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A Characterization of the Existence of Statistical Limit of Measurable Functions

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We introduce the concept of the statistical limit (at ∞) of a measurable function in several variables and recall the concept of the statistical convergence of a multiple sequence. Then we extend a classical theorem of Schoenberg (which characterizes statistical convergence) from single to multiple sequences, and prove an analogous theorem on statistical limit.

1. Introduction and Background

We recall that a sequence $(x_k : k = 1, 2, \dots)$ of complex numbers is said to be *statistically convergent* if there exists a number ξ such that for every $\varepsilon > 0$,

$$\lim_{\ell \to \infty} \ell^{-1} |\{1 \le k \le \ell : |x_k - \xi| \ge \varepsilon\}| = 0, \tag{1}$$

where $|\mathcal{S}|$ means the number of the integers in a subset $\mathcal{S} \subset \mathbb{N}_+ := \{1, 2, \dots\}$. Clearly, ξ in (1) is uniquely determined. In symbol, we shall write

$$\text{st} - \lim_{k \to \infty} x_k = \xi.$$

It is also obvious that the statistical convergence relation possesses the properties of additivity and homogeneity. Furthermore, the ordinary convergence of (x_k) implies statistical convergence; the converse statement is false. Statistical convergence of a sequence implies only statistical boundedness of its terms. In fact, we have

$$\lim_{\ell \to \infty} \ell^{-1} |\{1 \le k \le \ell : |x_k| \ge B\}| = 0,$$

for every $B > |\xi|$ (cf. (1)).

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Example 1. Let $(p_j : j = 1, 2, \dots)$ be an arbitrary nondecreasing sequence of positive integers such that $\lim_{j\to\infty} p_j = \infty$. Define the sequence (x_k) as follows:

$$x_k := \begin{cases} k, & \text{if } k = jp_j \text{ for some } j \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, (x_k) is divergent, but it is statistically convergent to 0.

The term "statistical convergence" first appeared in [1] by Fast, where he attributed this concept to Steinhaus [8], who used the term "asymptotic convergence". In fact, a root of the concept of statistical convergence can be detected in [9, item # 2 on p. 264]; see also [10, Vol. II on pp. 181 and 188], where the term "almost convergence" is used. The intensive study of the concept of statistical convergence was initiated by Fridy [2] in 1985.

The following characterization of statistical convergence is due to Schoenberg [6], who used the term "D-convergence".

Theorem 0. *Let* $(x_k) : \mathbb{N}_+ \to \mathbb{C}$. *For* $\mathrm{st} - \lim_{k\to\infty} x_k = \xi$ *it is necessary and sufficient that for every* $t \in \mathbb{R}$,

$$\lim_{\ell\to\infty} \ell^{-1} \sum_{k=1}^{\ell} e^{itx_k} = e^{it\xi}.$$

In [4] Móricz introduced a non-discrete version of statistical convergence and called it "statistical limit". Let $f(u)$ be a complex-valued function defined and measurable in Lebesgue's sense on some interval (a, ∞), where $a \geq 0$. We say that f has a statistical limit at ∞ if there exists a number ξ such that for every $\varepsilon > 0$,

$$\lim_{b\to\infty} (b-a)^{-1} |\{a \leq u \leq b : |f(u) - \xi| \geq \varepsilon\}| = 0, \tag{2}$$

where this time by $|\{\cdot\}|$ we denote the Lebesgue measure of the set defined in $\{\cdot\}$. Clearly, ξ in (2) is uniquely determined. In symbol, we shall write $\mathrm{st} - \lim_{u\to\infty} f(u) = \xi$. The statistical limit relation also enjoys the properties of additivity and homogeneity. Furthermore, the particular choice of the left endpoint a of the definition domain of f is indifferent in (2). That is, if (2) is satisfied for some $a \geq 0$ and $a_1 > a$, then it is also satisfied for a_1 in place of a; and vice versa. For the sake of simplicity in writing, in the sequel we shall assume that $a = 0$.

Clearly, the ordinary limit of $f(u)$ at ∞ implies the existence of statistical limit; the converse statement is false. Statistical limit implies only statistical boundedness:

$$\lim_{b\to\infty} b^{-1} |\{0 \leq u \leq b : |f(u)| \geq B\}| = 0,$$

where any $B > |\xi|$ will be appropriate (cf. (2)).

2. Main Results

We shall consider a complex-valued function f defined on $\mathbb{R}_+^n$, $\mathbb{R}_+ := [0, \infty)$, which is measurable in the sense of the n-dimensional Lebesgue measure, where $n \geq 1$ is a fixed integer. Motivated by (2), we say that the function $f(\mathbf{u}) := f(u_1, u_2, \dots, u_n)$ has a statistical limit at ∞ if there exists a number ξ such that for every $\varepsilon > 0$,

$$\lim_{\mathbf{b}\to\infty} |\mathbf{b}|^{-1} |\{\mathbf{0} \leq \mathbf{u} \leq \mathbf{b} : |f(\mathbf{u}) - \xi| \geq \varepsilon\}| = 0, \tag{3}$$

where we agree that

$$\mathbf{b} := (b_1, b_2, \dots, b_n) \to \infty \quad \text{means} \quad \min_{1\leq j\leq n} b_j \to \infty, \quad |\mathbf{b}| := b_1 b_2 \dots b_n,$$

and

$$\mathbf{0} \leq \mathbf{u} \leq \mathbf{b} \quad \text{means} \quad 0 \leq u_j \leq b_j \quad \text{for each} \quad j = 1, 2, \dots, n.$$

If this is the case, then we shall write $\text{st} - \lim_{\mathbf{u}\to\infty} f(\mathbf{u}) = \xi$. Clearly, ξ is uniquely determined and this limit relation also possesses the properties of additivity and homogeneity.

We recall that a function f is said to have a limit at ∞ in Pringsheim's sense if there exists a number ξ such that for every $\varepsilon > 0$ there exists a number $v_0 = v_0(\varepsilon) \geq 0$ such that

$$|f(\mathbf{u}) - \xi| < \varepsilon \quad \text{whenever} \quad \min_{1\leq j\leq n} u_j \geq v_0.$$

If $n \geq 2$, then the boundedness of a function f does not follow from the existence of the limit at ∞ in Pringsheim's sense. On the other hand, the statistical boundedness of a function f does follow from the existence of the statistical limit at ∞. In fact, we have

$$\lim_{\mathbf{b}\to\infty} |\mathbf{b}|^{-1} |\{\mathbf{0} \leq \mathbf{u} \leq \mathbf{b} : |f(\mathbf{u})| \geq B\}| = 0$$

for every $B > |\xi|$ (cf. (3)). We note that the concept of the statistical limit of measurable functions in two variables ($n = 2$) was defined and studied in [5].

Now, our main result is formulated in the following

Theorem 1. *Let $f : \mathbb{R}_+^n \to \mathbb{C}$ be a measurable function, where $n \geq 1$ is a fixed integer. For*

$$\text{st} - \lim_{\mathbf{u}\to\infty} f(\mathbf{u}) = \xi \tag{4}$$

to hold it is necessary and sufficient that for every $t \in \mathbb{R}$,

$$\lim_{\mathbf{b}\to\infty} \frac{1}{|\mathbf{b}|} \int_0^{b_1} \dots \int_0^{b_n} e^{itf(\mathbf{u})} du_1 \dots du_n = e^{it\xi}. \tag{5}$$

We also extend Theorem 0 from single to multiple sequences of complex numbers. To this effect, let $\mathbb{N}_+^n$ be the set of n-tuples $\mathbf{k} := (k_1, k_2, \dots, k_n)$ with positive integers for the coordinates k_j, where $n \geq 1$ is a fixed integer. Two tuples $\mathbf{k}$ and $\boldsymbol{\ell} := (\ell_1, \ell_2, \dots, \ell_n)$ are said to be distinct if $k_j \neq \ell_j$ for at least one j. $\mathbb{N}_+^n$ is partially ordered by agreeing that

$$\mathbf{k} \leq \boldsymbol{\ell} \quad \text{means} \quad k_j \leq \ell_j \quad \text{for each} \quad j = 1, 2, \dots, n.$$

We shall consider an n-multiple sequence $(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}_+^n)$ of complex numbers. We say that $(x_{\mathbf{k}})$ is statistically convergent if there exists a number ξ such that for every $\varepsilon > 0$,

$$\lim_{\boldsymbol{\ell} \to \infty} |\boldsymbol{\ell}|^{-1} |\{\mathbf{1} \leq \mathbf{k} \leq \boldsymbol{\ell} : |x_{\mathbf{k}} - \xi| \geq \varepsilon\}| = 0, \tag{6}$$

where

$$|\boldsymbol{\ell}| = |(\ell_1, \ell_2, \dots, \ell_n)| = \ell_1 \ell_2 \dots \ell_n \quad \text{and} \quad \mathbf{1} := (1, 1, \dots, 1).$$

If this is the case, we shall write that $\text{st} - \lim_{\mathbf{k} \to \infty} x_{\mathbf{k}} = \xi$.

We recall that a sequence $(x_{\mathbf{k}})$ is said to be convergent in Pringsheim's sense if there exists a number ξ such that for every $\varepsilon > 0$ there exists a number $v_0 = v_0(\varepsilon) > 0$ such that

$$|x_{\mathbf{k}} - \xi| < \varepsilon \quad \text{whenever} \quad \min_{1 \leq j \leq n} k_j \geq v_0.$$

If $n \geq 2$, then the boundedness of the terms of the sequence $(x_{\mathbf{k}})$ does not follow from the existence of the limit in Pringsheim's sense. On the other hand, the statistical boundedness of the terms of $(x_{\mathbf{k}})$ does follow from the existence of the statistical limit. In fact, we have

$$\lim_{\boldsymbol{\ell} \to \infty} |\boldsymbol{\ell}|^{-1} |\{\mathbf{1} \leq \mathbf{k} \leq \boldsymbol{\ell} : |x_{\mathbf{k}}| \geq B\}| = 0,$$

for every $B > |\xi|$ (cf. (6)).

Now, the extension of Theorem 1 to multiple sequences reads as follows.

Theorem 2. *Let $x_{\mathbf{k}} : \mathbb{N}_+^n \to \mathbb{C}$, where $n \geq 1$ is a fixed integer. For*

$$\text{st} - \lim_{\mathbf{k} \to \infty} x_{\mathbf{k}} = \xi \tag{7}$$

it is necessary and sufficient that for every $t \in \mathbb{R}$,

$$\lim_{\boldsymbol{\ell} \to \infty} \frac{1}{|\boldsymbol{\ell}|} \sum_{\mathbf{1} \leq \mathbf{k} \leq \boldsymbol{\ell}} e^{itx_{\mathbf{k}}} = e^{it\xi}.$$

3. Auxiliary Results

We need the following three lemmas in the proof of Theorem 1.

Lemma 1. *Let $f : \mathbb{R}^n_+ \to \mathbb{C}$ be a measurable function such that* (4) *holds true. If a function $g : \mathbb{C} \to \mathbb{C}$ is continuous at ξ, then*

$$\mathrm{st} - \lim_{\mathbf{u}\to\infty} g(f(\mathbf{u})) = g(\xi). \tag{8}$$

Proof. By the continuity of g at ξ, for every $\eta > 0$ there exists $\varepsilon = \varepsilon(\eta) > 0$ such that

$$|g(z) - g(\xi)| < \eta \quad \text{whenever} \quad |z - \xi| < \eta,$$

whence it follows that

$$|z - \xi| \geq \varepsilon \quad \text{whenever} \quad |g(z) - g(\xi)| \geq \eta.$$

In particular, we have

$$|f(\mathbf{u}) - \xi| \geq \varepsilon \quad \text{whenever} \quad |g(f(\mathbf{u})) - g(\xi)| \geq \eta.$$

By this and (3), we conclude that

$$|\mathbf{b}|^{-1}|\{\mathbf{0} \leq \mathbf{u} \leq \mathbf{b} : |g(f(\mathbf{u})) - g(\xi)| \geq \eta\}|$$
$$\leq |\mathbf{b}|^{-1}|\{\mathbf{0} \leq \mathbf{u} \leq \mathbf{b} : |f(\mathbf{u}) - \xi| \geq \varepsilon\}| \to 0 \quad \text{as} \quad \mathbf{b} \to \infty.$$

Since $\eta > 0$ is arbitrary, this proves (8). □

Lemma 2. *Let $f : \mathbb{R}^n_+ \to \mathbb{C}$ be a measurable function such that* (4) *holds true. If f is bounded on $\mathbb{R}^n_+$, say by $B > 0$, then*

$$\lim_{\mathbf{b}\to\infty} \frac{1}{|\mathbf{b}|} \int_0^{b_1} \cdots \int_0^{b_n} f(\mathbf{u}) du_1 \ldots du_n = \xi. \tag{9}$$

Proof. For a given $\varepsilon > 0$ and $\mathbf{b} \in \mathbb{R}^n_+$, we shall denote by $\mathcal{S}_\varepsilon(\mathbf{b})$ the set defined between the braces $\{\cdot\}$ in (3); that is, let

$$\mathcal{S}_\varepsilon(\mathbf{b}) := \{\mathbf{0} \leq \mathbf{u} \leq \mathbf{b} : |f(\mathbf{u}) - \xi| \geq \varepsilon\}, \tag{10}$$

and let

$$[\mathbf{0}, \mathbf{b}] := \{\mathbf{u} \in \mathbb{R}^n_+ : \mathbf{0} \leq \mathbf{u} \leq \mathbf{b}\}.$$

Clearly, we have $|[\mathbf{0},\mathbf{b}]| = |\mathbf{b}|$ and estimate as follows:

$$\begin{aligned}
&\left| \frac{1}{|\mathbf{b}|} \int_0^{b_1} \cdots \int_0^{b_n} f(\mathbf{u})\, du_1 \dots du_n - \xi \right| \\
&\qquad \le \frac{1}{|\mathbf{b}|} \int_0^{b_1} \cdots \int_0^{b_n} |f(\mathbf{u}) - \xi|\, du_1 \dots du_n \\
&\qquad \le \frac{1}{|\mathbf{b}|} \left\{ \int_{\mathcal{S}_\varepsilon(\mathbf{b})} + \int_{[\mathbf{0},\mathbf{b}]\setminus \mathcal{S}_\varepsilon(\mathbf{b})} \right\} |f(\mathbf{u}) - \xi|\, du_1 \dots du_n \\
&\qquad \le |\mathbf{b}|^{-1}[(B + |\xi|)|\mathcal{S}_\varepsilon(\mathbf{b})| + \varepsilon|\mathbf{b}|] \\
&\qquad \le (B + |\xi|)|\mathbf{b}|^{-1}|\mathcal{S}_\varepsilon(\mathbf{b})| + \varepsilon \\
&\qquad \le 2\varepsilon,
\end{aligned}$$

whenever $\min_{1\le j\le n} b_j$ is large enough, due to (3). Since $\varepsilon > 0$ is arbitrary, this proves (9). □

Following Schoenberg [6], we introduce the following auxiliary function:

$$h(t) := \begin{cases} 0, & \text{if } t < -1 \\ 1 + t, & \text{if } -1 \le t < 0 \\ 1 - t, & \text{if } 0 \le t < 1 \\ 0, & \text{if } t \ge 1. \end{cases}$$

Lemma 3. *For every $t \in \mathbb{R}$,*

$$h(t) := \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{\sin(s/2)}{s/2} \right)^2 e^{its}\, ds. \tag{11}$$

In particular, we have

$$\frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{\sin(s/2)}{s/2} \right)^2 ds = 1. \tag{12}$$

Proof. It is routine to check that the Fourier transform $\hat{h}$ of h is of the following form:

$$\hat{h}(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(t) e^{-ist} dt = \frac{1}{\sqrt{2\pi}} \left(\frac{\sin(s/2)}{s/2} \right)^2, \quad s \in \mathbb{R}. \tag{13}$$

Since both h and $\hat{h}$ are integrable (in Lebesgue's sense) and continuous on the whole real line $\mathbb{R}$, the inversion formula

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{h}(s) e^{its}\, ds, \quad t \in \mathbb{R}, \tag{14}$$

applies (see, e.g., [7, p. 11]). Taking into account (13) and (14) yields (11). □

The next two lemmas are the discrete counterparts of Lemmas 1 and 2, and they are needed in the proof of Theorem 2.

Lemma 4. *Let $x_{\mathbf{k}} : \mathbb{N}_+^n \to \mathbb{C}$ be such that* (7) *holds true. If a function $g : \mathbb{C} \to \mathbb{C}$ is continuous at ξ, then*

$$\mathrm{st} - \lim_{\mathbf{k}\to\infty} g(x_{\mathbf{k}}) = g(\xi).$$

Lemma 5. *If $x_{\mathbf{k}} : \mathbb{N}_+^n \to \mathbb{C}$ is such that* (7) *holds true and the set of its terms is bounded, then*

$$\lim_{\ell\to\infty} \frac{1}{|\ell|} \sum_{\mathbf{1}\le\mathbf{k}\le\ell} x_{\mathbf{k}} = \xi.$$

The proofs of Lemma 4 and 5 follow, with appropriate modifications, those of Lemmas 1 and 2, respectively. We leave them to the reader.

4. Proofs of Theorems 1 and 2

***Proof of Theorem* 1.** *Necessity.* Assume that (4) holds true. By Lemma 1, for every $t \in \mathbb{R}$ we have

$$\mathrm{st} - \lim_{\mathbf{u}\to\infty} e^{itf(\mathbf{u})} = e^{it\xi}.$$

Then the application of Lemma 2 gives (5) for every $t \in \mathbb{R}$.

Sufficiency. Assume the fulfillment of (5) for every $t \in \mathbb{R}$. Both (4) and (5) can be equivalently rewritten in the following forms:

$$\mathrm{st} - \lim_{\mathbf{u}\to\infty} [f(\mathbf{u}) - \xi] = 0$$

and

$$\lim_{\mathbf{b}\to\infty} \frac{1}{|\mathbf{b}|} \int_0^{b_1} \dots \int_0^{b_n} e^{it[f(\mathbf{u})-\xi]} du_1 \dots du_n = 1.$$

Thus, without loss of generality, we may assume that $\xi = 0$ in both (4) and (5).

Let $\varepsilon > 0$ be given. We shall make use of Lemma 3. By (11), an appropriate change of variables gives

$$h\left(\frac{f(\mathbf{u})}{\varepsilon}\right) = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{\sin(\varepsilon s/2)}{\varepsilon s/2}\right)^2 e^{isf(\mathbf{u})}\, ds.$$

By Fubini's theorem, we have

$$\frac{1}{|\mathbf{b}|} \int_0^{b_1} \dots \int_0^{b_n} h\left(\frac{f(\mathbf{u})}{\varepsilon}\right) du_1 \dots du_n$$
$$= \frac{\varepsilon}{2\pi} \int_{\mathbb{R}} \left(\frac{\sin \varepsilon s/2}{\varepsilon s/2}\right)^2 \left\{\frac{1}{|\mathbf{b}|} \int_0^{b_1} \dots \int_0^{b_n} e^{isf(\mathbf{u})} du_1 \dots du_n\right\} ds.$$

Taking into account (5) (with $\xi = 0$) and the fact that

$$\left| \frac{1}{|\mathbf{b}|} \int_0^{b_1} \cdots \int_0^{b_n} e^{isf(\mathbf{u})}\, du_1 \ldots, du_n \right| \leq 1$$

for every $s \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}_+^n$, Lebesgue's dominated convergence theorem applies. As a result, we obtain

$$\lim_{\mathbf{b}\to\infty} \frac{1}{|\mathbf{b}|} \int_0^{b_1} \cdots \int_0^{b_n} h\left(\frac{f(\mathbf{u})}{\varepsilon}\right) du_1 \ldots du_n = \frac{\varepsilon}{2\pi} \int_{\mathbb{R}} \left(\frac{\sin(\varepsilon s/2)}{\varepsilon s/2}\right)^2 dt = h(0) = 1, \quad (15)$$

due to (12).

Consider the set $\mathcal{S}_\varepsilon(\mathbf{b})$ defined in (10) with $\xi = 0$. By the definition of h, we have

$$\int_0^{b_1} \cdots \int_0^{b_n} h\left(\frac{f(\mathbf{u})}{\varepsilon}\right) du_1 \ldots du_n = \int \cdots \int_{[\mathbf{0},\mathbf{b}]\setminus \mathcal{S}_\varepsilon(\mathbf{b})} h\left(\frac{f(\mathbf{u})}{\varepsilon}\right) du_1 \ldots du_n \leq |\mathbf{b}| - |\mathcal{S}_\varepsilon(\mathbf{b}|,$$

where $[\mathbf{0}, \mathbf{b}] := \{\mathbf{u} : \mathbf{0} \leq \mathbf{u} \leq \mathbf{b}\}$. Hence, we conclude that

$$\frac{|\mathcal{S}_\varepsilon(\mathbf{b})|}{|\mathbf{b}|} \leq 1 - \frac{1}{|\mathbf{b}|} \int_0^{b_1} \cdots \int_0^{b_n} h\left(\frac{f(\mathbf{u})}{\varepsilon}\right) du_1 \ldots du_n. \quad (16)$$

Combining (15) and (16) yields

$$\lim_{\mathbf{b}\to\infty} |\mathbf{b}|^{-1} |\mathcal{S}_\varepsilon(\mathbf{b})| = 0,$$

which is (3) in a different notation (cf. (10)). Since $\varepsilon > 0$ is arbitrary, this proves (4) (with $\xi = 0$).

The proof of Theorem 1 is complete. □

***Proof of Theorem* 2.** It goes essentially along the same lines as the proof of Theorem 1, while this time we make use of Lemmas 4 and 5 in place of Lemmas1 and 2. We leave the details to the reader. □

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