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Interpolation of Rational Functions on a Geometric Mesh

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We discuss the Newton-Gregory interpolation process based on the geometric mesh $1, q, q^2, \dots$, with a quotient $q \in \mathbb{C}$, $|q| < 1$, for rational functions with a single pole $\zeta \in \mathbb{C}$. It is shown that the sequence of interpolating polynomials converges in the disc $\{z \ : \ |z| < |\zeta|\}$.

1. Introduction

Let $p_m(z) = z(z-1)\dots(z-m+1)/m!$, $m > 0$, $p_0(z) \equiv 1$, be the binomial polynomials and

$$\Delta^k f(0) := \sum_{j=0}^{k} (-1)^{k+j} \binom{k}{j} f(j), \quad k = 0, 1, 2, \dots ,$$

be the finite differences of the function f. Then the Newton-Gregory interpolating polynomials

$$N_n(f; z) = \sum_{k=0}^{n} \Delta^k f(0)\, p_k(z)$$

satisfy $N_n(f; k) = f(k), k = 0, \dots , n$.

The problem about convergence of the interpolation process has been completely solved when the interpolated function is an entire function [1, 2]. A rather curious observation about interpolation of simple rational functions was made in [3]. It was shown that, when $f(z)$ is a rational function with a single pole γ that does not coincide with a non-negative integer, the sequence of interpolating polynomials $\{N_n(f; z)\}$ converge to the interpolated function $f(z)$ for every $z \in \mathbb{C}$ with $\Re z > \Re \gamma$. Observe that the convergence holds not only on the real line, where the function is interpolated, but on the right semi-plane of the complex plane determined by the vertical line $\Re\gamma$, no matter where γ is

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located. It is really surprising, especially when the real part of γ is a negative number with large modulus. It is worth mentioning that the proof furnished in [3] implies that the polynomials $N_n(f;z)$ converge not only pointwise but also locally uniformly to $f(z)$ in the semi-plane. This means that the convergence is uniform in every compact subset of $\Re z > \Re\gamma$. Also, the result can be extended to convergence of the interpolation process not only for rational functions but for quotients of Gamma functions. Summarizing, we may state a more general result than the one proved in [3].

Theorem 1. *Let $b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots,$ and*

$$f(z) = \frac{\Gamma(z+c-b)}{\Gamma(z+c)}.$$

Then the sequence $\{N_n(f;z)\}$ converges locally uniformly in $\Re z > \Re(b-c)$.

In this short note we discuss the question of interpolation of rational functions when the interpolation nodes coincide with geometric mesh $1, q, q^2, \dots,$ where the quotient q of the progression is in the unit disc $D = \{z \ : \ |z| < 1\}$. Let $f(z)$ be any function defined at $q^k, k = 0, 1, \dots$. Denote by $N_{n,q}(f;z)$ the polynomial of degree n which interpolates $f(z)$ at $1, q, \dots, q^n$.

Theorem 2. *Let $q \in D$, and $f(z)$ be a rational function with a single pole ζ, where $\zeta \neq q^k$, $k = 0, 1, \dots$. Then the sequence $\{N_{n,q}(f;z)\}$ converges locally uniformly in $D_\zeta = \{z \ : \ |z| < |\zeta|\}$ to $f(z)$, as n goes to infinity.*

It is surprising again that the convergence holds not only in the unit disc D but in the disc D_ζ, no matter how large its radius is, i.e., how far the pole of $f(z)$ is located.

2. Proofs

***Proof of Theorem* 1.** The proof in [3] was based on the Gauss identity about the hypergeometric function, defined by

$$F(a,b;c;z) \equiv {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$, $k > 0$, $(\alpha)_0 = 1$ is the Pochhammer symbol. Gauss [6] (see also [4, p.103]) proved that the hypergeometric series $F(a,b;c;z)$ is absolutely convergent for $|z| = 1$ if $\Re(c-a-b) > 0$, $c \neq 0, -1, \dots,$ and in this case

$$F(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}. \tag{1}$$

Observe that in the terminating case $a = -n, n \in \mathbb{N}$ it reduces to the Chu-Vadredmond formula

$$F(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n}$$

which holds for any value of the parameters b and c, with $c \neq 0, -1, -2, \ldots$.

If we consider $F(-z, b; c; 1)$, by the Gauss theorem, this series converges absolutely to

$$g(z) = \frac{\Gamma(c)\,\Gamma(z+c-b)}{\Gamma(c-b)\,\Gamma(z+c)}$$

when $\Re z > \Re(b-c)$. Thus, the convergence is uniform in every compact subset of this semi-plane.

Denote by $g_n(z)$ the nth partial sum of $F(-z, b; c; 1)$,

$$g_n(z) = \sum_{k=0}^{n} (-1)^k \frac{(b)_k}{(c)_k}\, p_k(z).$$

It remains to prove that $N_n(g; z) \equiv g_n(z)$ for every non-negative integer n. Thus, the theorem will be established if we show that

$$\Delta^k g(0) = (-1)^k \frac{(b)_k}{(c)_k}, \quad k = 0, 1, \ldots, n. \tag{2}$$

In order to this, observe that

$$g(j) = \frac{\Gamma(c)\,\Gamma(j+c-b)}{\Gamma(j+c)\,\Gamma(c-b)} = \frac{(c-b)_j}{(c)_j}.$$

Hence

$$\begin{aligned}
\Delta^k g(0) &= \sum_{j=0}^{k} (-1)^{k+j} \binom{k}{j} \frac{(c-b)_j}{(c)_j} \\
&= (-1)^k \sum_{j=0}^{k} \frac{(c-b)_j}{(c)_j} \frac{(-k)_j}{j!} \\
&= (-1)^k F(-k, c-b; c; 1) \\
&= (-1)^k (b)_k/(c)_k,
\end{aligned}$$

where we used simple properties of the Pochhammer symbols and the Chu-Vandermond formula. Thus, we proved (2), and this completes the proof of Theorem 1. □

Observe that if we set $b = 1$ and $c = 1 - \gamma$, we obtain immediately the uniform convergence of $N_n(f; z)$ to the interpolated function $f(z) = \gamma/(\gamma - z)$, in the compact subsets of $\Re z > \Re\gamma$.

The proof of Theorem 2 uses the so-called q-analogue of Gauss' summation formula that was established by Heine in 1847. We need some definitions results from the book of Gasper and Rahman [5]. Let

$$(a;q)_k = \begin{cases} 1, & k=0 \\ (1-a)(1-aq)\cdots(1-aq^{k-1}), & k\in\mathbb{N}. \end{cases}$$

be the q-shifted factorial. Then the basic hypergeometric series is defined by

$$\phi(a,b;c;q;z) \equiv {}_2\phi_1(a,b;c;q;z) = \sum_{k=0}^{\infty} \frac{(a;q)_k(b;q)_k}{(c;q)_k(q;q)_k} z^k.$$

If $(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$, then obviously

$$(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

Heine [7] (see also [5, (1.5.1)]) proved that

$$\phi(a,b;c;q;c/(ab)) = \frac{(c/a;q)_\infty(c/a;q)_\infty}{(c;q)_\infty(c/(ab);q)_\infty} \quad \text{when} \quad |c/(ab)|<1. \tag{3}$$

It is the q-analogue of Gauss' summation formula (1). In the terminating case $a=q^{-m}$, $m\in\mathbb{N}\cup\{0\}$, Heine's formula reduces to the following q-anlogue of the Chu-Vandermond formula:

$$\phi(q^{-m},b;c;q;cq^m/b) = \frac{(c/b;q)_m}{(c;q)_m}. \tag{4}$$

***Proof of Theorem* 2.** Setting $1/a=z$ and $b=q$ and $c=q/\zeta$ in the left-hand side of Heine's formula and using the the expression for the basic hypergeometric series, we obtain

$$\begin{aligned}\phi(1/z,q;q/\zeta;q;z/\zeta) &= \sum_{k=0}^{\infty} \frac{(1/z;q)_k(q;q)_k}{(q/\zeta;q)_k(q;q)_k}\frac{z^k}{\zeta^k} \\ &= 1+\sum_{k=1}^{\infty}\frac{(z-1)(z-q)\cdots(z-q^{k-1})}{(\zeta-q)(\zeta-q^2)\cdots(\zeta-q^k)}.\end{aligned} \tag{5}$$

It is clear that this series converges absolutely when $q\in D$ and $|z|<|\zeta|$. Hence it converges locally uniformly in D_ζ. On the other hand, Heine's formula (3) implies

$$\phi(1/z,q;q/\zeta;q;z/\zeta) = \frac{(qz/\zeta;q)_\infty(1/\zeta;q)_\infty}{(q/\zeta;q)_\infty(z/\zeta;q)_\infty} \quad \text{for} \quad |z/\zeta|<1.$$

Denote by $h(z)$ the function that appears on the right-hand side of the latter identity. Then

$$\begin{aligned} h(z) &= \prod_{j=0}^{\infty} \frac{(1 - zq^{j+1}/\zeta)(1 - q^j/\zeta)}{(1 - q^{j+1}/\zeta)(1 - zq^j/\zeta)} \\ &= \prod_{j=0}^{\infty} \frac{(\zeta - q^j)(\zeta - zq^{j+1})}{(\zeta - q^{j+1})(\zeta - zq^j)} \\ &= \frac{\zeta - 1}{\zeta - z}. \end{aligned}$$

Let

$$h_n(z) = 1 + \sum_{k=1}^{n} \frac{(z-1)(z-q)\cdots(z-q^{k-1})}{(\zeta - q)(\zeta - q^2)\cdots(\zeta - q^k)}$$

be the nth partial sum of (5).Obvously $h_n(z)$ is algebraic polynomial of degree n. Moreover, (4) implies that $h_n(q^m) = h(q^m)$ for $m = 0, 1, \ldots, n$. Therefore $h_n(z)$ coincides with $N_{n,q}(h; z)$, the Newton-Gregory polynomial that interpolates $h(z)$ at $1, q, \ldots, q^n$. This completes the proof of Theorem 2. $\square$

3. Some Graphs

We provide some graphs which illustrate the results of Theorem 1 and Theorem 2. The error function $R_4(f, z) = |f(z) - N_4(f, z)|$ for the rational function $f(z) = 1/(\gamma - z)$, with the pole $\gamma = i - 1$, is shown in the first two graphs. On the first one $R_4(f, z)$ is shown as a bivariate function, of the real and imaginary part of z. The second one shows the pole γ together with the level curves of $R_4(f, z)$, where, the darker the region, the smaller the value of $R_4(f, z)$ is. It is seen that already for $n = 4$ the error function is small for $\Re z > \Re\gamma = -1$, at least close to the real axes.

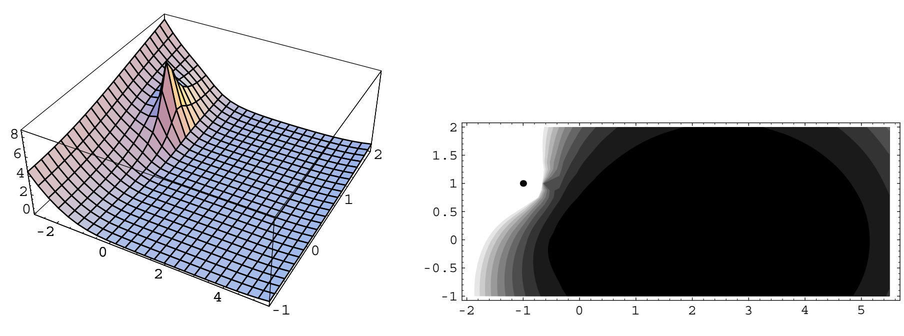

Figure 1. The error function $R_4(f; z)$ and its level curves.

Next figures illustrate the result of Theorem 2. The following two graphs show the error function $R_{6,q}(f;z)$ for the same rational function $f(z)=1/(\gamma-z)$, $\gamma=i-1$. Here the nodes coincide with the geometric progression $1,q,\ldots,q^6$, with quotient $q=(1-i)/2$. Observe that $|q|=1/\sqrt{2}$.

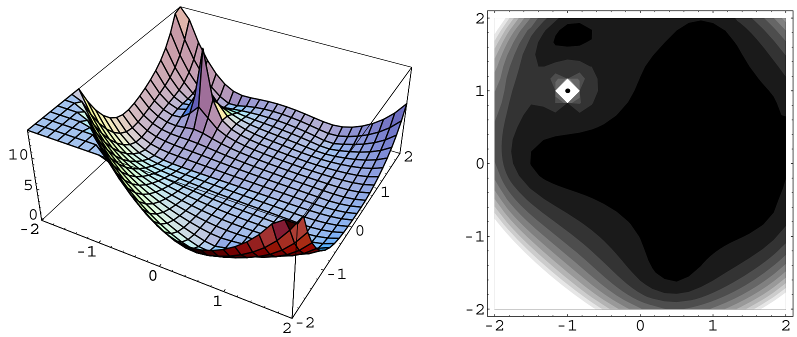

Figure 2. The error function $R_{6,q}(f;z)$ and its level curves, for $q=(1-i)/2$.

In the previous example $|q|=1/\sqrt{2}$. It is natural to expect that when the quotient q of the geometric mesh close to one, the interpolating polynomials would approximate the rational function better than for q close to the origin. The last two graphs of the error function $R_{6,q}(f;z)$ for the interpolation of the same rational function $f(z)=1/(\gamma-z)$, $\gamma=i-1$ but with $q=0.9(i-1)/\sqrt{2}$. This means that q has the same direction as the pole γ and modulus 0.9. It might be of interest to investigate how the speed of convergence of the Newton-Gregory interpolation process of a fixed rational function, based on the geometric mesh with quotient q, depends on q itself. It is pretty natural to expect that this speed will be faster for quotients close to the unit circumference. It is not clear what should be the choice of the direction of q, though we might guess that a natural choice could be such that q has the same argument as the pole γ.

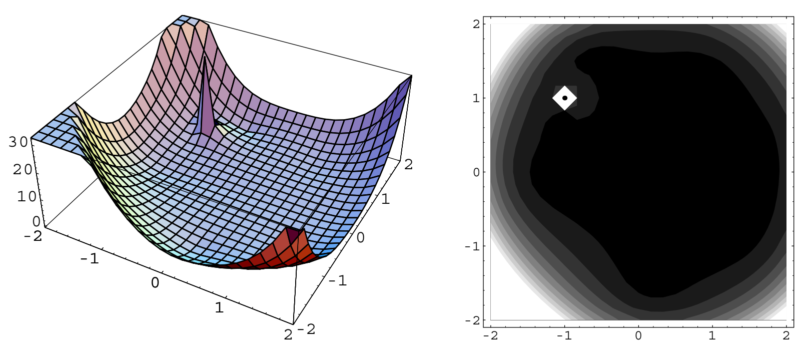

Figure 3. The error function $R_{6,q}(f;z)$ and its level curves, for $q=0.9(i-1)/\sqrt{2}$.

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