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LULU Operators and Locally δ-monotone Approximations

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The LULU operators, well known in the non-linear multiresolution analysis of sequences, are extended to functions defined on continuous domain, namely, a real interval $\Omega \subseteq \mathbb{R}$. Similar to their discrete counterparts, for a given $\delta > 0$ the operators L_δ and U_δ form a fully ordered semi-group of four elements. It is shown that the compositions $L_\delta \circ U_\delta$ and $U_\delta \circ L_\delta$ provide locally δ-monotone approximations for the bounded real functions defined on Ω. The error of approximation is estimated in terms of the modulus of non-monotonicity.

1. Introduction

The LULU operators remove impulsive noise before a signal extraction from a sequence. They are computationally convenient and conceptually simpler compared to the the median smoothers usually considered to be the "basic" smoothers. The LULU operators have particular properties, e.g. they are fully trend preserving, [3], preserve the total variation, [2], etc., which make them an essential tool for multiresolution analysis of sequences. Furthermore, it was demonstrated during the last decade or so that these operators, being specific cases of morphological filters, [6], have a critical role in the analysis and comparison of non-linear smoothers, [4].

We extend the LULU theory from sequences to functions on a continuous domain, namely, a real interval Ω. The existing LULU theory can be considered as a particular case in this new development, since the discrete LULU operators can be equivalently formulated for splines of order 1 or order 2 on the integer partition of real line, the one-to-one mapping being given by the B-spline basis.

Given a sequence $\xi = (\xi_i)_{i\in\mathbb{N}}$ and $n \in \mathbb{N}$ the operators L_n and U_n are defined as follows

$$(L_n\xi)_i = \max\{\min\{\xi_{i-n}, ..., \xi_i\}, ..., \min\{\xi_i, ..., \xi_{i+n}\}\}, \quad i \in \mathbb{N},$$
$$(U_n\xi)_i = \min\{\max\{\xi_{i-n}, ..., \xi_i\}, ..., \max\{\xi_i, ..., \xi_{i+n}\}\}, \quad i \in \mathbb{N}.$$

In analogy with the above discrete LULU operators, for a given $\delta > 0$, the basic smoothers L_δ and U_δ in the LULU theory are defined for functions on Ω through the concepts of the so called lower and upper δ-envelopes of these functions. These definitions are given in Section 2, where it is also shown that the operators L_δ and U_δ preserve essential properties of their discrete counterparts. In particular, the operators L_δ and U_δ generate through composition a fully ordered four element semi-group also called a strong LULU structure. This issue is dealt with in Section 3. In Section 4 we define the concept of local δ-monotonicity and show that the compositions $L_\delta \circ U_\delta$ and $U_\delta \circ L_\delta$ are smoothers in the sense that the resulting functions are locally δ-monotone. The errors of approximation of real functions f by the these compositions are estimated in terms of the modulus of non-monotonicity $\mu(f, \delta)$.

2. The Basic Smoothers L_δ and U_δ

Let $\mathcal{A}(\Omega)$ denote the set of all bounded real functions defined on a real interval $\Omega \subseteq \mathbb{R}$. Let $B_\delta(x)$ denote the closed δ-neighborhood of x in Ω, that is, $B_\delta(x) = \{y \in \Omega : |x - y| \leq \delta\}$. The pair of mappings I, $S : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ defined by

$$I(f)(x) = \sup_{\delta>0} \inf\{f(y) : y \in B_\delta(x)\}, \quad x \in \Omega, \tag{1}$$

$$S(f)(x) = \inf_{\delta>0} \sup\{f(y) : y \in B_\delta(x)\}, \quad x \in \Omega, \tag{2}$$

are called lower Baire, and upper Baire operators, respectively, [5]. We consider on $\mathcal{A}(\Omega)$ the point-wise defined partial order, that is, for any $f, g \in \mathcal{A}(\Omega)$,

$$f \leq g \Longleftrightarrow f(x) \leq g(x), \quad x \in \Omega. \tag{3}$$

Then the lower and upper Baire operators can be defined in the following equivalent way. For every $f \in \mathcal{A}(\Omega)$ the function $I(f)$ is the maximal lower semi-continuous function which is not greater than f. Hence, it is also called lower semi-continuous envelope. In a similar way, $S(f)$ is the smallest upper semi-continuous function which is not less than f and is called the upper semi-continuous envelope of f. In analogy with $I(f)$ and $S(f)$ we call the functions

$$I_\delta(f)(x) = \inf\{f(y) : y \in B_\delta(x)\}, \quad x \in \Omega, \tag{4}$$

$$S_\delta(f)(x) = \sup\{f(y) : y \in B_\delta(x)\}, \quad x \in \Omega. \tag{5}$$

a lower δ-envelope of f and an upper δ-envelope of f, respectively.

It is easy to see from (4) and (5) that for every $\delta_1, \delta_2 > 0$

$$I_{\delta_1} \circ I_{\delta_2} = I_{\delta_1+\delta_2}, \qquad S_{\delta_1} \circ S_{\delta_2} = S_{\delta_1+\delta_2}. \tag{6}$$

Furthermore, the operators I_δ and S_δ, $\delta > 0$, as well as I and S are all monotone increasing with respect to the order (3), that is, for every $f, g \in \mathcal{A}(\Omega)$,

$$f \le g \Longrightarrow I_\delta(f) \le I_\delta(g), \quad S_\delta(f) \le S_\delta(g), \quad I(f) \le I(g), \quad S(f) \le S(g). \quad (7)$$

The following operators can be considered as continuous analogues of the discrete $LULU$ operators given in the Introduction:

$$L_\delta = S_{\frac{\delta}{2}} \circ I_{\frac{\delta}{2}}, \qquad U_\delta = I_{\frac{\delta}{2}} \circ S_{\frac{\delta}{2}}.$$

We will show that these operators have similar properties to their discrete counterparts. Let us note that they inherit monotonicity with respect of the functional argument from the operators I_δ and S_δ, (7), that is, for $f, g \in \mathcal{A}(\Omega)$,

$$f \le g \Longrightarrow L_\delta(f) \le L_\delta(g), \qquad U_\delta(f) \le U_\delta(g). \quad (8)$$

Theorem 1. *For every $f \in \mathcal{A}(\Omega)$ and $\delta > 0$ we have $L_\delta(f) \le f$, $U_\delta(f) \ge f$.*

Proof. Let $f \in \mathcal{A}(\Omega)$, $\delta > 0$. For any $x \in \Omega$ it follows from the definition of I_δ that $I_{\frac{\delta}{2}}(f)(y) \le f(x)$, $y \in B_{\frac{\delta}{2}}(x)$. Therefore

$$L_\delta(f)(x) = S_{\frac{\delta}{2}}(I_{\frac{\delta}{2}}(f))(x) = \sup\{I_{\frac{\delta}{2}}(f)(y) : y \in B_{\frac{\delta}{2}}(x)\} \le f(x), \quad x \in \Omega.$$

The second inequality in the theorem is proved in a similar way. □

Theorem 2. *The operator L_δ is monotone increasing on δ while the operator U_δ is monotone decreasing on δ, that is, for any $f \in \mathcal{A}(\Omega)$ and $0 < \delta_1 \le \delta_2$ we have $L_{\delta_1}(f) \le L_{\delta_2}(f)$, $U_{\delta_1}(f) \ge U_{\delta_2}(f)$.*

Proof. Let $\delta_2 > \delta_1 > 0$. Using properties (6) the operator L_{δ_2} can be represented in the form

$$L_{\delta_2} = S_{\frac{\delta_2}{2}} \circ I_{\frac{\delta_2}{2}} = S_{\frac{\delta_1}{2}} \circ S_{\frac{\delta_2-\delta_1}{2}} \circ I_{\frac{\delta_2-\delta_1}{2}} \circ I_{\frac{\delta_1}{2}} = S_{\frac{\delta_1}{2}} \circ L_{\delta_2-\delta_1} \circ I_{\frac{\delta_1}{2}}.$$

It follows from Theorem 1 that for every $f \in \mathcal{A}(\Omega)$ we have $L_{\delta_2-\delta_1}(I_{\frac{\delta_1}{2}}(f)) \le I_{\frac{\delta_1}{2}}(f)$. Hence using the monotonicity of the operator S_δ given in (7) we obtain

$$L_{\delta_2}(f) = S_{\frac{\delta_1}{2}}(L_{\delta_2-\delta_1}(I_{\frac{\delta_1}{2}}(f))) \le S_{\frac{\delta_1}{2}}(I_{\frac{\delta_1}{2}}(f)) = L_{\delta_1}(f), \quad f \in \mathcal{A}(\Omega).$$

The inequality $U_{\delta_1}(f) \ge U_{\delta_2}(f)$ is proved in a similar way. □

The next lemma is useful in dealing with compositions of I_δ and S_δ.

Lemma 1. *We have $I_\delta \circ S_\delta \circ I_\delta = I_\delta$, $S_\delta \circ I_\delta \circ S_\delta = S_\delta$.*

Proof. Using the monotonicity of I_δ, see (7), and Theorem 1 for $f \in \mathcal{A}(\Omega)$ we have

$$(I_\delta \circ S_\delta \circ I_\delta)(f) = I_\delta(L_{2\delta}(f)) \le I_\delta(f).$$

On the other side, applying Theorem 1 to $U_{2\delta}$ we obtain

$$(I_\delta \circ S_\delta \circ I_\delta)(f) = U_{2\delta}(I_\delta(f)) \ge I_\delta(f).$$

Therefore

$$(I_\delta \circ S_\delta \circ I_\delta)(f) = I_\delta(f), \quad f \in \mathcal{A}(\Omega).$$

The second equality is proved similarly. □

Theorem 3. *For every $\delta_1, \delta_2 > 0$ we have $L_{\delta_1} \circ L_{\delta_2} = L_{\max\{\delta_1,\delta_2\}}$ and $U_{\delta_1} \circ U_{\delta_2} = U_{\max\{\delta_1,\delta_2\}}$.*

Proof. We will only prove the first equality since the proof of the second one is done in a similar manner. Let first $\delta_2 > \delta_1 > 0$. Using property (6) and Lemma 1 we obtain

$$\begin{aligned} L_{\delta_1} \circ L_{\delta_2} &= (S_{\frac{\delta_1}{2}} \circ I_{\frac{\delta_1}{2}}) \circ (S_{\frac{\delta_2}{2}} \circ I_{\frac{\delta_2}{2}}) = (S_{\frac{\delta_1}{2}} \circ I_{\frac{\delta_1}{2}} \circ S_{\frac{\delta_1}{2}}) \circ (S_{\frac{\delta_2-\delta_1}{2}} \circ I_{\frac{\delta_2}{2}}) \\ &= S_{\frac{\delta_1}{2}} \circ S_{\frac{\delta_2-\delta_1}{2}} \circ I_{\frac{\delta_2}{2}} = S_{\frac{\delta_2}{2}} \circ I_{\frac{\delta_2}{2}} = L_{\delta_2}. \end{aligned}$$

If $\delta_1 > \delta_2 > 0$ in a similar way we have

$$\begin{aligned} L_{\delta_1} \circ L_{\delta_2} &= (S_{\delta_1} \circ I_{\delta_1}) \circ (S_{\delta_2} \circ I_{\delta_2}) = (S_{\delta_1} \circ I_{\delta_1-\delta_2}) \circ (I_{\delta_2} \circ S_{\delta_2} \circ I_{\delta_2}) \\ &= S_{\delta_1} \circ I_{\delta_1-\delta_2} \circ I_{\delta_2} = S_{\delta_1} \circ I_{\delta_1} = L_{\delta_1}. \end{aligned}$$

The proof in the case when $\delta_2 = \delta_1 > 0$ follows from either of the above identities where $S_{\delta_2-\delta_1}$ or $I_{\delta_1-\delta_2}$, respectively, are replaced by the identity operator. □

Important properties of smoothing operators are their idempotence and co-idempotence. Hence the significance of the next theorem.

Theorem 4. *The operators L_δ and U_δ are both idempotent and co-idempotent, that is,*

$$\begin{aligned} L_\delta \circ L_\delta &= L_\delta, \qquad (id - L_\delta) \circ (id - L_\delta) = id - L_\delta, \\ U_\delta \circ U_\delta &= U_\delta, \qquad (id - U_\delta) \circ (id - U_\delta) = id - U_\delta, \end{aligned}$$

where id denotes the identity operator.

Proof. The idempotence of L_δ and U_δ follows directly from Theorem 3. The co-idempotence of the operator L_δ is equivalent to $L_\delta \circ (id - L_\delta) = 0$. Using the first inequality in Theorem 1 one can easily obtain $L_\delta \circ (id - L_\delta) \ge 0$. Hence, for the co-idempotence of L_δ it remains to show that $L_\delta \circ (id - L_\delta) \le 0$.

Assume the opposite. Namely, there exists a function $f \in \mathcal{A}(\Omega)$ and $x \in \Omega$ such that

$$(L_\delta \circ (id - L_\delta))(f)(x) > 0.$$

Let $\varepsilon > 0$ be such that

$$(L_\delta \circ (id - L_\delta))(f)(x) > \varepsilon > 0.$$

Using the definition of L_δ the above inequality implies that there exists $y \in B_{\frac{\delta}{2}}(x)$ such that for every $z \in B_{\frac{\delta}{2}}(y)$ we have $(id - L_\delta)(f)(z) > \varepsilon$, or, equivalently,

$$f(z) > L_\delta(f)(z) + \varepsilon, \quad z \in B_{\frac{\delta}{2}}(y). \tag{9}$$

For every $z \in B_{\frac{\delta}{2}}(y)$ we also have

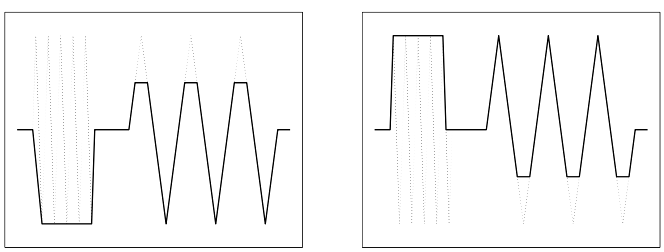
$$L_\delta(f)(z) \geq I_{\frac{\delta}{2}}(f)(y) = \inf\{f(t) : t \in B_{\frac{\delta}{2}}(y)\}.$$

Hence, there exists $t \in B_{\frac{\delta}{2}}(y)$ such that

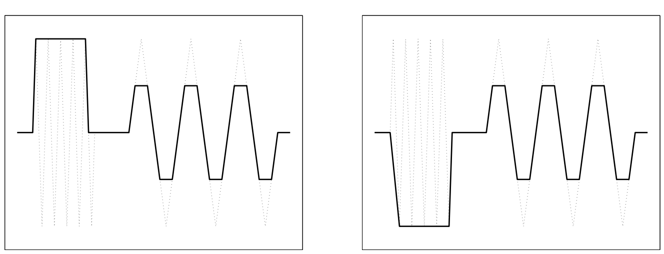
$$f(t) < I_{\frac{\delta}{2}}(f)(y) + \varepsilon \leq L_\delta(f)(z) + \varepsilon, \quad z \in B_{\frac{\delta}{2}}(y).$$

Taking $z = t$ in the above inequality we obtain $f(t) < L_\delta(f)(t) + \varepsilon$, which contradicts (9). The co-idempotence of U_δ is proved in a similar way. □

Example 1. The figures below illustrate graphically the smoothing effect of the operators L_δ, U_δ and their compositions. The graph of function f is given by dotted lines.



The functions $L_\delta(f)$ and $U_\delta(f)$



The functions $(L_\delta \circ U_\delta)(f)$ and $(U_\delta \circ L_\delta)(f)$

The operator L_δ smoothes the function f from above by removing sharp picks while the operator U_δ smoothes the function f from below by removing deep depressions. The smoothing effect of the compositions $L_\delta \circ U_\delta$ and $U_\delta \circ L_\delta$ can be described in terms of the local δ-monotonicity discussed in Section 4. Note that $L_\delta \circ U_\delta$ and $U_\delta \circ L_\delta$ resolve ambiguities in a different way; $L_\delta \circ U_\delta$ treats oscillations of length less then δ as upward impulses and removes them while $U_\delta \circ L_\delta$ considers such oscillations as downward impulses which are accordingly removed. The inequality $(U_\delta \circ L_\delta)(f) \le (L_\delta \circ U_\delta)(f)$ which is observed here will be proved in the next section for $f \in \mathcal{A}(\Omega)$.

3. The LULU Semi-group

In this section we consider the set of operators L_δ and U_δ and their compositions. For operators on $\mathcal{A}(\Omega)$ we consider the point-wise defined partial order. Namely, for operators P, Q on $\mathcal{A}(\Omega)$ we have

$$P \le Q \iff P(f) \le Q(f), \quad f \in \mathcal{A}(\Omega).$$

Then the inequalities in Theorem 1 can be represented in the form

$$L_\delta \le id \le U_\delta, \tag{10}$$

where id denotes the identity operator on $\mathcal{A}(\Omega)$.

Theorem 5. *For any $\delta > 0$ we have $U_\delta \circ L_\delta \le L_\delta \circ U_\delta$.*

Proof. Let $f \in \mathcal{A}(\Omega)$ and let $x \in \Omega$. For the sake of simplicity, we denote

$$p = (L_\delta \circ U_\delta)(f)(x) = S_{\frac{\delta}{2}}(I_\delta(S_{\frac{\delta}{2}}(f)))(x).$$

Let ε be an arbitrary positive number. For every $y \in B_{\frac{\delta}{2}}(x)$ we have

$$I_\delta(S_{\frac{\delta}{2}}(f))(y) \le p < p + \varepsilon. \tag{11}$$

<u>*Case 1.*</u> There exists $z \in B_{\frac{\delta}{2}}(x)$ such that $S_{\frac{\delta}{2}}(f)(z) < p + \varepsilon$. Then $f(t) < p + \varepsilon$ for $t \in B_{\frac{\delta}{2}}(z)$, which implies that

$$I_{\frac{\delta}{2}}(f)(t) < p + \varepsilon \quad \text{for } t \in B_\delta(z).$$

Hence $S_\delta(I_{\frac{\delta}{2}}(f))(z) \le p + \varepsilon$. Then

$$(U_\delta \circ L_\delta)(f)(t) = I_{\frac{\delta}{2}}(S_\delta(I_{\frac{\delta}{2}}(f)))(t) \le p + \varepsilon \quad \text{for } t \in B_{\frac{\delta}{2}}(z).$$

Since $x \in B_{\frac{\delta}{2}}(z)$, see the case assumption, from the above inequality we have $(U_\delta \circ L_\delta)(f)(x) \le p + \varepsilon$.

<u>*Case 2.*</u> For every $z \in B_{\frac{\delta}{2}}(x)$ we have $S_{\frac{\delta}{2}}(f)(z) \geq p+\varepsilon$. Let us denote

$$D = \left\{z \in \Omega : S_{\frac{\delta}{2}}(f)(z) < p+\varepsilon\right\}.$$

We will show that for every $z \in B_\delta(x)$ we have

$$B_\delta(z) \cap D \neq \emptyset. \tag{12}$$

Due to the inequality (11) we have that (12) holds for every $z \in B_{\frac{\delta}{2}}(x)$. Let $z \in B_\delta(x)$ and let $z > x+\frac{\delta}{2}$. This implies that $x+\frac{\delta}{2} \in \Omega$. Using the inequality (11) for $y = x+\frac{\delta}{2}$ as well as the case assumption we obtain that the set $\left(x+\frac{\delta}{2}, x+\frac{3\delta}{2}\right] \cap D$ is not empty. Then

$$B_\delta(z) \cap D \supset \left(x+\frac{\delta}{2}, x+\frac{3\delta}{2}\right] \cap D \neq \emptyset.$$

For $z < x-\frac{\delta}{2}$ condition (12) is proved in a similar way. Hence (12) holds for all $z \in B_\delta(x)$. Let $z \in B_\delta(x)$ and $v \in B_\delta(y) \cap D$. Since $v \in D$, we have $f(t) < p+\varepsilon$ for $t \in B_{\frac{\delta}{2}}(v)$. Using that $B_{\frac{\delta}{2}}(z) \cap B_{\frac{\delta}{2}}(v) \neq \emptyset$ we obtain

$$I_{\frac{\delta}{2}}(f)(z) < p+\varepsilon, \quad z \in B_\delta(x).$$

Therefore $S_\delta(I_{\frac{\delta}{2}}(f))(x) \leq p+\varepsilon$. Then

$$(U_\delta \circ L_\delta)(f)(x) = I_{\frac{\delta}{2}}(S_\delta(I_{\frac{\delta}{2}}(f)))(x) \leq S_\delta(I_{\frac{\delta}{2}}(f))(x) \leq p+\varepsilon.$$

Combining the results of Case 1 and Case 2 we have $(U_\delta \circ L_\delta)(f)(x) \leq p+\varepsilon$. Since ε is arbitrary this implies that

$$(U_\delta \circ L_\delta)(f)(x) \leq p = (L_\delta \circ U_\delta)(f)(x). \qquad \square$$

Theorem 6. *For a given $\delta > 0$ the operators $L_\delta \circ U_\delta$ and $U_\delta \circ L_\delta$ are both idempotent.*

The proof is an immediate application of Lemma 1.

Theorem 7. *We have $U_\delta \circ L_\delta \circ U_\delta = L_\delta \circ U_\delta$, $L_\delta \circ U_\delta \circ L_\delta = U_\delta \circ L_\delta$, $\delta > 0$.*

Proof. Using the inequalities (10) and the monotonicity of the operators L_δ, U_δ, see (8), we obtain

$$U_\delta \circ L_\delta \circ U_\delta \geq id \circ L_\delta \circ U_\delta = L_\delta \circ U_\delta.$$

For the proof of the inverse inequality we use Theorem 5 and the idempotence of U_δ as follows:

$$U_\delta \circ L_\delta \circ U_\delta = (U_\delta \circ L_\delta) \circ U_\delta \leq (L_\delta \circ U_\delta) \circ U_\delta = L_\delta \circ (U_\delta \circ U_\delta) = L_\delta \circ U_\delta.$$

Therefore $U_\delta \circ L_\delta \circ U_\delta = L_\delta \circ U_\delta$. The second equality is proved in a similar way. □

It follows from Theorems 6 and 7 that for a fixed $\delta > 0$ every composition involving finite number of the operators L_δ and U_δ is an element of the set $\{L_\delta, U_\delta, U_\delta \circ L_\delta, L_\delta \circ U_\delta\}$. Hence, the operators L_δ and U_δ form a semi-group with a composition table as follows:

	L_δ	U_δ	$U_\delta \circ L_\delta$	$L_\delta \circ U_\delta$
L_δ	L_δ	$L_\delta \circ U_\delta$	$U_\delta \circ L_\delta$	$L_\delta \circ U_\delta$
U_δ	$U_\delta \circ L_\delta$	U_δ	$U_\delta \circ L_\delta$	$L_\delta \circ U_\delta$
$U_\delta \circ L_\delta$	$U_\delta \circ L_\delta$	$L_\delta \circ U_\delta$	$U_\delta \circ L_\delta$	$L_\delta \circ U_\delta$
$L_\delta \circ U_\delta$	$U_\delta \circ L_\delta$	$L_\delta \circ U_\delta$	$U_\delta \circ L_\delta$	$L_\delta \circ U_\delta$

Furthermore, an easy application of Theorem 5 shows that this semi-group is completely ordered. Namely, we have

$$L_\delta \le U_\delta \circ L_\delta \le L_\delta \circ U_\delta \le U_\delta.$$

4. Locally δ-monotone Approximations

Definition 1. Let $\delta > 0$ and a function $f \in \mathbb{A}(\Omega)$ be given.

(i) The function f is called upwards δ-monotone if for every interval $[x, y] \subset \Omega$ with $y - x \le \delta$ we have

$$\sup_{z \in [x,y]} f(z) = \max\{f(x), f(y)\}.$$

(ii) The function f is called downwards δ-monotone if for every interval $[x, y] \subset \Omega$ with $y - x \le \delta$ we have

$$\inf_{z \in [x,y]} f(z) = \min\{f(x), f(y)\}.$$

(iii) The function f is called locally δ-monotone if it is both downwards δ-monotone and upwards δ-monotone.

The name locally δ-monotone reflects the following characterization:

f is locally δ-monotone	$\Longleftrightarrow$	On any interval $[x, y] \subseteq \Omega$, $y - x \le \delta$, f is either monotone increasing or monotone decreasing.	(13)

Theorem 8. *For every $\delta > 0$ and $f \in \mathcal{A}(\Omega)$ the function $S_{\frac{\delta}{2}}(f)$ is upwards δ-monotone while the function $I_{\frac{\delta}{2}}(f)$ is downwards δ-monotone.*

Proof. Let $\delta > 0$, $f \in \mathcal{A}(\Omega)$ and $[x,y] \subseteq \Omega$, $y - x \leq \delta$. Denote $g = S_{\frac{\delta}{2}}(f)$. It is easy to see that for every $z \in [x,y]$ we have $B_{\frac{\delta}{2}}(z) \subseteq B_{\frac{\delta}{2}}(x) \cup B_{\frac{\delta}{2}}(y)$. Therefore

$$\begin{aligned} g(z) &= \sup\left\{f(t) : t \in B_{\frac{\delta}{2}}(z)\right\} \leq \sup\left\{f(t) : t \in B_{\frac{\delta}{2}}(x) \cup B_{\frac{\delta}{2}}(y)\right\} \\ &= \max\left\{ \sup\{f(t) : t \in B_{\frac{\delta}{2}}(x)\}, \sup\{f(t) : t \in B_{\frac{\delta}{2}}(z)\}\right\} \\ &= \max\{g(x), g(y)\}, \end{aligned}$$

which shows that function g is upwards δ-monotone. The downwards δ-monotonicity of $U_\delta(f)$ is proved in a similar way. □

Corollary 1. *For every $\delta > 0$ and $f \in \mathcal{A}(\Omega)$ the function $L_\delta(f)$ is upwards δ-monotone while the function $U_\delta(f)$ is downwards δ-monotone.*

Theorem 9. *For every $\delta > 0$ and $f \in \mathcal{A}(\Omega)$ the functions $(U_\delta \circ L_\delta)(f)$ and $(L_\delta \circ U_\delta)(f)$ are both locally δ-monotone.*

The proof follows from Corollary 1 and the composition table in the preceding section. Theorem 9 shows that the operators $U_\delta \circ L_\delta$ and $L_\delta \circ U_\delta$ provide locally δ-monotone approximations to the functions in $\mathcal{A}(\Omega)$. The error of the approximation can be estimated in terms of the modulus of non-monotonicity. Let us recall the definition.

Definition 2. Let $f \in \mathcal{A}(\Omega)$. The mapping $\mu(f,\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ given by

$$\mu(f,\delta) = \frac{1}{2} \sup_{\substack{x_{1,2} \in \Omega \\ 0 < x_2 - x_2 \leq \delta}} \sup_{x \in [x_1,x_2]} \left(|f(x_1) - f(x)| + |f(x_2) - f(x)| - |f(x_1) - f(x_2)|\right)$$

is called modulus of non-monotonicity of f.

The locally δ-monotone functions can be conveniently characterized through the modulus of non-monotonicity. For any $f \in \mathcal{A}(\Omega)$ and $\delta > 0$ we have

$$f \text{ is locally } \delta\text{-monotone} \iff \mu(f,\delta) = 0 \qquad (14)$$

We will derive error estimates first in the case when $\Omega = \mathbb{R}$. It will prove useful to consider the upper semi-continuous envelope of the modulus of non-monotonicity. Let $f \in \mathcal{A}(\Omega)$. Using that $\mu(f,\delta)$ is monotone increasing with respect to δ the upper semi-continuous envelope of μ can be represented as

$$\hat{\mu}(f,\delta) = S(\mu(f,\cdot))(\delta) = \lim_{\varepsilon \to 0^+} \mu(f, \delta + \varepsilon).$$

Theorem 10. *Let $f \in \mathcal{A}(\mathbb{R})$ and $\delta > 0$. Then*

$$f(x) - L_\delta(f)(x) \leq \hat{\mu}(f,\delta), \quad U_\delta(f)(x) - f(x) \leq \hat{\mu}(f,\delta), \quad x \in \Omega.$$

Proof. Let $x \in \mathbb{R}$. Denote $p = L_\delta(f)(x)$. If $p = f(x)$, the first inequality of the theorem holds. Assume that $p < f(x)$. Let $\eta > 0$ be such that $p + \eta < f(x)$. Then we have

$$I_{\frac{\delta}{2}}(f)(y) < p + \eta, \quad y \in B_{\frac{\delta}{2}}(x). \tag{15}$$

Denote

$$\begin{aligned} z_1 &= \sup D_1, \quad D_1 = \{z \le x : f(z) < p + \eta\}, \\ z_2 &= \sup D_2, \quad D_2 = \{z \ge x : f(z) < p + \eta\}. \end{aligned}$$

Using the inequality (15) with $y = x - \frac{\delta}{2}$ and $y = x + \frac{\delta}{2}$ we obtain that D_1, and respectively D_2, are not empty and that $x - \delta \le z_1 \le x \le z_2 \le x + \delta$. Therefore

$$z_3 = \frac{z_1 + z_2}{2} \in \left[\frac{x - \delta + x}{2}, \frac{x + x + \delta}{2}\right] = B_{\frac{\delta}{2}}(x).$$

Then the inequality (15) implies that $B_{\frac{\delta}{2}} \cap (D_1 \cup D_2) \neq \emptyset$. Hence

$$z_2 - z_3 = z_3 - z_1 \le \frac{\delta}{2}.$$

Let $\varepsilon > 0$ be arbitrary. The neighborhood $B_{\frac{\delta+\varepsilon}{2}}(z_3)$ has nonempty intersections with both D_1 and D_2. Let $t_1 \in B_{\frac{\delta+\varepsilon}{2}} \cap D_1$ and $t_2 \in B_{\frac{\delta+\varepsilon}{2}} \cap D_2$. We have $t_2 - t_1 < \delta + \varepsilon$ and $x \in [t_1, t_2]$. From the definition of the modulus of non-monotonicity we have

$$|f(t_1) - f(x)| + |f(t_2) - f(x)| - |f(t_1) - f(t_2)| \le 2\mu(f, \delta + \varepsilon).$$

On the other side

$$\begin{aligned} |f(t_1) - f(x)| &+ |f(t_2) - f(x)| - |f(t_1) - f(t_2)| \\ &= 2f(x) - f(t_1) - f(t_2) - |f(t_1) - f(t_2)| \\ &= 2f(x) - \max\{f(t_1), f(t_2)\} \\ &> 2f(x) - 2(p + \eta). \end{aligned}$$

Therefore $f(x) - p - \eta < \mu(f, \delta + \varepsilon)$. Going with ε to 0 we obtain $f(x) - p \le \hat{\mu}(f, \delta) + \eta$. Since η is arbitrary small this implies the first inequality of the theorem. The second inequality is proved in a similar manner. □

Using Theorem 10 as well as (13) and Corollary 1, we have the following characterization of the fixed points of operators L_δ and U_δ. For any $f \in \mathcal{A}(\mathbb{R})$,

$$\hat{\mu}(f, \delta) = 0 \implies (L_\delta(f) = f,\ U_\delta(f) = f) \implies \mu(f, \delta) = 0$$

Theorem 11. *Let $\delta > 0$ and $f \in \mathcal{A}(\mathbb{R})$. Then*

$$\begin{aligned} \mu(L_\delta(f), \delta) \le \mu(f, \delta), &\qquad \hat{\mu}(L_\delta(f), \delta) \le \hat{\mu}(f, \delta), \\ \mu(U_\delta(f), \delta) \le \mu(f, \delta), &\qquad \hat{\mu}(U_\delta(f), \delta) \le \hat{\mu}(f, \delta). \end{aligned}$$

Proof. We will prove the inequalities for $L_\delta(f)$ since the ones for $U_\delta(f)$ are proved in a similar way. Denote $g = L_\delta(f)$. Let $[x_1, x_2]$ be an arbitrary interval of length at most δ and let $x \in [x_1, x_2]$. We consider the number

$$q = |f(x_1) - f(x)| + |f(x_2) - f(x)| - |f(x_1) - f(x_2)|.$$

According to Corollary 1 the function g is upper δ monotone, which implies that $g(x) \leq \max\{g(x_1), g(x_2)\}$. If we also have $g(x) \geq \min\{g(x_1), g(x_2)\}$, then the number q is zero and the first inequality of the theorem is trivially satisfied. Let $g(x) < \min\{g(x_1), g(x_2)\}$ and let $\eta > 0$ be such that $g(x) + \eta < \min\{g(x_1), g(x_2)\}$. The number q can then be represented in the form

$$q = 2(\min\{g(x_1), g(x_2)\} - g(x)).$$

If we assume that $f(y) \geq g(x) + \eta$ for all $y \in B_{\frac{\delta}{2}}(x)$, then

$$g(x) \geq I_{\frac{\delta}{2}}(f)(x) \geq g(x) + \eta,$$

which is a contradiction. Therefore, there exists $y \in B_{\frac{\delta}{2}}(x)$ such that $f(y) < g(x) + \eta$. If $y < x_1$ then using that $B_{\frac{\delta}{2}}(x_1) \subseteq B_{\frac{\delta}{2}}(y) \cup B_{\frac{\delta}{2}}(x)$ we obtain $I_{\frac{\delta}{2}}(z) < g(x) + \eta$ for all $z \in B_{\frac{\delta}{2}}(x_1)$, which implies

$$g(x_1) = S_{\frac{\delta}{2}}(I_{\frac{\delta}{2}}(f))(x_1) \leq g(x) + \eta < g(x_1).$$

This contradiction shows that $y \geq x_1$. In a similar way we show that $y \leq x_2$. Using also the first inequality of Theorem 1 we have

$$\begin{aligned} q &= 2\big(\min\{g(x_1), g(x_2)\} - g(x)\big) \\ &\leq 2(\min\{f(x_1), f(x_2)\} - f(y) - 2\eta) \\ &\leq \mu(f, \delta) - 2\eta. \end{aligned}$$

Using that η can be arbitrary small we obtain $q \leq \mu(f, \delta)$. Since the interval $[x_1, x_2]$ of length at most δ and $x \in [x_1, x_2]$ are arbitrary this implies that $\mu(g, \delta) \leq \mu(f, \delta)$. The inequality $\hat{\mu}(g, \delta) \leq \hat{\mu}(f, \delta)$ follows from the monotonicity of the operator S, see (7). $\square$

In the next theorem we give estimates for the error of approximation of a function $f \in \mathcal{A}(\mathbb{R})$ in terms of the supremum norm denoted here by $\|\cdot\|$.

Theorem 12. *Let $\delta > 0$ and $f \in \mathcal{A}(\mathbb{R})$. Then*

$$\|f - (L_\delta \circ U_\delta)(f)\| \leq \hat{\mu}(f, \delta), \qquad \|f - (U_\delta \circ L_\delta)(f)\| \leq \hat{\mu}(f, \delta).$$

Proof. Applying Theorems 10 and 11 we obtain

$$(L_\delta \circ U_\delta)(f)(x) \leq U_\delta(f)(x) \leq f(x) + \hat{\mu}(f, \delta)$$

and

$$(L_\delta \circ U_\delta)(f)(x) \geq U_\delta(f)(x) - \hat{\mu}(L_\delta(f), \delta) \geq f(x) - \hat{\mu}(f, \delta),$$

which implies the first inequality of the theorem. The second inequality is proved in a similar way. □

Error estimates similar to Theorem 12 can be derived in case of Ω being finite or semi-finite interval using a modification of the modulus of non-monotonicity. For simplicity we will only consider the case $\Omega = [a, b]$, $a, b \in \mathbb{R}$.

Definition 3. Let $f \in \mathcal{A}(\Omega)$. The mapping $\tilde{\mu}(f, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ given by

$$\tilde{\mu}(f, \delta) = \sup \left\{ \hat{\mu}(f, \delta), \sup_{x_{1,2} \in [a, a+\frac{\delta}{2}]} (|f(x_1) - f(x_2)|, \sup_{x_{1,2} \in [b-\frac{\delta}{2}, b]} (|f(x_1) - f(x_2)| \right\}$$

is called modified modulus of non-monotonicity of f.

This modulus is similar to the corrected modulus of non-motonicity in [1].

Theorem 13. *Let $f \in \mathcal{A}(\mathbb{R})$ and $\delta > 0$. Then*

$$f(x) - L_\delta(f)(x) \leq \tilde{\mu}(f, \delta), \quad U_\delta(f)(x) - f(x) \leq \tilde{\mu}(f, \delta), \quad x \in \Omega.$$

The proof is similar to the proof of Theorem 10.

It is easy to see that for any $f \in \mathcal{A}[a, b]$ the functions $L_\delta(f)$ and $U_\delta(f)$ are constants on each of the intervals $[a, a + \frac{\delta}{2}]$ and $[b - \frac{\delta}{2}]$. Therefore, using also Theorem 11 we have

$$\tilde{\mu}(L_\delta(f), \delta) = \hat{\mu}(L_\delta(f), \delta) \leq \hat{\mu}(f, \delta) \leq \tilde{\mu}(f, \delta).$$

In the same way we obtain

$$\tilde{\mu}(U_\delta(f), \delta) \leq \tilde{\mu}(f, \delta).$$

Hence the modified modulus satisfies similar inequalities to the ones given in Theorem 11 for μ and $\hat{\mu}$. Using these inequalities and Theorem 13, we obtain the error estimates in the next theorem, which are similar to the ones in Theorem 12.

Theorem 14. *Let $\delta > 0$ and $f \in \mathcal{A}(\mathbb{R})$. Then*

$$\|f - (L_\delta \circ U_\delta)(f)\| \leq \tilde{\mu}(f, \delta), \qquad \|f - (U_\delta \circ L_\delta)(f)\| \leq \tilde{\mu}(f, \delta).$$

5. Conclusion

In this paper we extended the LULU operators from sequences to real functions defined on a real interval using the lower and upper δ-envelopes of functions. The obtained structure, although more general than the well known LULU structure of the discrete operators, retains some of its essential properties. For a fixed $\delta > 0$ the compositions $L_\delta \circ U_\delta$ and $U_\delta \circ L_\delta$ provide locally δ-monotone approximations for real functions, the error of approximation being estimated in terms of the modulus of non-monotonicity of the functions. Further properties of the LULU operators for functions on continuous domains, e.g. trend preservation, will be investigated in the future. Generalizing the theory to functions on multidimensional domains is still an open problem.

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