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Averaged Moduli of Smoothness on the Plane and Runge-Kutta Methods

A. ANDREEV AND M. NIKOLTJEVA-HEDBERG

We estimate the error in widely used in practice Runge-Kutta method of 3^{rd} order of convergence

$$\begin{aligned} &\tilde{y}_{i+1} = \tilde{y}_i + \tfrac{1}{6}(k_1 + 4k_2 + k_3),\\ &k_1 = hf(x_i, \tilde{y}_i),\\ &k_2 = hf(x_{i+\frac{1}{2}}, \tilde{y}_i + \tfrac{1}{2}k_1),\\ &k_3 = hf(x_{i+1}, \tilde{y}_i - k_1 + 2k_2),\\ &\tilde{y}_0 = y_0, \quad i = 0, 1, ..., n-1, \end{aligned} \tag{1}$$

for solving the Cauchy problem

$$\begin{cases} y' = f(x, y), \quad 0 \le x \le A \\ y(0) = y_0, \end{cases} \tag{2}$$

when the function f satisfies only the Lipschitz condition

$$|f(x, v) - f(x, z)| \le K|v - z|. \tag{3}$$

The so called averaged moduli of smoothness on the plane are used. By the properties of these moduli all rates of convergence follow under additional assumptions for smoothness (as a rule weaker than the well-known ones) of the function f, respectively y.

1. Introduction

The error estimates for numerical solution of (2) using Runge-Kutta (RK) methods of first and second order of convergence are given in [1]. The description of RK methods and different type of estimations can be found in [4],

[5], [6]. Other RK method of 3rd order of convergence, which is widely used in practice, is

$$\begin{aligned}
&\tilde{y}_{i+1} = \tilde{y}_i + \tfrac{1}{4}k_1 + \tfrac{3}{4}k_3,\\
&k_1 = hf(x_i, \tilde{y}_i),\\
&k_2 = hf(x_{i+\frac{1}{3}}, \tilde{y}_i + \tfrac{1}{3}k_1),\\
&k_3 = hf(x_{i+\frac{2}{3}}, \tilde{y}_i + \tfrac{2}{3}k_2),\\
&\tilde{y}_0 = y_0,\ x_i = ih,\ h = A/n,\ y_{i+\alpha} = y(x_i + \alpha h),\ i = 0, 1, ..., n-1.
\end{aligned} \tag{4}$$

It is proved in [2] that for the method (4) we have the following error estimate:

$$e \le c(A, K) \sum_{i=1}^{3} h^{3-i} \tau_i(y'; h)_L.$$

Here,

$$e_i = y_i - \tilde{y}_i, \qquad e = \{\max |e_i| : 0 \le i \le n\},$$

the averaged modulus of smoothness for a function g defined on $[a, b]$ is given by

$$\tau_k(g; \delta)_{L_p} = \|\omega_k(g, \cdot\,; \delta)\|_{L_p} = \Big(\int_a^b \omega_k^p(g, x; \delta)\, dx \Big)^{1/p},$$

where $k \ge 1$ is an integer number and (see also [3])

$$\omega_k(g, x; \delta) = \sup \Big\{ |\Delta_h^k g(t)| : t, t + kh \in [x - \tfrac{k\delta}{2}, x + \tfrac{k\delta}{2}] \cap [a, b] \Big\},$$

$$\Delta_h^k g(t) = \sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} g(t + mh).$$

To get an error estimation for the RK method (1), based solely on the smoothness assumption (3), we use the modulus introduced in the next section.

2. Averaged Moduli on the Plane

2.1. Local Modulus of Smoothness at a Point

Let the function f be defined in $D = [a, b] \times [c, d]$. Then the function

$$\begin{aligned}
f_h^{k,s}(x, y) = (-h)^{-(k+s)} \underbrace{\int_0^h \dots \int_0^h}_{k+s} \sum_{\substack{i+j>0\\ 0\le i\le k\\ 0\le j\le s}} (-1)^{k+s+i+j} \binom{k}{i} \binom{s}{j}\\
\times f(x + i\theta_x, y + j\theta_y)\, dt_1 \dots dt_k du_1 \dots du_s,
\end{aligned} \tag{5}$$

where

$$\theta_x = \frac{t_1 + \cdots + t_k}{k} - \frac{x-a}{b-a}\,h, \quad \theta_y = \frac{u_1 + \cdots + u_s}{s} - \frac{y-c}{d-c}\,h,$$

is the modified Steklov function of (k,s)-order with step h and

$$\begin{aligned} \omega_{k,s}\big(f,(x,y);\delta\big) &= \sup\big\{\big|\Delta^k_{h_1}\big(\Delta^s_{h_2} f(t,u)\big)\big| : t, t+kh_1 \in \\ &\quad [x - \tfrac{\bar{k}\delta}{2}, x + \tfrac{\bar{k}\delta}{2}] \cap [a,b], u, u+sh_2 \in [y - \tfrac{\bar{s}\delta}{2}, y + \tfrac{\bar{s}\delta}{2}] \cap [c,d]\big\}, \\ \bar{k} &= \max(1,k), \quad \bar{s} = \max(1,s), \end{aligned} \tag{6}$$

is the local modulus of smoothness of the function f of order (k,s) at the point (x,y). From (5) and (6) we derive the following properties of $f_h^{k,s}$:

(i)

$$\begin{aligned} &\frac{\partial^{r+t}}{\partial x^r \partial y^t} f_h^{k,s}(x,y) \\ &\quad = h^{-k-s} \underbrace{\int_0^h \cdots \int_0^h}_{k+s-r-t} \sum_{\substack{i+j>0 \\ 0\le i\le k \\ 0\le j\le s}} (-1)^{i+j} \binom{k}{i}\binom{s}{j} \big(1 - \tfrac{ih}{b-a}\big)^r \big(1 - \tfrac{jh}{d-c}\big)^t \\ &\quad \times \left(\frac{k}{i}\right)_+^r \left(\frac{s}{j}\right)_+^t \Delta^r_{\frac{i}{k}h}\,\Delta^t_{\frac{j}{s}h} f\big(x + i\,\tfrac{t_1+\cdots+t_{k-r}}{k} - i\,\tfrac{x-a}{b-a}h, \\ &\quad y + j\,\tfrac{u_1+\cdots+u_{s-t}}{s} - j\,\tfrac{y-c}{d-c}h\big)\, dt_1 \ldots dt_{k-r} du_1 \ldots du_{s-t} \end{aligned}$$

where

$$\left(\frac{a}{b}\right)_+ = \begin{cases} 1, & b = 0 \\ \frac{a}{b}, & b > 0, \end{cases}, \quad r \le k,\ t \le s.$$

As a consequence

$$\begin{aligned} \Big|\frac{\partial^{r+t}}{\partial x^r \partial y^t} f_h^{k,s}(x,y)\Big| &\le \omega_{r,t}\big(f,(x,y);4(k+s)h\big)\, h^{-r-t}\, k^r\, s^t \sum_{\substack{0\le i\le k \\ 0\le j\le s}} \binom{k}{i}\binom{s}{j} \\ &\le 2^{k+s}\, k^r\, s^t\, h^{-r-t}\, \omega_{r,t}\big(f,(x,y);4(k+s)h\big). \end{aligned}$$

(ii)

$$\begin{aligned} \Big|f_h^{k,s}(x,y) - f(x,y)\Big| &\le h^{-k-s} \underbrace{\int_0^h \cdots \int_0^h}_{k+s} \big|\Delta^k_{\theta_x}\Delta^s_{\theta_y} f(x,y)\big| dt_1 \ldots dt_k du_1 \ldots du_s \\ &\le \omega_{k,s}\big(f,(x,y);2h\big). \end{aligned} \tag{7}$$

2.2. Averaged Modulus of Smoothness along a Line

If the function $y(x)$ is defined on $[a,b]$ and $f(x,y)$ is a measurable and bounded function on $D=[a,b]\times[c,d]$, then we define

$$\tau_{k,s}(f;\delta)_{L_p}=\Big(\int_a^b \omega_{k,s}^p\big(f,(x,y(x));\delta\big)\,dx\Big)^{1/p},\quad 1\le p\le\infty.$$

We call $\tau_{k,s}(f;\delta)_{L_p}$ the averaged modulus of smoothness of f along the line $y(x)$. The main properties of $\tau_{k,s}(f;\delta)_{L_p}$ are:

(i) $\tau_{k,s}(f;\delta)_{L_p}\le\delta\tau_{k-1,s}\big(f'_x;\tilde{k}\delta\big)_{L_p},\quad ,\tilde{k}=\begin{cases}1, & k=1,\\ \frac{k}{k-1}, & k>1.\end{cases}.$

We have $\Delta_h^k\big(\Delta_h^s f(t_1,t_2)\big)=\displaystyle\int_0^h\Delta_h^{k-1}(\Delta_h^s f'_x(t_1+u,t_2))\,du$ and

$$\begin{aligned}&\sup\big\{\big|\Delta_h^k\big(\Delta_h^s f(t_1,t_2)\big)\big| : t_1,t_1+kh\in[x-\tfrac{k\delta}{2},x+\tfrac{k\delta}{2}]\cap[a,b],\\ &\qquad t_2,t_2+sh\in[y(x)-\tfrac{s\delta}{2},y(x)+\tfrac{s\delta}{2}]\cap[c,d]\big\}\\ \le\ &\sup\Big\{\int_0^h\big|\Delta_h^{k-1}\big(\Delta_h^s f'_x(t_1+u,t_2)\big)\big|\,du : t_1,t_1+kh\in[x-\tfrac{k\delta}{2},x+\tfrac{k\delta}{2}]\cap[a,b],\\ &\qquad t_2,t_2+sh\in[y(x)-\tfrac{s\delta}{2},y(x)+\tfrac{s\delta}{2}]\cap[c,d]\Big\}.\end{aligned}$$

If $t_1,t_1+kh\in[x-\frac{k\delta}{2},x+\frac{k\delta}{2}]$, then

$$t_1+u,t_1+u+(k-1)h\in[x-\tfrac{k\delta}{2},x+\tfrac{k\delta}{2}]$$

and we get

$$\big|\Delta_h^{k-1}\big(\Delta_h^s f'_x(t_1+u,t_2)\big)\big|\le\omega_{k-1,s}\big(f'_x,(x,y(x));\tilde{k}\delta\big),$$

i.e.,

$$\omega_{k,s}\big(f,(x,y(x));\delta\big)\le\delta\omega_{k-1,s}\big(f'_x,(x,y(x));\tilde{k}\delta\big),$$

which proves (i).

(ii) Similarly $\tau_{k,s}\big(f,(x,y(x));\delta)\big)_{L_p}\le\delta\tau_{k,s-1}\big(f'_y,(x,y(x));\tilde{s}\delta\big)_{L_p}.$

(iii) $\tau_{1,0}(f;\delta)_L\le\delta\|S(f'_x;\delta)\|_L$, where

$$S(f,(x,y);\delta)=\sup\big\{f(t_1,t_2):|t_1-x|\le\tfrac{\delta}{2},|t_2-y|\le\tfrac{\delta}{2}\big\}.$$

We have

$$\begin{aligned}\omega_{1,0}\big(f,(x,y(x));\delta)\big) &= \sup\big\{\big|\Delta_h f(t_1,t_2)\big| : t_1, t_1+h \in [x-\tfrac{\delta}{2}, x+\tfrac{\delta}{2}]\cap[a,b],\\ &\qquad t_2\in[y(x)-\tfrac{\delta}{2}, y(x)+\tfrac{\delta}{2}]\cap[c,d]\big\}\\ &\le \sup\Big\{\Big|\int_{t'}^{t''} f'_x(t,t_2)\,dt\Big| : t',t''\in[x-\tfrac{\delta}{2},x+\tfrac{\delta}{2}]\cap[a,b],\\ &\qquad t_2\in[y(x)-\tfrac{\delta}{2},y(x)+\tfrac{\delta}{2}]\cap[c,d]\Big\}\\ &\le \sup\Big\{\int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} |f'_x(t,t_2)|\,dt : t_2\in\\ &\qquad [y(x)-\tfrac{\delta}{2},y(x)+\tfrac{\delta}{2}]\cap[c,d]\Big\}\\ &\le \delta\big|S\big(f'_x,(x,y(x));\delta\big)\big|,\end{aligned}$$

which proves (iii).

(iv) Similarly we have $\tau_{0,1}(f;\delta)_L \le \delta\|S(f'_y;\delta)\|_L$.

3. Main Result

Let $f(x,y)$ be defined on $D=[a,b]\times[c,d]$, where $[c,d]$ is large enough and contains all values that appear as a second argument of f and (3) is fulfilled. Denote by $\varphi(x,y)$ the function $f_h^{3,3}(x,y)$. For $x\in[x_i,x_{i+1}]$ let us consider the equation

$$\begin{cases} u' = \varphi(x,u)\\ u(x_i)=y_i.\end{cases} \tag{8}$$

Then

$$\begin{aligned}u(x)-y(x) &= \int_{x_i}^{x}\big(u'(t)-y'(t)\big)dt = \int_{x_i}^{x}\big(\varphi(t,u(t))-f(t,y(t))\big)dt\\ &= \int_{x_i}^{x}\big(\varphi(t,u(t))\pm\varphi(t,y(t))-f(t,y(t))\big)dt\end{aligned}$$

and

$$\begin{aligned}|u(x)-y(x)| \le &\int_{x_i}^{x_{i+1}}\big|\varphi(t,u(t))-\varphi(t,y(t))\big|dt\\ &+\int_{x_i}^{x_{i+1}}\big|\varphi(t,y(t))-f(t,y(t))\big|dt = A+B.\end{aligned}$$

It follows from (7) and (8) that for $x_i\le\eta\le x_{i+1}$ we have:

$$B \le h\omega_{3,3}(f,(\eta,y(\eta));2h) \tag{9}$$

and

$$A = \int_{x_i}^{x_{i+1}} \Big|(-h)^{-6} \underbrace{\int_0^h \cdots \int_0^h}_{6} \sum_{\substack{0\le k,l\le 3\\ k+l>0}} (-1)^{6+k+l} \binom{3}{k}\binom{3}{l} \big(f(t+k\theta_t,$$
$$u(t)+l\theta_{u(t)}) - f(t+k\theta_t, y(t)+l\theta_{y(t)})\big)\Big| dt_1 dt_2 dt_3 du_1 du_2 du_3 dt$$
$$\le \int_{x_i}^{x_{i+1}} \sum_{\substack{0\le k,l\le 3\\ k+l>0}} \binom{3}{k}\binom{3}{l} K|u(t)+l\theta_{u(t)} - y(t) - l\theta_{y(t)}|dt$$
$$\le KC \int_{x_i}^{x_{i+1}} \|y-u\|_{[x_i,x_{i+1}]}\, dt$$
$$= KCh\, \|y-u\|_{[x_i,x_{i+1}]},$$

where C is a constant. If $KhC < \frac{1}{2}$, then the above estimation and (9) give

$$\|u-y\|_{[x_i,x_{i+1}]} \le 2h\omega_{3,3}\big(f,(\eta,y(\eta));2h\big). \tag{10}$$

For the method (1) we prove the following

Theorem 1. *Under the smoothness assumptions (3), the numerical solution of (2) by the RK method (1) satisfies the error estimate*

$$e \le C_0\Big(\sum_{k=0}^{2} h^k \tau_{3-k}(y';h)_L + \tau_{3,3}(f;C_1h)_L + h^3\Big),$$

where C_i are constants.

Proof. From (1), (2), and (3) we get

$$e_{i+1} = y_{i+1} - \tilde y_{i+1} = y_{i+1} - \tilde y_i - \tfrac{1}{6}(k_1 + 4k_2 + k_3)$$
$$= y_{i+1} \pm y_i - \tilde y_i - \tfrac{h}{6} f(x_i,\tilde y_i) - \tfrac{2h}{3} f\big(x_i + \tfrac{h}{2}, \tilde y_i + \tfrac{h}{2} f(x_i,\tilde y_i)\big)$$
$$- \tfrac{h}{6} f\big(x_i + h, \tilde y_i - hf\big(x_i, \tilde y_i + 2hf(x_i + \tfrac{h}{2}, \tilde y_i + \tfrac{h}{2} f(x_i,\tilde y_i))\big)\big)$$
$$\pm \tfrac{h}{6} f(x_i,y_i) \pm \tfrac{2h}{3}\big(x_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} f(x_i,y_i)\big)$$
$$\pm \tfrac{h}{6} f\big(x_i + h, y_i - hf\big(x_i, y_i + 2hf(x_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} f(x_i,y_i))\big)\big)$$

and

$$|e_{i+1}| \le \big(1 + \tfrac{Kh}{6}\big)|y_i - \tilde y_i| + \tfrac{2h}{3} K\big|\tilde y_i + \tfrac{h}{2} f(x_i,\tilde y_i) - y_i - \tfrac{h}{2} f(x_i,y_i)\big|$$
$$+ \tfrac{hK}{6}\big|\tilde y_i - hf(x_i,\tilde y_i) + 2hf\big(x_i + \tfrac{h}{2}, \tilde y_i + \tfrac{h}{2} f(x_i,\tilde y_i) - y_i + hf(x_i,y_i)\big)$$
$$- 2hf\big(x_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} f(x_i,y_i)\big)\big| + |C_2|$$
$$\le \big(1 + \tfrac{hK}{6} + \tfrac{2hK}{6}\big)|y_i - \tilde y_i| + \tfrac{h^2K^2}{3}|y_i - \tilde y_i| + \tfrac{hK}{6}|y_i - \tilde y_i|$$
$$+ \tfrac{h^2K^2}{6}|y_i - \tilde y_i| + \tfrac{hK}{6} 2hK\big|y_i - \tilde y_i + \tfrac{h}{2} f(x_i,\tilde y_i) - \tfrac{h}{2} f(x_i,y_i)\big| + |C_2|$$
$$\le \big(1 + hK + \tfrac{5h^2K^2}{6}\big)|y_i - \tilde y_i| + \tfrac{hK}{6}(hK)^2|y_i - \tilde y_i| + |C_2|$$
$$= \big(1 + hK + \tfrac{5h^2K^2}{6} + \tfrac{(hK)^3}{6}\big)|y_i - \tilde y_i| + |C_2|, \tag{11}$$

where

$$\begin{aligned}C_2 &= y_{i+1} - y_i - \tfrac{h}{6} f(x_i, y_i) - \tfrac{2h}{3}\, f\big(x_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} f(x_i, y_i)\big)\\ &\quad - \tfrac{h}{6} f\big(x_i + h, y_i - h f(x_i, y_i) + 2h f(x_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} f(x_i, y_i)\big)\\ &= y_{i+1} - y_i - \tfrac{h}{6} f(x_i, y_i) - \tfrac{2h}{3}\, f\big(x_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} f(x_i, y_i)\big)\\ &\quad - \tfrac{h}{6} f\big(x_i + h, y_i - h f(x_i, y_i) + 2h f\big(x_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} f(x_i, y_i)\big)\big)\\ &\quad \pm \tfrac{h}{6} f\big(x_i + h, y_i - h f(x_i, y_i) + 2h f\big(x_i + \tfrac{h}{2}, y_{i+\frac{1}{2}}\big)\big)\\ &= A_1 + A_2.\end{aligned}$$

We estimate the expressions A_1 and A_2 as follows:

$$\begin{aligned}A_2 &= \tfrac{h}{6} f\big(x_i + h, y_i - h f(x_i, y_i) + 2h f\big(x_i + \tfrac{h}{2}, y_{i+\frac{1}{2}}\big)\big)\\ &\quad - \tfrac{h}{6} f\big(x_i + h, y_i - h f(x_i, y_i) + 2h f\big(x_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} f(x_i, y_i)\big)\big),\end{aligned}$$

whence

$$\begin{aligned}|A_2| &\le \tfrac{hk}{6}\, 2h \big| f\big(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\big) - f\big(x_{i+\frac{1}{2}}, y_i + \tfrac{h}{2} f(x_i, y_i)\big)\big|\\ &\le \tfrac{h^2 K^2}{3} \big| y_{i+\frac{1}{2}} - y_i - \tfrac{h}{2} y_i' \big| = \tfrac{h^3 K^2}{6} \Big| \tfrac{y_{i+\frac{1}{2}} - y_i}{\frac{h}{2}} - y_i' \Big|\\ &\le \tfrac{h^3 K^2}{6}\, \omega(y', x_i; h).\end{aligned} \tag{12}$$

For the estimation of A_1 we get

$$\begin{aligned}A_1 &= y_{i+1} - y_i - \tfrac{h}{6} y_i' - \tfrac{2h}{3} f\big(x_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} f(x_i, y_i)\big)\\ &\quad - \tfrac{h}{6} f\big(x_i + h, y_i - h f(x_i, y_i) + 2h f\big(x_i + h2, y_{i+\frac{1}{2}}\big)\big)\\ &\quad \pm \tfrac{h}{6} f(x_i + h, y_i + h y_{i+1}')\\ &= A_3 + A_4,\end{aligned}$$

where

$$A_4 = \tfrac{h}{6} f(x_i + h, y_i + h y_{i+1}') - \tfrac{h}{6} f\big(x_i + h, y_i - h f(x_i, y_i) + 2h f\big(x_i + \tfrac{h}{2}, y_{i+\frac{1}{2}}\big)\big)$$

and

$$|A_4| \le \tfrac{h^2 K}{6} |y_{i+1}' - 2y_{i+\frac{1}{2}}' + y_i'| \le \tfrac{h^2 K}{6} \omega_2(y', x_i, h). \tag{13}$$

Since $A_3 = A_1 - A_4$ it follows that

$$\begin{aligned}A_3 &= y_{i+1} - y_i - \tfrac{h}{6} y_i' - \tfrac{2h}{3} f\big(x_i + \tfrac{h}{2}, y_i + \tfrac{h}{2} f(x_i, y_i)\big)\\ &\quad - \tfrac{h}{6} f\big(x_i + h, y_i + h f(x_{i+1}, y_{i+1})\big) \pm \tfrac{2h}{3} f\big(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\big) \pm \tfrac{h}{6} f(x_{i+1}, y_{i+1})\\ &= A_5 + A_6.\end{aligned}$$

Here $A_5 = y_{i+1} - y_i - \frac{h}{6} y_i' - \frac{2h}{3} f\big(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\big) - \frac{h}{6} f(x_{i+1}, y_{i+1})$,

$$|A_5| = |y_{i+1} - y_i - \tfrac{h}{6} y_i' - \tfrac{2h}{3} y_{i+\frac{1}{2}}' - \tfrac{h}{6} y_{i+1}'| \le C_{11} h\, \omega_3(y', x_i; h), \tag{14}$$

and

$$\begin{aligned} A_6 &= \tfrac{2h}{3} f\big(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\big) - \tfrac{2h}{3} f\big(x_{i+\frac{1}{2}}, y_i + \tfrac{h}{2} f(x_i, y_i)\big) \\ &\quad + \tfrac{h}{6} f(x_{i+1}, y_{i+1}) - \tfrac{h}{6} f\big(x_i + h, y_i + h f(x_{i+1}, y_{i+1})\big) \\ &= h\big(\tfrac{2}{3} R_1 - \tfrac{2}{3} R_2 + \tfrac{1}{6} R_3 - \tfrac{1}{6} R_4\big). \end{aligned} \tag{15}$$

In the next we estimate R_i, $i = 1, ..., 4$.

$$\begin{aligned} R_1 &= f\big(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\big) = f\big(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\big) \pm \varphi\big(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\big) \pm \varphi\big(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}\big) \\ &= \varphi\big(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}\big) + R_{11} + R_{12}, \end{aligned}$$

where

$$\begin{aligned} R_{11} &= f\big(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\big) - \varphi\big(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\big), \\ R_{12} &= \varphi\big(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\big) - \varphi\big(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}\big), \end{aligned}$$

and for $\eta \in [x_i, x_{i+1}]$ it follows from (7) and (10) that

$$|R_{11}| \le \omega_{3,3}\big(f, (x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}); 2h\big), \tag{16}$$

$$|R_{12}| \le C_{11} K \|y - u\|_{[x_i, x_{i+1}]} \le C_{12}\, \omega_{3,3}\big(f, (\eta, y(\eta)); 2h\big). \tag{17}$$

Now $R_3 = f\big(x_{i+1}, y_{i+1}\big)$ can be estimated in the same way as R_1. It is only necessary to change $\omega_{3,3}\big(f, (x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}); 2h\big)$ to $\omega_{3,3}\big(f, (x_{i+1}, y_{i+1}); 2h\big)$.

$$\begin{aligned} R_2 &= f\big(x_{i+\frac{1}{2}}, y_i + \tfrac{h}{2} f(x_i, y_i)\big) \\ &= f\big(x_{i+\frac{1}{2}}, y_i + \tfrac{h}{2} f(x_i, y_i)\big) \pm \varphi\big(x_{i+\frac{1}{2}}, u_i + \tfrac{h}{2} \varphi(x_i, u_i)\big) \\ &\quad \pm f\big(x_{i+\frac{1}{2}}, u_i + \tfrac{h}{2} \varphi(x_i, u_i)\big) \\ &= \varphi\big(x_{i+\frac{1}{2}}, u_i + \tfrac{h}{2} \varphi(x_i, u_i)\big) + R_{21} + R_{22} \end{aligned}$$

and using $y_i = u_i$ we get

$$\begin{aligned} R_{21} &= f\big(x_{i+\frac{1}{2}}, y_i + \tfrac{h}{2} f(x_i, y_i)\big) - f\big(x_{i+\frac{1}{2}}, y_i + \tfrac{h}{2} \varphi(x_i, y_i)\big), \\ R_{22} &= f\big(x_{i+\frac{1}{2}}, y_i + \tfrac{h}{2} \varphi(x_i, y_i)\big) - \varphi\big(x_{i+\frac{1}{2}}, y_i + \tfrac{h}{2} \varphi(x_i, y_i)\big), \end{aligned}$$

which leads to

$$|R_{21}| \le \tfrac{Kh}{2} |f(x_i, y_i) - \varphi(x_i, y_i)| \le \tfrac{Kh}{2}\, \omega_{3,3}(f, (x_i, y_i); 2h) \tag{18}$$

and

$$\begin{aligned} |R_{22}| &\le \omega_{3,3}\big(f, \big(x_{i+\frac{1}{2}}, y_i + \tfrac{h}{2} \varphi(x_i, y_i)\big); 2h\big) \\ &= \omega_{3,3}\big(f, (x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}} + \big(y_i - y_{i+\frac{1}{2}} + \tfrac{h}{2} \varphi(x_i, y_i)\big)); 2h\big) \\ &= \omega_{3,3}\big(f, (x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}} + \tfrac{h}{2}(y'(\xi) + \varphi(x_i, y_i))); 2h\big) \\ &\le \omega_{3,3}\big(f, (x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}); C_{21} h \, \|f\|_D\big), \end{aligned} \tag{19}$$

because of

$$|y'(\xi)| = |f(\xi, y(\xi))| \le \|f\|_D, \quad |\varphi(x_i, y_i)| \le C_{22}\|f\|_D, \quad \xi \in [x_i, x_{i+\frac{1}{2}}].$$

$$\begin{aligned} R_4 &= f\big(x_{i+1}, y_i + hf(x_{i+1}, y_{i+1})\big) \\ &= f\big(x_{i+1}, y_i + hf(x_{i+1}, y_{i+1})\big) \pm \varphi\big(x_{i+1}, u_i + h\varphi(x_{i+1}, u_{i+1})\big) \\ &\quad \pm f\big(x_{i+1}, u_i + h\varphi(x_{i+1}, y_{i+1})\big) \pm f\big(x_{i+1}, u_i + h\varphi(x_{i+1}, u_{i+1})\big) \\ &= \varphi\big(x_{i+1}, u_i + h\varphi(x_{i+1}, u_{i+1})\big) + R_{41} + R_{42} + R_{43}, \end{aligned}$$

where

$$\begin{aligned} R_{41} &= f\big(x_{i+1}, y_i + hf(x_{i+1}, y_{i+1})\big) - f\big(x_{i+1}, y_i + h\varphi(x_{i+1}, y_{i+1})\big), \\ R_{42} &= f\big(x_{i+1}, y_i + h\varphi(x_{i+1}, y_{i+1})\big) - f\big(x_{i+1}, y_i + h\varphi(x_{i+1}, u_{i+1})\big), \\ R_{43} &= f\big(x_{i+1}, y_i + h\varphi(x_{i+1}, u_{i+1})\big) - \varphi\big(x_{i+1}, y_i + h\varphi(x_{i+1}, u_{i+1})\big), \end{aligned}$$

and it follows from (5), (7), (8), and (10) that:

$$|R_{41}| \le Kh\,\omega_{3,3}\big(f, (x_{i+1}, y_{i+1}); 2h\big), \tag{20}$$

$$\begin{aligned} |R_{42}| &\le Kh|\varphi(x_{i+1}, y_{i+1}) - \varphi(x_{i+1}, u_{i+1})| \\ &\le C_{41}Kh\,\omega_{3,3}\big(f, (\eta, y(\eta)); 2h\big), \end{aligned} \tag{21}$$

$$\begin{aligned} |R_{43}| &\le \omega_{3,3}\big(f, (x_{i+1}, y_i + h\varphi(x_{i+1}, u_{i+1})); 2h\big) \\ &= \omega_{3,3}\big(f, (x_{i+1}, y_{i+1} - (y_{i+1} - y_i - h\varphi(x_{i+1}, u_{i+1}) \pm h\varphi(x_{i+1}, y_{i+1}))); 2h\big). \end{aligned}$$

Combining the last inequality and

$$\begin{aligned} &|y_{i+1} - y_i - h\big(\varphi(x_{i+1}, u_{i+1}) - \varphi(x_{i+1}, y_{i+1})\big) - h\varphi(x_{i+1}, y_{i+1})| \\ &\qquad \le h|y'(\xi)| + h\,\omega_{3,3}\big(f, (\eta, y(\eta)); 2h\big) + hC_{42}\|f\|_D \le C_{43}h\|f\|_D, \end{aligned}$$

we obtain

$$|R_{43}| \le C_{44}\,\omega_{3,3}\big(f, (x_{i+1}, y_{i+1}); C_{15}h\,\|f\|_D\big), \quad \eta, \xi \in [x_i, x_{i+1}]. \tag{22}$$

From the above estimates of R_i, $i = 1, \dots, 4$, it follows that for the estimation of A_6 it is necessary to estimate the expression

$$\begin{aligned} R &= \tfrac{2}{3}\,\varphi\big(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}\big) - \tfrac{2}{3}\,\varphi\big(x_{i+\frac{1}{2}}, u_i + \tfrac{h}{2}\,\varphi(x_i, u_i)\big) \\ &\quad + \tfrac{1}{6}\,\varphi(x_{i+1}, u_{i+1}) - \tfrac{1}{6}\,\varphi\big(x_{i+1}, u_i + h\varphi(x_{i+1}, u_{i+1})\big) \\ &= \tfrac{2}{3}\,A - \tfrac{2}{3}\,B + \tfrac{1}{6}\,C - \tfrac{1}{6}\,D. \end{aligned} \tag{23}$$

Using Taylor formula at the point $(x_i, u_i) = (x_i, y_i)$ we get:

$$A = \varphi\big(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}\big) = u'_{i+\frac{1}{2}} = u'_i + \tfrac{h}{2}u''_i + \tfrac{h^2}{8}u'''_i + \tfrac{h^3}{48}u^{IV}(\xi_1),$$

where $\xi_1 \in (x_i, x_{i+\frac{1}{2}})$,

$$\begin{aligned} B &= \varphi\big(x_{i+\frac{1}{2}}, u_i + \tfrac{h}{2}u_i'\big) \\ &= \varphi(x_i,u_i) + \big(\tfrac{h}{2}\tfrac{\partial}{\partial x} + \tfrac{h}{2}u_i'\tfrac{\partial}{\partial y}\big)\varphi(x_i,u_i) \\ &\quad + \tfrac{1}{2}\big(\tfrac{h}{2}\tfrac{\partial}{\partial x} + \tfrac{h}{2}u_i'\tfrac{\partial}{\partial y}\big)^2\varphi(x_i,u_i) + \tfrac{1}{6}\big(\tfrac{h}{2}\tfrac{\partial}{\partial x} + \tfrac{h}{2}u_i'\tfrac{\partial}{\partial y}\big)^3\varphi(\xi_2,\eta_2) \\ &= u_i' + \tfrac{h}{2}\varphi_x'(x_i,u_i) + \tfrac{h}{2}u_i'\varphi_u'(x_i,u_i) + \tfrac{h^2}{8}\varphi''_{x^2}(x_i,u_i) \\ &\quad + \tfrac{h^2}{4}u_i'\varphi_{xu}''(x_i,u_i) + \tfrac{h^2}{8}u'^2_i\varphi''_{u^2}(x_i,u_i) + \tfrac{h^3}{48}\varphi'''_{x^3}(\xi_2,\eta_2) \\ &\quad + \tfrac{h^3}{16}u_i'\varphi'''_{x^2u}(\xi_2,\eta_2) + \tfrac{h^3}{16}u'^2_i\varphi'''_{xu^2}(\xi_2,\eta_2) + \tfrac{h^3}{48}u'^3_i\varphi'''_{u^3}(\xi_2,\eta_2), \end{aligned}$$

where $\xi_2 \in (x_i, x_{i+\frac{1}{2}})$, $\eta_2 \in (u_i, u_i + \frac{h}{2}u_i')$,

$$C = u_{i+1}' = u_i' + hu_i'' + \tfrac{h^2}{2}u_i''' + \tfrac{h^3}{6}u^{IV}(\xi_3),$$

where $\xi_3 \in (x_i, x_{i+1})$,

$$\begin{aligned} D &= \varphi(x_{i+1}, u_i + hu_{i+1}') \\ &= \varphi(x_i,u_i) + \big(h\tfrac{\partial}{\partial x} + hu_{i+1}'\tfrac{\partial}{\partial y}\big)\varphi(x_i,u_i) \\ &\quad + \tfrac{1}{2}\big(h\tfrac{\partial}{\partial x} + hu_{i+1}'\tfrac{\partial}{\partial y}\big)^2\varphi(x_i,u_i) + \tfrac{1}{6}\big(h\tfrac{\partial}{\partial x} + hu_{i+1}'\tfrac{\partial}{\partial y}\big)^3\varphi(\xi_4,\eta_4) \\ &= u_i' + h\varphi_x'(x_i,u_i) + hu_{i+1}'\varphi_u'(x_i,u_i) + \tfrac{h^2}{2}\varphi''_{x^2}(x_i,u_i) \\ &\quad + h^2u_{i+1}'\varphi''_{xu}(x_i,u_i) + \tfrac{h^2}{2}u'^2_{i+1}\varphi''_{u^2}(x_i,u_i) + \tfrac{h^3}{6}\varphi'''_{x^3}(\xi_4,\eta_4) \\ &\quad + \tfrac{h^3}{2}u_{i+1}'\varphi'''_{x^2u}(\xi_4,\eta_4) + \tfrac{h^3}{2}u'^2_{i+1}\varphi'''_{xu^2}(\xi_4,\eta_4) + \tfrac{h^3}{6}u'^3_{i+1}\varphi'''_{u^3}(\xi_4,\eta_4), \end{aligned}$$

where $\xi_4 \in (x_i, x_{i+1})$, $\eta_4 \in (u_i, u_i + hu_{i+1}')$.

The expressions for A, B, C, D from above and (23) give

$$R = \tfrac{2}{3}A - \tfrac{2}{3}B + \tfrac{1}{6}C - \tfrac{1}{6}D = h^3G(\varphi), \tag{24}$$

where $G(\varphi)$ is a combinations of values of φ and its derivatives. It follows from inequalities (11)–(23) and (24) that

$$|e_{i+1}| \le |e_i| + h\Big(c_1\sum_{k=1}^{3} h^{3-k}\omega_k(y', x_i; h) + c_2\,\omega_{3,3}(f,(x_i,y_i); c_3h) + c_4h^3\Big).$$

Applying recursively the last inequality and using that $e_0 = 0$ we get

$$\begin{aligned} |e_{i+1}| &\le h\sum_{s=0}^{i}\Big(c_1\sum_{k=1}^{3} h^{3-k}\omega_k(y', x_s; h) + c_2\omega_{3,3}(f,(x_s,y_s); c_3h) + c_4h^3\Big) \\ &= \sum_{s=0}^{i}\int_{x_s}^{x_{s+1}}\Big(c_1\sum_{k=1}^{3} h^{3-k}\omega_k(y', x_s; h) + c_2\omega_{3,3}(f,(x_s,y_s); c_3h) + c_4h^3\Big)dx \end{aligned}$$

$$\leq \sum_{s=0}^{i} \int_{x_s}^{x_{s+1}} \Big(c_1 \sum_{k=1}^{3} h^{3-k} \omega_k(y', x; 2h) + c_2 \omega_{3,3}(f, (x, y(x)); c_5 h) + c_4 h^3 \Big) dx$$

$$\leq \int_{A}^{B} \Big(c_1 \sum_{k=1}^{3} h^{3-k} \omega_k(y', x; 2h) + c_2 \omega_{3,3}(f, (x, y(x)); c_5 h) + c_4 h^3 \Big) dx,$$

which proves the theorem. □

From Theorem 1 and the properties of $\tau_k(y', \delta)_L$ and $\tau_{k,s}(f; \delta)_L$ various orders of convergence follow for the method (1) under the additional assumption for smoothness of y, respectively $f(x, y)$.

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Andrey Andreev

Institute of Mathematics and Informatics of BAS
Acad. G. Bonchev str., bl. 8
1113 Sofia
BULGARIA
E-mail: aandreev@math.bas.bg

M. Nikoltjeva-Hedberg

Universitetet i Linköping
Matematiska Institutionen
S-581 83 Linköping
SWEDEN
E-mail: manik@mai.liu.se