_{x,r}$ in (2) we consider the truncated Newton series $P^{[h]}_{x,r}$ defined by

$$\left(P^{[h]}_{x,r} f\right)(t) = \sum_{j=0}^{r} \frac{1}{j!} \left(\frac{x-t}{h}\right)^{\underline{j}} \Delta^j_h f(t).$$

Let $\alpha : I \to \mathbb{R}$ be a function with $\alpha(x) \neq 0$ $(x \in I)$. For a given linear operator $L_n : C(I) \to C(I)$, we define the new operator $L^{\Delta}_{n,r,\alpha}$ of $r$-th order $(r = 0, 1, 2, \ldots)$ by

$$\left(L^{\Delta}_{n,r,\alpha} f\right)(x) = \left(L_n P^{[\alpha(x)/n]}_{x,r} f\right)(x) \tag{4}$$

or, in a more explicit form, with $\psi_x(t) := t - x$

$$\left(L^{\Delta}_{n,r,\alpha} f\right)(x) = \sum_{j=0}^{r} \frac{1}{j!} \sum_{\ell=0}^{j} S^{\ell}_j \left(\frac{n}{\alpha(x)}\right)^{\ell} \left(L_n \psi^{\ell}_x \Delta^j_{\alpha(x)/n} f\right)(x),$$

where $S^{\ell}_j$ denotes the Stirling number of the first kind. However, there arises the problem, that the computation of the operators $L^{\Delta}_{n,r,\alpha}(f; x)$ may require the evaluation of $f(t)$ for some $t \notin I$. Let us consider the case of a finite closed interval $I = [a, b]$. One way to overcome this difficulty is a continuous continuation of $f$ to $\mathbb{R}$ by the definition $f(x) = f(a)$ $(x < a)$ and $f(x) = f(b)$ $(x > b)$, which we shall assume in the following.

Usually this does not influence the asymptotic properties of $\left(L^{\Delta}_{n,r,\alpha} f\right)(x)$, since, for most approximation operators, $(L_n f)(x)$ is essentially given by values of $f(t)$ for arguments $t$ close to $x$.

However, it can affect the good approximation properties of $L^{\Delta}_{n,r,\alpha}$, for moderate values of $n$. Therefore, we choose the function $\alpha$ in a proper manner, for example,

$$\alpha(x) = \begin{cases} b - x, & \text{if } a \le x \le (a+b)/2 \\ a - x, & \text{if } (a+b)/2 < x \le b. \end{cases}$$

The operators $L_{n,r,\alpha}^{\Delta}$ improve the degree of approximation of the operators $L_n$ in a completely analogous fashion as it do the operators $L_{n,r}$. To be precise, we have the following result.

**Theorem 2** ([1]). *Suppose the linear operators $L_n : C(I) \to C(I)$ satisfy, for $x \in \operatorname{int} I$, an asymptotic expansion*

$$(L_n f)(x) = f(x) + \sum_{k=1}^{q} n^{-k} \sum_{\ell=\ell_k}^{L_k} g_{k,\ell}(x) f^{(\ell)}(x) + o\left(n^{-q}\right) \qquad (n \to \infty)$$

*with integers $L_k \geq \ell_k \geq 1$ and certain values $g_{k,\ell}(x)$ independent of $n$. Then, the operators $L_{n,r,\alpha}^{\Delta}$, as defined in* (4), *possess the asymptotic expansion*

$$\left(L_{n,r,\alpha}^{\Delta} f\right)(x) = f(x) + \sum_{k=1}^{q} n^{-k} \sum_{m=0}^{k-1} \alpha^m(x) \sum_{\ell=\ell_{k-m}}^{L_{k-m}} g_{k-m,\ell}(x) f^{(\ell+m)}(x)$$
$$\times \sum_{j=0}^{r} (-1)^j \binom{\ell}{j} \frac{j!}{(m+j)!} \sum_{\mu=0}^{r-j} S_{j+\mu}^{j} \sigma_{j+m}^{j+\mu} + o\left(n^{-q}\right)$$

*as $n \to \infty$, where $S_m^j$ and $\sigma_m^j$ denote the Stirling numbers of the first and second kind, respectively.*

**Remark 2.** As in Remark 1 we have in the case $r \leq 2q-1$ for operators with $L_k = 2k$ $(k \in \mathbb{N})$

$$\left(L_{n,r,\alpha}^{\Delta} f\right)(x) = f(x) + O\left(n^{-(\lfloor r/2 \rfloor + 1)}\right) \qquad (n \to \infty).$$

## 2. Approximation of Truncated Functions by Szász-Mirakjan-Durrmeyer Operators

### 2.1. Introduction

The Szász-Mirakjan-Durrmeyer operators $S_n$ $(n = 0, 1, 2, \ldots)$ associate with each locally integrable function $f$ on $I = (0, \infty)$ the power-series

$$(S_n f)(x) = n e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_0^{\infty} e^{-nt} \frac{(nt)^k}{k!} f(t)\, dt$$

$(x \in I)$, provided the expressions are senseful. If $f$ is locally integrable on $I$ and of exponential growth, $S_n f$ is well-defined for sufficiently large $n$. For $\alpha > 0$, let $W_\alpha(I)$ denote the class of all locally integrable functions $f$ on $I$ satisfying the estimate $|f(t)| \leq M e^{\alpha t}$ $(t \in I)$. Furthermore, put $W(I) = \bigcup_{\alpha > 0} W_\alpha(I)$. Note that $S_n f$ $\;(f \in W_\alpha(I))$ exists for all $n > \alpha$.

It is well-known that $(S_n f)(x) \to f(x)$ as $n \to \infty$ in each continuity point of $f \in W(I)$. Here we study the asymptotic approximation of the truncated functions $f\chi_{(0,x]}$ and $f\chi_{(x,\infty)}$, where $x \in I$ is a fixed point and $\chi$ denotes the characteristic function of the respective intervals. To this end we consider the decomposition

$$S_n = S_n^- + S_n^+$$

defined by

$$\left(S_n^- f\right)(x) = S_n\left(f\chi_{(0,x]}\right), \qquad \left(S_n^+ f\right)(x) = S_n\left(f\chi_{(x,\infty)}\right).$$

The key result is the following representation of $S_n f$ as a double integral.

**Theorem 3** ([3]). *Let $\alpha > 0$ and $n \geq \alpha$. For each $f \in W_{\alpha(I)}$ and $x \in I$, the SMD operators $S_n$ possess the representation*

$$(S_n f)(x) = \frac{n}{2\pi} \int_0^\infty \int_{-\pi}^{\pi} f(t) e^{-n(x+t-2\sqrt{xt}\cos s)}\, ds\, dt. \tag{5}$$

**Remark 3.** We note that (5) allows to extend the definition of $S_n$ for all complex numbers $n$ with $\operatorname{Re} n > 0$.

### 2.2. Commutativity and Powers of SMD Operators

In 1987 Heilmann [6] showed that SMD operators commute, i.e., $S_m S_n = S_n S_m$ for all $(m, n = 0, 1, 2, \dots)$. In this section we give a concise integral representation for $S_m S_n$ showing the symmetry in the parameters $m$ and $n$ for real numbers $m, n > 0$. This obviously implies the commutativity of the SMD operators. Surprisingly, it turns out that the product $S_m S_n$ is again an SMD operator.

**Theorem 4** ([3]). *For real numbers $m, n > 0$ there holds the relation*

$$S_m S_n = S_{mn/(m+n)}.$$

The explicit representation of $S_{mn/(m+n)}$ as a double integral follows by (5). By mathematical induction, Theorem 4 can easily be generalized to arbitrary finite products of SMD operators.

**Corollary 1.** *Let $k \in \mathbb{N}$. For any real numbers $n_1, \dots, n_k > 0$ there holds the representation*

$$\prod_{j=1}^{k} S_{n_j} = S_{k^{-1}H(n_1,\dots,n_k)},$$

*where $H(n_1, \dots, n_k) = k\left(\sum_{j=1}^k n_j^{-1}\right)^{-1}$ is the harmonic mean of the numbers $n_1, \dots, n_k$.*

In the case when all $n_j$ coincide, we have $S_n^k = S_{n/k}$. For the convergence of the powers of SMD operators we immediately obtain the following result:

**Theorem 5.** *Let $(k_n)_n$ be a sequence of positive reals and $\lambda > 0$. Then there holds*

$$\lim_{n\to\infty} S_n^{k_n} = S_\lambda \quad \text{if and only if} \quad \lim_{n\to\infty} \frac{n}{k_n} = \lambda.$$

### 2.3. Asymptotic Expansions

In the following we write $\mp$ for one of the signs $-$ or $+$.

**Proposition 1** ([3]). *Let $x \in I$ and suppose that the function $f \in W(I)$ satisfies the asymptotic relation*

$$f(t) \sim |t-x|^\gamma \qquad (t \to x\mp)$$

*with* $\operatorname{Re}\gamma > -1$. *Then there holds*

$$\left(S_n^\mp f\right)(x) \sim \frac{(2\sqrt{x})^\gamma}{2\sqrt{\pi}}\,\Gamma\Big(\frac{\gamma+1}{2}\Big) n^{-\gamma/2} \qquad (n\to\infty).$$

*Moreover, if*

$$f(t) = |t-x|^\gamma + o\left(|t-x|^{\gamma+1}\right) \qquad (t\to x\mp),$$

*then there holds the Voronovskaja type relation*

$$\lim_{n\to\infty} \sqrt{n}\left(n^{\gamma/2}\left(S_n^\mp f\right)(x) - \frac{(2\sqrt{x})^\gamma}{2\sqrt{\pi}}\,\Gamma\Big(\frac{\gamma+1}{2}\Big)\right) = \mp \frac{(2\sqrt{x})^\gamma(\gamma+1)}{4\sqrt{\pi x}}\,\Gamma\Big(\frac{\gamma}{2}+1\Big).$$

**Remark 4.** Proposition 1 can be generalized to functions satisfying the asymptotic relation

$$f(t) \sim \sum_{\nu=0}^\infty a_\nu^\mp |t-x|^{\gamma_\nu} \qquad (t\to x\mp)$$

with complex exponents satisfying $-1 < \operatorname{Re}\gamma_0 < \operatorname{Re}\gamma_1 < \operatorname{Re}\gamma_2 < \cdots$ and $a_\nu^\mp \in \mathbb{C}$. In order to avoid complicated notations, we restrict ourselves to the case

$$f(t) \sim \sum_{\nu=0}^\infty \frac{1}{\nu!}\, f_\mp^{(\nu)}(x)(t-x)^\nu \qquad (t\to x\mp),$$

where we set $f_\mp(x) = \lim_{t\to x\mp} f(t)$ and $f_\mp^{(\nu)}(x)$ denotes the left, resp. right-hand derivative.

**Theorem 6** ([3]). *Let $x \in I$, $r \in \mathbb{N}$, and suppose that the function $f \in W(I)$ satisfies the asymptotic relation*

$$f(t) = \sum_{\nu=0}^{r} \frac{1}{\nu!} f_{\mp}^{(\nu)}(x)(t-x)^{\nu} + O\big(|t-x|^{r+\varepsilon}\big) \qquad (t \to x\mp)$$

*with $0 < \varepsilon < 1$. Then there holds*

$$\left(S_n^{\mp} f\right)(x) = \sum_{k=0}^{r} (\mp)^k \frac{c_k(f_{\mp}; x)}{n^{k/2}} + O\big(n^{-(r+\varepsilon)}\big) \qquad (n \to \infty),$$

*where the coefficients $c_k(f; x)$ are given by*

$$c_k(f; x) = \frac{1}{2\pi} \sum_{j=0}^{k} b_{k-j}^{[j]} \frac{f^{(j)}(x)}{j!} (2x)^j \qquad (k = 0, 1, 2, \dots) \tag{6}$$

*with*

$$b_k^{[\gamma]} = \sum_{(i,j,\ell) \in I_{k-1} \cup I_k} (-1)^{j+\ell} 2^{-i-2\ell} \frac{1}{j!} \binom{\gamma}{i} \binom{-1/2}{\ell} \times \Gamma\Big(\frac{\gamma + k - 2\ell + 1}{2}\Big) \times \Gamma\Big(j + \ell + \frac{1}{2}\Big) \tag{7}$$

*and the index set $I_k = \big\{(i,j,\ell) \in \mathbb{N}_0^3 \;\big|\; i + j + 2\ell = k\big\}$.*

It seems to be possible to give a nicer expression for the $b_k^{[\gamma]}$ in (7).

**Conjecture 1** ([3]). *For all $k \in \mathbb{N}$ there holds*

$$b_k^{[\gamma]} = \sqrt{\pi} \binom{\frac{\gamma+k}{2}}{k} \Gamma\Big(\frac{\gamma+1}{2}\Big) \Gamma\Big(\frac{\gamma+2}{2}\Big) \Gamma\Big(\frac{\gamma-k}{2} + 1\Big)^{-1}, \tag{8}$$

*where the expression is to be read as 0 when it happens that $(\gamma - k)/2$ is a negative integer.*

**Remark 5.** Equality (8) is valid for $k = 1, \dots, 6$. We conjecture that this formula is true in general.

For each $k \in \mathbb{N}$ for which (8) is valid, the coefficients as given in (6) satisfy

$$c_k(f; x) = \frac{1}{2\Gamma\left(\frac{k}{2} + 1\right)} \sum_{j=0}^{k} \binom{k/2}{j} (x^{k/2})^{(j)} f^{(k-j)}(x).$$

For even indices, this implies by Leibniz rule $c_{2k}(f; x) = (2 \cdot k!)^{-1} \left(x^k f^{(k)}(x)\right)^{(j)}$. In the special case when $f_-^{(\nu)}(x) = f_+^{(\nu)}(x) = f^{(\nu)}(x)$ $(\nu = 0, \dots, r)$ the coefficients of the odd powers of $n^{-1/2}$ in the expansion for $S_n^- + S_n^+$ cancel out and Theorem 6 yields the following result.

**Corollary 2.** *Let $x \in I$, $r \in \{0, \ldots, 6\}$, and suppose that the function $f \in W(I)$ satisfies the asymptotic relation*

$$f(t) = \sum_{\nu=0}^{r} \frac{1}{\nu!} f^{(\nu)}(x)(t-x)^{\nu} + O\big(|t-x|^{r+\varepsilon}\big) \qquad (t \to x).$$

*Then the SMD operators possess the complete asymptotic expansion*

$$(S_n f)(x) = \sum_{k=0}^{r} \frac{1}{k!\, n^k} \big(x^k f^{(k)}(x)\big)^{(j)} + O\big(n^{-(r+\varepsilon)}\big) \qquad (n \to \infty). \tag{9}$$

Formula (9) with $O\left(n^{-(r+\varepsilon)}\right)$ replaced by $o(n^{-r})$ is valid for all $r \in \mathbb{N}$, provided $f \in W(I)$ possesses a derivative of order $2r$ in $x$ [2, Corollary 2.4].

### 2.4. Proofs

After the talk some colleagues asked for the proofs of Theorems 3 and 4. Therefore, we present them below.

***Proof of Theorem* 3.** We can write

$$(S_n f)(x) = \int_0^{\infty} K_n(x,t) f(t)\, dt$$

with the kernel function

$$K_n(x,t) = ne^{-n(x+t)} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \frac{(nt)^k}{k!} = ne^{-n(x+t)} I_0\big(2n\sqrt{xt}\big),$$

where $I_0$ denotes the modified Bessel function given by $I_0(z) = \sum_{j=0}^{\infty} (j!)^{-2} (z^2/4)^j$. By the well-known formula

$$I_0(a) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{a \cos s}\, ds \qquad (a > 0) \tag{10}$$

Theorem 3 follows. □

***Proof of Theorem* 4.** Using the notation of the preceding proof, there holds

$$\begin{aligned}(S_m S_n f)(x) &= \frac{mn}{(2\pi)^2}\, e^{-mx} \int_0^{\infty} K_m(x,t) \int_0^{\infty} K_n(t,u)\, f(u)\, du\, dt \\ &= \frac{mn}{(2\pi)^2}\, e^{-mx} \int_0^{\infty} e^{-(m+n)t}\, I_0(2m\sqrt{xt}) \int_0^{\infty} e^{-nu}\, I_0(2n\sqrt{tu})\, f(u)\, du\, dt \\ &= mne^{-mx} \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{m^{2\mu} x^{\mu}}{(\mu!)^2} \frac{n^{2\nu}}{(\nu!)^2} \frac{(\mu+\nu)!}{(m+n)^{\mu+\nu+1}} \int_0^{\infty} f(u)\, e^{-nu} u^{\nu}\, du.\end{aligned}$$

In the following we make use of the identity

$$\sum_{\nu=0}^{\infty} \frac{a^\nu}{(\nu!)^2}\,(\mu+\nu)! = e^a \sum_{j=0}^{\nu} \binom{\nu}{j} \frac{\nu!}{j!}\, a^j \qquad (\mu = 0,1,2,\dots),$$

which can be verified by observing that either side is equal to $(\partial/\partial z)^\mu (z^\nu e^{az})$ evaluated at $z=1$. Thus, we obtain

$$\sum_{\nu=0}^{\infty} \frac{(n^2u)^\nu}{(\nu!)^2}\, \frac{(\mu+\nu)!}{(m+n)^\nu} = e^{n^2u/(m+n)} \sum_{j=0}^{\mu} \binom{\mu}{j} \frac{\mu!}{j!} \Big(\frac{n^2u}{m+n}\Big)^j .$$

Hence

$$\begin{aligned}(S_mS_nf)(x) &= \frac{mn}{m+n}\, e^{-mx} \sum_{\mu=0}^{\infty} (\mu!)^{-2} \Big(\frac{m^2x}{m+n}\Big)^\mu \sum_{j=0}^{\mu} \binom{\mu}{j} \frac{\mu!}{j!} \\ &\quad \times \int_0^\infty f(u)\, e^{-nu} \Big(\frac{n^2u}{m+n}\Big)^j e^{n^2u/(m+n)}\, du \\ &= \frac{mne^{-mx}}{m+n}\, e^{m^2x/(m+n)} \\ &\quad \times \int_0^\infty f(u)\, e^{-mnu/(m+n)} \sum_{j=0}^{\infty} \frac{1}{(j!)^2} \Big(\frac{m^2xn^2u}{(m+n)^2}\Big)^j du \\ &= \frac{mn}{m+n} e^{-mnx/(m+n)} \int_0^\infty f(u)\, e^{-mnu/(m+n)}\, I_0\big(\frac{2mn}{m+n}\sqrt{xu}\big)\, du.\end{aligned}$$

Application of Formula (10) yields

$$\begin{aligned}&(S_mS_nf)(x) \\ &\quad = \frac{1}{2\pi}\, \frac{mn}{m+n} \int_0^\infty f(u)\, e^{-mn(x+u)/(m+n)} \int_{-\pi}^{\pi} e^{(2mn\sqrt{xu}/(m+n))\cos s}\, ds\, du\end{aligned}$$

which is the desired formula. This completes the proof of Theorem 4. □

## References

[1] U. Abel and B. DellaVecchia, Enhanced asymptotic approximation by linear operators, *Facta Univ., Ser. Math. Inf.* **19** (2004), 37–51.

[2] U. Abel, B. Della Vecchia and M. Ivan, Simultaneous asymptotic approximation by Jakimovski-Leviatan-Durrmeyer operators and their linear combinations, to appear.

[3] U. Abel, One-sided asymptotic approximation by Szász-Mirakjan-Durrmeyer operators, Friedberg, September 2004.

[4] U. Abel and M. Ivan, Asymptotic expansion of the Jakimovski-Leviatan operators and their derivatives, *in* "Functions, Series, Operators" (L. Leindler, F. Schipp, and J. Szabados, Eds.), pp. 103–119, Budapest, 2002.

[5] U. Abel and M. Ivan, The asymptotic expansion for approximation operators of Favard-Szász type, *Friedberger Hochschulschriften* **2** (1999).

[6] M. Heilmann, Commutativity of operators from Baskakov-Durrmeyer type, *in* "Constructive Theory of Functions '87" (B. Sendov, P. Petrushev, K. Ivanov, and R. Maleev, Eds.), pp. 197–206, Publishing House of the Bulgarian Academy of Sciences, Sofia, 1988.

[7] M. Heilmann, Erhöhung der Konvergenzgeschwindigkeit bei der Approximation von Funktionen mit Hilfe von Linearkombinationen spezieller positiver linearer Operatoren, Habilitationsschrift, Universität Dortmund, 1991.

[8] M. Heilmann and M. W. Müller, Direct and converse results on simultaneous approximation by the method of Bernstein-Durrmeyer operators, *in* "Algorithms for Approximation II" (J. C. Mason and M. G. Cox, Eds.), pp. 107–116, Chapman & Hall, London, New York, 1989.

[9] G. H. Kirov, A generalization of the Bernstein polynomials, *Math. Balkanica (N. S.)* **6** (1992), 147–153.

[10] G. H. Kirov and L. Popova, A generalization of the linear positive operators, *Math. Balkanica (N. S.)* **7** (1993), 149–162.

Ulrich Abel

Fachbereich MND
Fachhochschule Giessen–Friedberg
University of Applied Sciences
Wilhelm-Leuschner-Straße 13
GERMANY
*E-mail:* ulrich.abel@mnd.fh-friedberg.de

Mircea Ivan

Department of Mathematics
Technical University of Cluj-Napoca
Str. C. Daicoviciu 15
400020 Cluj-Napoca
ROMANIA
*E-mail:* mircea.ivan@math.utcluj.ro