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Mean Square Approximation by Minimum Norm Interpolation

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The concept of periodic Hilbert spaces introduced by Babuska [1] (see also [1, 9, 10, 3, 4]) has been extended to the non-periodic case by the concept of harmonic Hilbert spaces ([5, 11]). In these spaces error estimates for cardinal interpolation were derived [6, 7]). In this paper we will derive error estimates for minimum norm interpolation in the mean square norm which extend the results obtained in [8]).

1. Cardinal Interpolation

We start with functions

$$f(x) = \int_{-\infty}^{\infty} F(t) e^{ixt} dt$$

with $F \in L_1(\mathbb{R})$. These functions are elements of the *Wiener algebra* $A(\mathbb{R})$. Let $b > 0$. The *periodization* of F is defined by

$$F_{2b}(t) = \sum_{k=-\infty}^{\infty} F(t + 2bk).$$

It is known [2] that $F_{2b} \in L_1([-b, b])$.

Let D be a bounded non-negative measurable function with associated periodization

$$D_{2b}(t) = \sum_{k=-\infty}^{\infty} D(t + 2bk).$$

Note that

$$\frac{D(t)}{D_{2b}(t)}$$

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is a bounded non-negative measurable function. We assume that

$$\frac{D(t)}{D_{2b}(t)} F_{2b}(t)$$

is absolutely integrable and we define

$$T_b^D(f)(x) := \int_{-\infty}^{\infty} \frac{D(t)}{D_{2b}(t)} F_{2b}(t) e^{ixt} dt.$$

Theorem 1. *Assume $f \in A(\mathbb{R})$ and $T_b^D(f) \in A(\mathbb{R})$. Then the interpolation conditions*

$$T_b^D(f)(r\frac{\pi}{b}) = f(r\frac{\pi}{b}), \quad r \in \mathbb{Z},$$

are satisfied.

Proof. We can conclude

$$\begin{aligned} T_b^D(f)(r\frac{\pi}{b}) &= \int_{-\infty}^{\infty} \frac{D(t)}{D_{2b}(t)} F_{2b}(t) e^{i(r\pi/b)t} dt \\ &= \sum_{k=-\infty}^{\infty} \int_0^{2b} \frac{D(t+2bk)}{D_{2b}(t)} F_{2b}(t) e^{i(r\pi/b)[t+2bk]} dt \\ &= \int_0^{2b} \frac{\sum_{k=-\infty}^{\infty} D(t+2bk)}{D_{2b}(t)} F_{2b}(t) e^{i(r\pi/b)t} dt \\ &= \int_0^{2b} F_{2b}(t) e^{i(r\pi/b)t} dt \\ &= \int_0^{2b} \sum_{k=-\infty}^{\infty} F(t+2bk) e^{i(r\pi/b)t} dt \\ &= \sum_{k=-\infty}^{\infty} \int_{2bk}^{2bk+2b} F(s) e^{i(r\pi/b)[s-2bk]} ds \\ &= \int_{-\infty}^{\infty} F(s) e^{i(r\pi/b)s} ds = f(r\frac{\pi}{b}). \end{aligned}$$

□

To establish uniqueness we consider the *harmonic Hilbert space* $H_D(\mathbb{R})$ related to defining function D (see [5]). It is the linear subspace of the Wiener algebra $A(\mathbb{R})$ of those functions

$$f(x) = \int_{-\infty}^{\infty} F(t) e^{ixt} dt$$

satisfying

$$\int_{-\infty}^{\infty} \frac{|F(t)|^2}{D(t)} dt < \infty.$$

The inner product of

$$f(x) = \int_{-\infty}^{\infty} F(t)e^{ixt}dt, \qquad g(x) = \int_{-\infty}^{\infty} G(t)e^{ixt}dt,$$

is given by

$$(f,g)_D = \int_{-\infty}^{\infty} \frac{F(t)\overline{G(t)}}{D(t)}\,dt.$$

We have shown in [7] that $H_D(\mathbb{R}) \subseteq L_2(\mathbb{R})$ and

$$\|f\|_2 \leq \sqrt{2\pi\|D\|_\infty}\,\|f\|_D.$$

Theorem 2 ([8]). *Let $f \in H_D(\mathbb{R})$. Then $T_b^D(f)$ is the unique function in the linear manifold of functions $g \in H_D(\mathbb{R})$ satisfying*

$$g(r\frac{\pi}{b}) = f(r\frac{\pi}{b}), \quad r \in \mathbb{Z},$$

which minimizes the norm $\|g\|_D$.

For the special case

$$D(t) = \chi_{-b,b}(t), \quad b > 0,$$

we write

$$T_b^D = T_b$$

and we obtain interpolation by functions of exponential-type (*cardinal interpolation*) ([12, 8]):

$$T_b(f)(x) := \int_{-\infty}^{\infty} \chi_{[-b,b]}(t)F_{2b}(t)e^{ixt}dt.$$

We also have

$$F_{2b} \in L_2([-b,b]).$$

The Fourier series is given by

$$F_{2b}(t) = \frac{1}{2b}\sum_{k=-\infty}^{\infty} f(kh)e^{(-ikht)}, \qquad h = \frac{\pi}{b}.$$

It was shown in [7] that this property

$$F_{2b} \in L_2([-b,b])$$

holds also for functions from the harmonic Hilbert space $H_D(\mathbb{R})$ if the defining function D is continuous with the additional properties

$$D(-t) = D(t) > 0, \quad D(t+h) \leq D(t) \quad (t \geq 0,\ h > 0).$$

In particular we have estblished the estimate

$$\|T_b(f)\|_2 \le \sqrt{2\pi\|D_{2b}\|_\infty}\,\|f\|_D.$$

We briefly recall the approximation properties of S_b and T_b in the mean square norm as established in ([7]).

Theorem 3 ([7]). *Let*

$$S_b(f)(x) = \int_{-\infty}^{\infty} \chi_{[-b,b]}(t)F(t)e^{ixt}dt, \quad b > 0,$$

denote the Fourier partial integral *of* $f \in H_D(\mathbb{R})$. *Then the estimate*

$$\|f - S_b(f)\|_2 \le \sqrt{D(b)}\,\sqrt{2\pi}\,\|f - S_b(f)\|_D$$

holds true.

Next we consider the approximation of the *Fourier partial integral* $S_b(f)$ of f by the cardinal interpolant $T_b(f)$ of f.

Theorem 4 ([7]). *Assume* $f \in H_D(\mathbb{R})$. *Then the estimate*

$$\|S_b(f) - T_b(f)\|_2 \le \sqrt{\sum_{r\ge 1} D(rb)}\,\sqrt{2\pi}\|f - S_b(f)\|_D$$

holds true. As a consequence we have

$$\|f - T_b(f)\|_2 \le 2\sqrt{\sum_{r\ge 1} D(rb)}\,\sqrt{2\pi}\|f - S_b(f)\|_D.$$

2. Minimum Norm Interpolation

Our main objective is to extend Theorems 3 and 4 from cardinal interpolation T_b to minimum norm interpolation T_b^D. We start with a property of periodization of the defining function D.

Lemma 1. *The estimate*

$$(D^2)_{2b}(t) \le D_{2b}^2(t) \tag{1}$$

is valid.

Proof. We have

$$(D^2)_{2b}(t) = \sum_{k=-\infty}^{\infty} D^2(t+2bk) \le \left(\sum_{k=-\infty}^{\infty} D(t+2bk)\right)^2 = D_{2b}^2(t). \qquad \square$$

We first establish an inequality for the minimum norm interpolant.

Theorem 5. *Assume $f \in H_D(\mathbb{R})$. Then the estimate*

$$\|T_b^D(f)\|_2 \le \|T_b(f)\|_2$$

holds true.

Proof. Taking into account (1) we get

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} |T_b^D(f)(x)|^2 dx &= \int_{-\infty}^{\infty} \left|\frac{F_{2b}(t)}{D_{2b}(t)}\right|^2 D^2(t)\,dt \\ &= \int_{-\infty}^{\infty} \chi_{[-b,b]}(t)\left|\frac{F_{2b}(t)}{D_{2b}(t)}\right|^2 (D^2)_{2b}(t)\,dt \\ &\le \int_{-\infty}^{\infty} \chi_{[-b,b]}(t)|F_{2b}(t)|^2 dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |T_b(f)(x)|^2 dx. \qquad \square\end{aligned}$$

The approximation power of minimum interpolation is first determined for functions of exponential-type:

Theorem 6. *Assume $f \in H_D(\mathbb{R})$ and $b > 0$ and consider $h = S_b(f)$. Then the estimate*

$$\|h - T_b^D(h)\|_2 \le \sqrt{2\sum_{r\ge 1} D(rb)}\,\sqrt{2\pi}\|h\|_D$$

holds true.

Proof. By definition the Fourier partial integral $h = S_b(f)$ of $f \in H_D(\mathbb{R})$ is given by

$$h(x) = \int_{-\infty}^{\infty} \chi_{[-b,b]}(t)F(t)e^{ixt}dt = \int_{-\infty}^{\infty} H(t)e^{ixt}dt.$$

The periodization of H satisfies

$$H_{2b}(t)\chi_{[-b,b]}(t) = \chi_{[-b,b]}(t)F(t).$$

We first obtain

$$\begin{aligned}&\|S_b(f) - T_b^D(S_b(f))\|_2^2 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |h(x) - T_b^D(h)(x)|^2 dx \\ &= \int_{-\infty}^{\infty} \left|\chi_{[-b,b]}(t)F(t) - \frac{H_{2b}(t)}{D_{2b}(t)}D(t)\right|^2 dt \\ &= \int_{-\infty}^{\infty} \left|\chi_{[-b,b]}(t)\Big(F(t) - \frac{F(t)}{D_{2b}(t)}D(t)\Big) - \big(1-\chi_{[-b,b]}(t)\big)\frac{H_{2b}(t)}{D_{2b}(t)}D(t)\right|^2 dt.\end{aligned}$$

Taking into account orthogonality we further get

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|h(x)-T_b^D(h)(x)\right|^2dx\\
&=\int_{-\infty}^{\infty}\chi_{[-b,b]}(t)\left|F(t)\Big(1-\frac{D(t)}{D_{2b}(t)}\Big)\right|^2dt\\
&\quad+\int_{-\infty}^{\infty}\big(1-\chi_{[-b,b]}(t)\big)\left|\frac{H_{2b}(t)D(t)}{D_{2b}(t)}\right|^2dt\\
&=\int_{-\infty}^{\infty}\chi_{[-b,b]}(t)|F(t)|^2\left\{\Big(1-\frac{D(t)}{D_{2b}(t)}\Big)^2-\Big(\frac{D(t)}{D_{2b}(t)}\Big)^2\right\}dt\\
&\quad+\int_{-\infty}^{\infty}\left|\frac{H_{2b}(t)}{D_{2b}(t)}D(t)\right|^2dt.
\end{aligned}$$

Now periodization yields

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|h(x)-T_b^D(h)(x)\right|^2dx\\
&=\int_{-\infty}^{\infty}\chi_{[-b,b]}(t)\big\{[D_{2b}(t)-D(t)]^2+[(D^2)_{2b}(t)-D(t)^2]\big\}\left|\frac{F(t)}{D_{2b}(t)}\right|^2dt.
\end{aligned}$$

Using (1) we can conclude

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|h(x)-T_b^D(h)(x)\right|^2dx\\
&\leq\int_{-\infty}^{\infty}\chi_{[-b,b]}(t)\left|\frac{F(t)}{D_{2b}(t)}\right|^2\big\{[D_{2b}(t)-D(t)]^2+[D_{2b}(t)^2-D(t)^2]\big\}dt\\
&=\int_{-\infty}^{\infty}\chi_{[-b,b]}(t)\left|\frac{F(t)}{D_{2b}(t)}\right|^2\big\{[2D_{2b}(t)^2-2D_{2b}(t)D(t)]^2\big\}dt\\
&=\int_{-\infty}^{\infty}\chi_{[-b,b]}(t)|F(t)|^2\frac{2}{D_{2b}(t)}[D_{2b}(t)-D(t)]\,dt.
\end{aligned}$$

Next we need the essential estimate

Lemma 2. *The relation*

$$0\leq D_{2b}(t)-D(t)\leq\sum_{k=1}^{\infty}D(bk)\tag{2}$$

holds true.

Proof. Using the additional properties of D we can conclude

$$\begin{aligned}
0 \le D_{2b}(t) - D(t) &= \sum_{k\neq 0} D(t+2bk) \\
&= \sum_{k=1}^{\infty} D(t+2bk) + \sum_{k=1}^{\infty} D(t-2bk) \\
&= \sum_{k=1}^{\infty} D(t+2bk) + \sum_{k=1}^{\infty} D(-t+2bk) \\
&\le \sum_{k=1}^{\infty} D(2bk) + \sum_{k=1}^{\infty} D(-b+2bk). \qquad \square
\end{aligned}$$

Using (2) we finally obtain

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|h(x) - T_b^D(h)(x)\right|^2 dx &\le 2\int_{-\infty}^{\infty} \chi_{[-b,b]}(t)\frac{|F(t)|^2}{D(t)}\left[D_{2b}(t) - D(t)\right] dt \\
&\le 2\sum_{r\ge 1} D(rb)\int_{-\infty}^{\infty} \chi_{[-b,b]}(t)\frac{|F(t)|^2}{D(t)}\, dt \\
&= 2\sum_{r\ge 1} D(rb)\|h\|_D^2. \qquad \square
\end{aligned}$$

Remark 1. For the case of classical cardinal interpolation $D(t) = \chi_{[-b,b]}(t)$ we have

$$T_b(S_b(f)) = S_b(f) \tag{3}$$

and the estimate becomes trivial.

Theorem 7. *Assume $f \in H_D(\mathbb{R})$. Then the estimate*

$$\|f - T_b^D(f)\|_2 \le 4\cdot\sqrt{\sum_{r\ge 1} D(rb)}\,\sqrt{2\pi}\,\|f\|_D$$

is true.

Proof. We have

$$\begin{aligned}
\|f - T_b^D(f)\|_2 &\le \|f - S_b(f)\|_2 + \|S_b(f) - T_b^D(S_b(f))\|_2 + \|T_b^D(S_b(f)) - T_b^D(f)\|_2 \\
&= \|f - S_b(f)\|_2 + \|S_b(f) - T_b^D(S_b(f))\|_2 + \|T_b^D[S_b(f) - f]\|_2.
\end{aligned}$$

Taking into account Theorem 5 we obtain

$$\|f - T_b^D(f)\|_2 \le \|f - S_b(f)\|_2 + \|S_b(f) - T_b^D(S_b(f))\|_2 + \|T_b[S_b(f) - f]\|_2$$

and (3) implies

$$\|f - T_b^D(f)\|_2 \le \|f - S_b(f)\|_2 + \|S_b(f) - T_b^D(S_b(f))\|_2 + \|S_b(f) - T_b(f)\|_2.$$

In view of Theorems 3, 4, and 6 we finally get

$$\begin{aligned}\|f - T_b^D(f)\|_2 \le \sqrt{D(b)}\,\|f - S_b(f)\|_D + 2 \cdot \sqrt{\sum_{r\ge 1} D(rb)}\,\|S_b(f)\|_D \\ + \sqrt{\sum_{r\ge 1} D(rb)}\,\|f - S_b(f)\|_D \\ \le 4 \cdot \sqrt{\sum_{r\ge 1} D(rb)}\,\|f\|_D.\end{aligned}$$

□

Example 1 (Sobolev space). The defining function is given by

$$D(t) = \frac{1}{1 + t^{2r}}$$

and the resulting harmonic Hilbert space is the Sobolev space:

$$H_D(R) = W^r(R).$$

Since

$$\sum_{k\ge 1} D(kb) \le \sum_{k\ge 1} (kb)^{-2r} = b^{-2r} \sum_{k\ge 1} (k)^{-2r},$$

application of Theorem 7 yields

$$\|f - T_b^D(f)\|_2 = O\left(b^{-r}\right).$$

Example 2 (Holomorphic Sobolev space). The defining function is given by

$$D(t) = e^{-\alpha|t|}$$

and the resulting harmonic Hilbert space is the holomorphic Sobolev space:

$$H_D(R) = H^\alpha(R).$$

Since

$$\sum_{k\ge 1} D(kb) \le \sum_{k\ge 1} (e^{-\alpha b})^k = -1 + \frac{1}{1 - e^{-\alpha b}} = \frac{e^{-\alpha b}}{-e^{-\alpha b} + 1}$$

an application of Theorem 7 yields

$$\|f - T_b^D(f)\|_2 = O\left(e^{-\alpha b/2}\right).$$

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