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Minimal Points and Contractive Projections

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This is a survey of results on contractive sets and optimal sets. Trivially, every contractive set is optimal but it is not known if every optimal set is contractive. We state results, in the setting of a reflexive, strictly convex, smooth Banach spaces, showing that this is true in many cases. The concept of minimal point is a special case of "Pareto optimal point," studied in economics. We interpret some of these results in an economics setting.

1. Definitions and Results

Let (X, d) be a metric space, and let $M \subseteq X$. We say that x is *minimal* with respect to M if

$$d(y, m) \leq d(x, m) \text{ for all } m \in M \Rightarrow y = x.$$

We let $\min M$ denote the set of all points that are minimal with respect to M.

Example 1. If M consists of two points in Hilbert space, $\min M$ is the line segment between them. More generally, for any set M in Hilbert space,

$$\min M = \overline{\text{conv } M}.$$

We say that M is *optimal* if $\min M = M$. We say that M is *contractive* if there is a projection P (in general non-linear) from X onto M such that

$$d(Px, Py) \leq d(x, y) \text{ for all } x, y \in X.$$

In this study, we often consider Banach spaces B, which are reflexive, strictly convex, and smooth. What are the contractive subsets of B? This problem arises naturally when studying isometric properties of Banach spaces. Classifying the contractive subsets is just a non-linear version of classifying subspaces

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for which there exists a linear projection of norm one. A complete characterization of Banach spaces which are the ranges of contractive linear projections in any space containing them was given by Nachbin [14], Goodner [9], and Kelley [11] for the real case and by Hasumi [10] for the complex case. Ando [1] characterized the subsets of L_p which are ranges of contractive, linear projections. The non-linear version has been studied in Beauzamy-Maurey [4], Beauzamy [3], Davis-Enflo [6], Enflo [7], [8], Lacruz [12], Westphal [15].

For subsets of B we have the obvious implications:

$$\text{contractive} \Rightarrow \text{optimal} \Rightarrow \text{closed and convex}.$$

To what extent are the reverse implications true? In Hilbert space, a closed, convex set is contractive since the "nearest point map" is a contractive projection.

Theorem 1 (Beauzamy-Maurey [4]). *Let B be a reflexive, strictly convex, smooth Banach space of dimension 3 or greater. If the unit ball of B is optimal, then B is a Hilbert space.*

Remark 1. Since l_∞ and $C(0,1)$ are not strictly convex, we cannot apply the Beauzamy-Maurey result. However, in $C(0,1)$ the unit ball is contractive. Just truncate the function $f(x)$ at ± 1, and we have a contractive projection onto the unit ball.

In a reflexive, strictly convex, smooth Banach space it is a long standing open problem if optimal $\Rightarrow$ contractive. In many cases it is true:

Theorem 2 (Beauzamy-Maurey [4]). *In a reflexive, strictly convex, smooth Banach space, an optimal subspace is contractive.*

Theorem 3 (Beauzamy [3]). *In a reflexive, strictly convex, smooth Banach space, an optimal set with interior points is contractive. Moreover, it is the intersection of optimal half-spaces.*

What about subsets of l_p and L_p? Here we also have optimal $\Rightarrow$ contractive. More precisely, we have

Theorem 4 (Davis-Enflo [6]). *For a subset $C \subset l_p$, $1 < p < \infty$, the following assertions are equivalent:*

1. *C is optimal;*
2. *C is contractive;*
3. *C is the intersection of a countable collection of half-spaces of the form $ax_i + bx_j \geq c$.*

The proof of this theorem uses Beauzamy's result. This technique will not work in L_p since there are no contractive half-spaces. Note that

$$\min B(L_p) = \delta_p B(L_p),$$

where $B(L_p)$ is the closed unit ball in L_p. Here $\delta_p > 1$ if $p \neq 2$, and $\delta_p \to 2$ if $p \to 1$ or if $p \to \infty$. In Larsson [13] it is proved that δ_p is continuous, and upper and lower estimates are given. So the only optimal set in L_p with interior points is the whole space. The structure of $\min B(l_p)$ is more complicated than $\min B(L_p)$. In Lacruz [12] it is proved that there exists an $\epsilon > 0$ such that if $1 \leq p < 1 + \epsilon$ then $B(l_p)$ is not convex. Since L_1 is not reflexive, nor strictly convex, nor smooth, we restrict ourselves to convex subsets. On L_1, we have

Theorem 5 (Enflo [7]). *For convex subsets C of $L_1(0,1)$ the following assertions are equivalent:*

1. *C is optimal;*
2. *C is contractive;*
3. *C is the intersection of a countable collection of cones of the form $C_{f_\alpha, T_\alpha, x_\alpha}$, $C_{E_\beta,+,x_\beta}$, $C_{E_\gamma,-,x_\gamma}$.*

Definition of the cones $C_{f,T,0}$. Let $f \in L_1(0,1)$, let E and F be subsets of $[0,1]$ with positive measure such that f has constant sign on E and constant sign on F and such that f and $1/f$ are bounded on $E \cup F$. Let T map subsets of E onto subsets of F such that for any subsets E_1, E_2 of E:

1. $T(E_1 \cup E_2) = T(E_1) \cup T(E_2)$;
2. $m(E_1) > 0 \Rightarrow mT(E_1) > 0$ and $m(E_1) = 0 \Rightarrow mT(E_1) = 0$;
3. T is left continuous. i.e., if E_n is an increasing sequence of sets then

$$T(\bigcup_n E_n) = \bigcup_n T(E_n).$$

$C_{f,T,0}$ consists of all $g \in L_1(0,1)$ such that

$$\operatorname{ess} \inf_{t \in E_1} \frac{g(t)}{f(t)} \leq \operatorname{ess} \inf_{t \in T(E_1)} \frac{g(t)}{f(t)},$$

for all $E_1 \subseteq E$. Set $C_{f,T,x} = x + C_{f,T,0}$. The cone $C_{E,+,0}$ consists of functions in $L_1(0,1)$ that are non-negative on E. Set $C_{E,+,x} = x + C_{E,+,0}$ and similarly define $C_{E,-,0}$.

The same theorem is true in $L_p(0,1)$ but the cones $C_{f,T,0}$ are defined slightly differently.

Theorem 6 (Enflo [8]). *For subsets M of $L_p(0,1)$, $1 < p < \infty$, $p \neq 2$, the following assertions are equivalent:*

1. *M is optimal;*

2. *M is contractive;*

3. *M is the intersection of a countable collection of cones of the form $C_{f_\alpha,T_\alpha,x_\alpha}$, $C_{E_\beta,+,x_\beta}$, $C_{E_\gamma,-,x_\gamma}$.*

Here the cones $C_{f,T,0}$ are defined similarly to the cones in the L_1 case. For $f \in L_p$, consider a measurable subset $E \subset [0,1]$ such that $f > 0$ on E or $f < 0$ on E. Set $F = [0,1]\backslash E$. If g is any real valued function on F define the subset G_g of $\mathbb{R}^2$ as:

$$\{(t,y) : t \in F,\ y \in [0,g(t)]\}.$$

(If $g(t) < 0$ then put $[a,b] = [b,a]$.) Assume that T is a map from subsets of E to G_g where

1. $g(t)$ is defined for almost all $t \in F$;

2. sign $g(t) =$ sign $f(t)$ for almost all $t \in F$;

3. $|g(t)| \leq |f(t)|$ for almost all $t \in F$.

We again assume the following

1. $T(E_1 \cup E_2) = T(E_1) \cup T(E_2)$;

2. $m(E_1) = 0 \Rightarrow mT(E_1) = 0$;

3. T is left continuous, i.e., if E_n is an increasing sequence of sets then

$$T(\bigcup_n E_n) = \bigcup_n T(E_n).$$

We let $g_{T(E_1)}$ denote the function such that $T(E_1) = G_g$. The cone $C_{f,T,0}$ consists of all $h \in L_p(0,1)$ such that

$$\operatorname{ess}\inf_{t\in E_1} \frac{h(t)}{f(t)} \leq \operatorname{ess}\inf_{t\in T(E_1)} \frac{h(t)}{g_{T(E_1)}(t)},$$

for all $E_1 \subseteq E$. For division by zero, we use the convention $\frac{a}{0} = \infty$ $(-\infty)$ if $a > 0$ $(a < 0)$. Set $C_{f,T,x} = x + C_{f,T,0}$. The cone $C_{E,+,0}$ consists of functions in $L_p(0,1)$ that are non-negative on E. Set $C_{E,+,x} = x + C_{E,+,0}$ and similarly define $C_{E,-,0}$.

2. Other Aspects of Minimal Points

In Hilbert space the point

$$s_{n2} = \frac{x + Tx + T^2x + \cdots T^n x}{n},$$

where T is a non-linear map, is the minimal point that minimizes

$$\Phi(z) = \sum_{m=0}^{n-1} \|z - T^m x\|^2.$$

We have the following result by Baillon [2]:

Theorem 7. *Let H be a Hilbert space and C a closed, convex bounded subset of H. Let $T : C \to C$ be a contraction. i.e., $\|Tx - Ty\| \leq \|x - y\|$. Then the points s_n converge weakly to a fixed point of T.*

In L_p or in l_p, one can look at the point s_{np} minimizing

$$\Phi(z) = \sum_{m=0}^{n-1} \|z - T^m x\|^p.$$

Theorem 7 generalizes to l_p in the following way: The points s_{np} converge weakly to a fixed point of T. We do not know if s_{np} converges in L_p but every weak accumulation point of s_{np} is a fixed point of T.

The concept of minimal point is a special case of the concept of *Pareto optimal point* from economics. Let $u_1(x), u_2(x), \ldots, u_N(x)$ be real-valued utility functions on a Banach space B. A point $x \in B$ is Pareto optimal if

$$u_i(y) \geq u_i(x) \text{ for all } i \Rightarrow y = x.$$

The utility functions which give minimal points are negative distance functions. Under very general conditions, we have

Theorem 8. *A Pareto optimal point for a (large) population is near to a Pareto optimal point for a randomly chosen subpopulation.*

Is there an "always isometric" theory?

Example 2. Let M be a subset of Hilbert space H such that there is a projection of Lipschitz norm $1 + \epsilon$ from H onto M. Then trivially, M is $(1 + \epsilon)$ convex in the sense that any two points x and y in M can be joined by an arc of length less than or equal to $(1 + \epsilon)\|x - y\|$. Is the converse true? i.e., if M is $(1 + \epsilon)$ convex is there a projection of Lipschitz norm less than or equal to $1 + \delta(\epsilon)$ from H onto M?

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