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# Interpolation of Besov and Sobolev Spaces in the Non-diagonal Case

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We will show how the new formula for calculating the result of interpolation of a tuple of vector-valued spaces $(l_{q_0}^{\vec{s}_0}(L_{p_0}), \ldots, l_{q_n}^{\vec{s}_n}(L_{p_n}))$ allows to calculate the result of interpolation of isotropic and anisotropic (with dominated mixed smoothness) Besov and Sobolev spaces in the non-diagonal case.

## 1. Introduction

The theory of interpolation of Banach couples is now both well developed and widely used. An extension of this theory to tuples of Banach spaces has been considered in a number of papers (see, for example, [13], [11], [9], [7], [1], etc.). However, it turns out that such a generalization of the theory of couples to the theory of tuples runs into considerable difficulties. The main reason for this situation is the absence of the equivalence theorem of $K$- and $J$-methods of the real interpolation theory.

However, it has been shown in [2] that in some cases we can construct a nice interpolation theory for tuples, for example when we consider tuples of functional Banach lattices or their retracts. Moreover, for a number of problems instead of considering a couple of spaces, it is more correct to consider tuples containing more than two spaces. For example, it is well-known that the interpolation space of a couple of Besov spaces $(B_{p_0}^{s_0}, B_{p_1}^{s_1})_{\theta,q}$ in the non-diagonal case, $\frac{1}{q} \neq \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, is out of the scale of Besov spaces if $p_0 \neq p_1$, $s_0 \neq s_1$. But the interpolation space of a triple of Besov spaces $(B_{p_0}^{s_0}, B_{p_1}^{s_1}, B_{p_2}^{s_2})_{\vec{\theta},q}$ belongs to the scale of Besov spaces even in the non-diagonal case if three points $(\frac{1}{p_i}, s_i)$, $i = 0, 1, 2$, do not lie on the same line (see [3]). However, the proof in [3] is rather complicated and does not work for the case of more than three spaces.

The object of the present paper is to give a new approach to the problem which not only generalizes the result of [3] for $n$-tuples of Besov spaces but also works in the case of tuples of anisotropic Besov and Sobolev spaces with dominated smoothness. This approach is based on the well-known possibility (see [10], [6], [11]) to obtain Besov spaces as retracts of some vector-valued weighted $l_p$ spaces and on our new results in vector-valued interpolation.

## 2. Basic Definitions

Let $A_0, A_1, \dots, A_n$ be $n+1$ Banach or quasi-Banach spaces. We shall say that they form a compatible collection or simply a collection $\vec{A} = (A_0, A_1, \dots, A_n)$ if they are linearly and continuously embedded in some (common for all) topological linear space with Hausdorff topology. Then we can, analogously to the case of couple, define the $K$-functional (see [11]) by the formula

$$K(\vec{t}, a; \vec{A}) = \inf(\|a_0\|_{A_0} + t_1\|a_1\|_{A_1} + \cdots + t_n\|a_n\|_{A_n}), \quad \vec{t} = (t_1, t_2, \dots, t_n) \in \mathbb{R}_n^+,$$

where the infimum is taken over all decompositions $a = a_0 + a_1 + \cdots + a_n$.

Let $\vec{\theta} = (\theta_0, \theta_1, \dots, \theta_n)$ be a parameter vector, i.e., $\theta_i > 0$, $i = 0, 1, \dots, n$, $\theta_0 + \cdots + \theta_n = 1$, and let $0 < q \leq \infty$. Then the interpolation space $\vec{A}_{\vec{\theta},q} = (A_0, A_1, \dots, A_n)_{\vec{\theta},q}$ (usually this space is defined by $\vec{A}_{\vec{\theta},q;K}$, but we will omit the index $K$ as we will consider only $K$-spaces) is defined by the norm (if $q \geq 1$) or quasinorm (if $0 < q \leq 1$)

$$\|a\|_{\vec{\theta},q} = \left( \int_{\mathbb{R}_+^n} \left(t_1^{-\theta_1} t_2^{-\theta_2} \dots t_n^{-\theta_n} K(\vec{t}, a; \vec{A})\right)^q \frac{dt_1}{t_1} \frac{dt_2}{t_2} \cdots \frac{dt_n}{t_n} \right)^{1/q} \tag{1}$$

with the usual modification for $q = \infty$.

We will also need the definition of a vector-valued space $l_q^{\vec{s}}(A)$. Let $A$ be a Banach or quasi-Banach space and $\vec{s} = (s_1, s_2, \dots, s_m) \in \mathbb{R}^m$, then the space $l_q^{\vec{s}}(A)$ is defined as a sequence of elements $\{a_{\vec{k}}\}_{\vec{k} \in \mathbb{Z}^m}$, $a_{\vec{k}} \in A$ such that

$$\left\| \{a_{\vec{k}}\}_{\vec{k} \in \mathbb{Z}^m} \right\|_{l_q^{\vec{s}}(A)} = \Big( \sum_{k \in \mathbb{Z}^m} (2^{-\vec{s} \cdot \vec{k}} \, \|a_{\vec{k}}\|_A)^q \Big)^{1/q} < \infty,$$

with the usual changes for $q = \infty$.

## 3. Vector-Valued Interpolation

Importance of vector-valued interpolation for interpolation of Besov and Sobolev spaces is well-known (see, for example, [6]), however, for interpolation in the non-diagonal case we will need some new formulas. Let us explain the situation.

The first vector-valued result for the real method was obtained by Lions and Peetre in 1964

$$(L_{q_0}(A_0), L_{q_1}(A_1))_{\theta,q} = L_q((A_0, A_1)_{\theta,q}), \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Attempts to generalize this result to the non-diagonal case, $\frac{1}{q} \neq \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, encounters considerable difficulties. The only known result of this type is the following:

$$(l_{q_0}^{s_0}(A), l_{q_1}^{s_1}(A))_{\theta,q} = l_q^{s_\theta}(A), \quad s_\theta = (1-\theta)s_0 + \theta s_1. \tag{2}$$

Note that in this couple the inner spaces are the same. Moreover, as was shown by Cwikel in [8] there is no reasonable generalization of (2) for different inner spaces. In particular, it means that in general the interpolation space

$$(l_{q_0}^{s_0}(A_0), l_{q_1}^{s_1}(A_1))_{\theta,q}$$

for $A_0 \neq A_1$ and $\frac{1}{q} \neq \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ is not a space of the type $\Phi((A_0, A_1)_{\theta,q})$, where $\Phi$ is some functional Banach lattice. However, in [4] it was shown that if instead of a couple $(l_{q_0}^{s_0}(A_0), l_{q_1}^{s_1}(A_1))$ we consider a tuple $(l_{q_0}^{s_0}(A_0), \dots, l_{q_n}^{s_n}(A_n))$ of more than two spaces, then the situation is different and we can obtain a formula analogous to (2).

Developing the ideas of the papers [4] and [5] allows us to obtain the following result.

**Theorem 1.** *Let $(A_0, A_1)$ be a couple of Banach spaces, $0 < \eta_i < 1$, $p_i > 0$, $q_i > 0$, $\vec{s}_i \in \mathbb{R}_i^m$. If the convex hull of $n+1$ vectors $(\vec{s}_0, \eta_0), (\vec{s}_1, \eta_1), \dots, (\vec{s}_n, \eta_n)$ contains some ball in $\mathbb{R}^{m+1}$ (therefore $n \geq m+1$), then*

$$(l_{q_0}^{\vec{s}_0}((A_0, A_1)_{\eta_0,p_0}), \dots, l_{q_n}^{\vec{s}_n}((A_0, A_1)_{\eta_n,p_n}))_{\vec{\theta},q} = l_q^{\vec{s}_{\vec{\theta}}}((A_0, A_1)_{\eta_{\vec{\theta}},q}),$$

*where*

$$(\vec{s}_{\vec{\theta}}, \eta_{\vec{\theta}}) = \theta_0(\vec{s}_0, \eta_0) + \theta_1(\vec{s}_1, \eta_1) + \dots + \theta_n(\vec{s}_n, \eta_n).$$

As for the scale of $L_p$ spaces we have

$$(L_{p_0}, L_{p_1})_{\theta,q} = L_{p_\theta,q}, \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

where $L_{p_\theta,q}$ is a Lorentz space equal to $L_{p_\theta}$ for $q = p_\theta$, therefore from Theorem 1 follows the following corollary.

**Corollary 1.** *Let $\vec{s}_0, \dots, \vec{s}_n$ be vectors in $\mathbb{R}^m$. Suppose that the convex hull of the vectors $(\vec{s}_0, \frac{1}{p_0}), \dots, (\vec{s}_n, \frac{1}{p_n})$ contains some ball in $\mathbb{R}^{m+1}$. Then the following formula is correct*

$$(l_{q_0}^{\vec{s}_0}(L_{p_0}), \dots, l_{q_n}^{\vec{s}_n}(L_{p_n}))_{\vec{\theta},q} = l_q^{\vec{s}_{\vec{\theta}}}(L_{p_{\vec{\theta}},q}), \tag{3}$$

*where*

$$\frac{1}{p_{\vec{\theta}}} = \theta_0 \frac{1}{p_0} + \dots + \theta_n \frac{1}{p_n}, \quad \vec{s}_{\vec{\theta}} = \theta_0 \vec{s}_0 + \dots + \theta_n \vec{s}_n.$$

## 4. Interpolation of Besov and Sobolev Spaces

It is well-known (see, for example, [6]) that isotropic Besov spaces $B^s_q(L_p)$, i.e., Besov spaces with the smoothness $s > 0$ constructed on the base of $L_p$ spaces, are retracts of the spaces $l^s_q(L_p)$. Therefore, the problem of interpolation of isotropic Besov spaces can be reduced to the problem of interpolation of a tuple $(l^{s_0}_{q_0}(L_{p_0}), \dots, l^{s_n}_{q_n}(L_{p_n}))$. Now we can apply formula (3) and obtain

**Theorem 2.** *If the convex hull of the vectors* $(s_0, \frac{1}{p_0}), \dots, (s_n, \frac{1}{p_n})$ *contains some ball in* $\mathbb{R}^2$*, i.e., all these points do not lie on the same line, then*

$$(B^{s_0}_{q_0}(L_{p_0}), \dots, B^{s_n}_{q_n}(L_{p_n}))_{\vec{\theta},q} = B^{s_{\vec{\theta}}}_q(L_{p_{\vec{\theta}},q}),$$

*where*

$$\frac{1}{p_{\vec{\theta}}} = \theta_0 \frac{1}{p_0} + \cdots + \theta_n \frac{1}{p_n}, \quad s_{\vec{\theta}} = \theta_0 s_0 + \cdots + \theta_n s_n, \quad \theta_0 = (1 - \theta_1 - \cdots - \theta_n).$$

Moreover, from the well-known embeddings

$$B^s_1(L_p) \subset W^s(L_p) \subset B^s_\infty(L_p)$$

for Sobolev spaces we immediately obtain that under the conditions of Theorem 2 we have

$$(W^{s_0}(L_{p_0}), \dots, W^{s_n}(L_{p_n}))_{\vec{\theta},q} = B^{s_{\vec{\theta}}}_q(L_{p_{\vec{\theta}},q}).$$

The same ideas could be used for anisotropic Besov and Sobolev spaces with dominated smoothness, the only difference is that we need to use a different retract.

Let us recall the history of the problem. Sparr showed in [11] that Besov spaces with dominated smoothness $\vec{s} \in \mathbb{R}^m$ constructed on the base of the space $L_p$ (we denote these spaces by $\tilde{B}^{\vec{s}}_q(L_p)$) are retracts of the space $l^{\vec{s}}_q(L_p)$. Therefore, analogously to the case of isotropic spaces, the problem of interpolation of a tuple $(\tilde{B}^{\vec{s}_0}_{q_0}(L_{p_0}), \tilde{B}^{\vec{s}_1}_{q_1}(L_{p_1}), \dots, \tilde{B}^{\vec{s}_n}_{q_n}(L_{p_n}))$ can be reduced to interpolation of a tuple of vector-valued spaces $(l^{\vec{s}_0}_{q_0}(L_{p_0}), l^{\vec{s}_1}_{q_1}(L_{p_1}), \dots, l^{\vec{s}_n}_{q_n}(L_{p_n}))$. Then he proved the following result

$$(l^{\vec{s}_0}_{q_0}(A), l^{\vec{s}_1}_{q_1}(A), \dots, l^{\vec{s}_n}_{q_n}(A))_{\vec{\theta},q} = l^{\vec{s}_{\vec{\theta}}}_q(A), \quad \vec{s}_{\vec{\theta}} = \theta_0 \vec{s}_0 + \theta_1 \vec{s}_1 + \cdots + \theta_n \vec{s}_n, \tag{4}$$

under the condition that

$$\operatorname{span}\{\vec{s}_1 - \vec{s}_0, \dots, \vec{s}_n - \vec{s}_0\} = \mathbb{R}^m. \tag{5}$$

Therefore, from (4) it follows that

$$(\tilde{B}^{\vec{s}_0}_{q_0}(L_p), \tilde{B}^{\vec{s}_1}_{q_1}(L_p), \dots, \tilde{B}^{\vec{s}_n}_{q_n}(L_p))_{\vec{\theta},q} = \tilde{B}^{\vec{s}_{\vec{\theta}}}_q(L_p), \quad \vec{s}_{\vec{\theta}} = \theta_0 \vec{s}_0 + \theta_1 \vec{s}_1 + \cdots + \theta_n \vec{s}_n, \tag{6}$$

under the condition (5).

G. Sparr also considered the question of interpolation of Sobolev spaces with dominated smoothness, which we denote by $\tilde{W}^{\vec{s}}(L_p)$. From (6) and the continuous embeddings

$$\tilde{B}_1^{\vec{s}}(L_p) \subset \tilde{W}^{\vec{s}}(L_p) \subset \tilde{B}_\infty^{\vec{s}}(L_p)$$

he obtained

$$(\tilde{W}^{\vec{s}_0}(L_p), \tilde{W}^{\vec{s}_1}(L_p), \ldots, \tilde{W}^{\vec{s}_n}(L_p))_{\vec{\theta},q} = \tilde{B}_q^{\vec{s}_{\vec{\theta}}}(L_p), \quad \vec{s}_{\vec{\theta}} = \theta_0\vec{s}_0 + \theta_1\vec{s}_1 + \cdots + \theta_n\vec{s}_n \tag{7}$$

with the same condition (5).

As we see in (6)-(7) Besov and Sobolev spaces were constructed on the base of the same space $L_p$. And therefore it is natural to ask whether it is possible to obtain an analogous result for the case when Besov and Sobolev spaces are constructed on the base of different $L_p$ spaces. In this case formula (3) can be applied and we obtain

**Theorem 3.** *If the convex hull of the vectors* $(\overrightarrow{s}_0, \frac{1}{p_0}), \ldots, (\overrightarrow{s}_n, \frac{1}{p_n})$ *contains some ball in* $\mathbb{R}^{m+1}$, *then*

$$(\tilde{B}_{q_0}^{\vec{s}_0}(L_{p_0}), \tilde{B}_{q_1}^{\vec{s}_1}(L_{p_1}), \ldots, \tilde{B}_{q_n}^{\vec{s}_n}(L_{p_n}))_{\vec{\theta},q} = \tilde{B}_q^{\vec{s}_{\vec{\theta}}}(L_{p_{\vec{\theta}},q})$$

*and*

$$(\tilde{W}^{\vec{s}_0}(L_{p_0}), \tilde{W}^{\vec{s}_1}(L_{p_1}), \ldots, \tilde{W}^{\vec{s}_n}(L_{p_n}))_{\vec{\theta},q} = \tilde{B}_q^{\vec{s}_{\vec{\theta}}}(L_{p_{\vec{\theta}},q})$$

*where*

$$\frac{1}{p_{\vec{\theta}}} = \theta_0\frac{1}{p_0} + \cdots + \theta_n\frac{1}{p_n}, \quad \overrightarrow{s}_{\vec{\theta}} = \theta_0\overrightarrow{s}_0 + \cdots + \theta_n\overrightarrow{s}_n, \quad \theta_0 = (1 - \theta_1 - \cdots - \theta_n),$$

*and* $\tilde{B}_q^{\vec{s}_{\vec{\theta}}}(L_{p_{\vec{\theta}},q})$ *is a Besov space constructed on the base of the Lorentz space* $L_{p_{\vec{\theta}},q}$.

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