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On Convergence Properties of Logarithmic Means of Walsh-Fourier Series

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The one-dimensional (Nörlund) logarithmic mean of the Fourier series of an integrable function f is:

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{n-k}, \quad \text{where } l_n := \sum_{k=1}^{n-1}\frac{1}{k}.$$

The aim of this paper is to give a resume of the recent developments concerning this logarithmic means of Walsh-Fourier series both in one and two-dimensional case.

In the literature, it is known the notion of the Riesz logarithmic means of a Fourier series. The n-th mean of the Fourier series of an integrable function f is defined by

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{k}.$$

This Riesz logarithmic mean with respect to the trigonometric system has been studied by many authors. We mention for instance the papers of Szász, and Yabuta [14, 16]. This mean with respect to the Walsh, Vilenkin system is discussed by Simon, and Gát [13, 1].

Let $\{q_k : k \geq 0\}$ be a sequence of non-negative numbers. The Nörlund means for the Fourier series of f are defined by

$$\frac{1}{Q_n}\sum_{k=1}^{n-1} q_{n-k}S_k(f),$$

where $Q_n := \sum_{k=1}^{n-1} q_k$. If $q_k = \frac{1}{k}$, then we get the (Nörlund) logarithmic means:

$$\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{S_k(f)}{n-k}.$$

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In this paper we call it (this will not cause any misunderstanding) logarithmic means. Although, it is a kind of "reverse" Riesz's logarithmic means. Móricz [10] investigates the approximation properties of some special Nörlund means of Walsh-Fourier series of L^p functions in norm. The case when $q_k = \frac{1}{k}$ is excluded, since the methods of Móricz are not applicable to logarithmic means. The question of Móricz [10] was: Does the norm convergence

$$\frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k(f)}{n-k} \to f$$

hold for every integrable function? The answer is in the negative [3]. On the other hand, in some papers we proved some interesting results, which led far beyond the fact of this "divergence result" above. We give a short résumé of the recent developments concerning the logarithmic means of Walsh-Fourier series both in one and two-dimensional case.

Let $\mathbb{N}$ denote the set of non-negative integers and $I = [0, 1)$ the unit interval. By a dyadic interval in I we mean one of the form $[l2^{-k}, (l+1)2^{-k})$ for some $k \in \mathbb{N}$, $0 \le l < 2^k$. For a given $k \in \mathbb{N}$ and $x \in I$, $I_k(x)$ denotes the dyadic interval of length 2^{-k} which contains the point x. Set $I_k(0) = I_k$ and $I_0(x) = I$.

Let $r_0(x)$ be the function defined by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in I_1 \\ -1, & \text{if } x \in I \backslash I_1, \end{cases} \qquad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad n \ge 1 \text{ and } x \in I.$$

Let $w_0, w_1, \ldots$ represent the Walsh functions, i.e., $w_0(x) = 1$ and if $k = 2^{n_1} + \cdots + 2^{n_s}$ is a positive integer with $n_1 > n_2 > \cdots > n_s \ge 0$, then

$$w_k(x) = r_{n_1}(x) \cdots r_{n_s}(x).$$

The idea of using products of Rademacher's functions to define the Walsh system originated from Paley [11].

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

It is well-known (see e.g. [12]) that

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n, \end{cases} \quad (n \in \mathbb{N}).$$

Suppose that f is a Lebesgue integrable ($f \in L(I)$) function on I and 1-periodic. Then its Walsh-Fourier series is defined by

$$\sum_{k=0}^{\infty} \hat{f}(k) w_k(x),$$

where

$$\hat{f}(k) = \int_0^1 f(t) w_k(t)\, dt$$

is called the k-th Walsh-Fourier coefficient of f.

Let us denote the n-th partial sum of the Walsh-Fourier series of the function f by $S_n(f,x)$. Namely

$$S_n(f,x) = \sum_{k=0}^{n-1} \hat{f}(k) w_k(x).$$

The (Nörlund) logarithmic mean of the Walsh-Fourier series is defined as follows

$$t_n(f,x) = \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k(f,x)}{n-k},$$

where

$$l_n = \sum_{k=1}^{n-1} \frac{1}{k}.$$

It is evident that

$$t_n(f,x) - f(x) = \int_0^1 [f(x \oplus t) - f(x)] F_n(t)\, dt,$$

where

$$F_n(t) = \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{D_k(t)}{n-k}$$

and $\oplus$ denotes dyadic addition [12, 9]. We say that the function $f \in L(I)$ belongs to the logarithm space $L \log^+ L(I)$ if the integral

$$\|f\|_{L \log^+ L} := \int_0^1 |f(x)| \log^+ |f(x)|\, dx$$

is finite.

In the joint paper of Gát and Goginava [3] it is proved that the answer to the question of Móricz is negative. We proved even more, that is, the maximal norm convergence space of the one-dimensional logarithmic means is $L \log^+ L$. Namely, let $\delta : [0, +\infty) \to [0, +\infty)$ be measurable, and $\delta(+\infty) = 0$. Then the following is true.

Theorem 1. *There exists a function $h \in L\log^+ L\delta(L)(I)$ for which $t_n(h)$ does not converge to h in L-norm. On the other hand, the L-norm convergence holds for all functions in $L\log^+ L$.*

It follows that the behavior of the logarithmic means is worse than the behavior of the Fejér (or $(C,1)$) means, and it is as "bad" as the behavior of the partial sums.

In the same paper [3] we proved the following. Denote by $X(I)$ either the space of continuous function on I $(C(I))$ or the space of integrable functions $(L(I))$. Let $f \in X(I)$. The expression

$$\omega(\delta, f)_X = \sup_{|h|\le\delta} \|f(\cdot \oplus h) - f(\cdot)\|_X$$

is called modulus of continuity of the function f with respect to the space X.

Let either $X = C$ or $X = L^1$, and let $f \in X(I)$,

$$\omega(1/2^n, f)_X = o(1/n).$$

Then we have

$$\|t_n(f) - f\|_X \to 0.$$

This is the best we can have. In other words, the following is true.

Theorem 2. *Let $X = C$ or $X = L^1$. Then there exists an $f \in X(I)$, such that*

$$\omega(1/2^n, f)_X = O(1/n),$$

and

$$\|t_n(f) - f\|_X \not\to 0.$$

This means that in the point of view of norm convergence the logarithmic means are closer to the partial sums then to the Fejér or Riesz means (for which the L-norm convergence holds for all integrable functions).

On the other hand, see the paper of Goginava [6].

Theorem 3. *Let $\{m_n : n \ge 1\}$ be a sequence of positive integers. If the inequality*

$$\sum_{n=1}^{\infty} \frac{\log^2(m_n - 2^{[\log m_n]} + 1)}{\log m_n} < \infty$$

is fulfilled, then

$$t_{m_n}(f) \to f \quad a.e. \quad (f \in L).$$

For instance, the sequence $m_n = 2^{n^{3+\delta}} + p_n$, where $p_n < 2^n$ and $\delta > 0$, is of this type.

Goginava and Tkebuchava [8] proved the following theorem.

Theorem 4. *Let either $X = C$ or $X = L$. Then there exists a sequence of positive integers (m_n) and $f \in X$ such that*

$$\lim_{n\to\infty} \|t_{m_n}(f) - f\|_X = 0, \qquad \overline{\lim_{n\to\infty}} \|S_{m_n}(f) - f\|_X > 0.$$

Next, we turn our attention to the two-dimensional case. We consider the double system $\{w_n(x) \times w_m(y) : n, m = 0, 1, 2, ...\}$ on the unit square $I^2 = [0,1) \times [0,1)$.

As usual, we denote by $L(I^2)$ the set of all measurable functions defined on I^2, for which

$$\|f\|_L = \int_0^1 \int_0^1 |f(x,y)|\, dx\, dy < \infty,$$

and by $C(I^2)$ the space of continuous functions on I^2, with the supremum norm

$$\|f\|_C = \sup_{x,y\in I} |f(x,y)| \quad (f \in C(I^2)).$$

Let a be a positive real. We say that the function $f \in L(I^2)$ belongs to the logarithm space $L(\log^+ L)^a(I^2)$ if the integral

$$\|f\|_{L(\log^+ L)^a} := \int_0^1 \int_0^1 |f(x,y)|(\log^+ |f(x,y)|)^a\, dx\, dy$$

is finite. Let $X = X(I^2)$ denote either the space $L(I^2)$, or the space of continuous functions $C(I^2)$. The corresponding norm is denoted by $\|\,.\,\|_X$. The total modulus of continuity, when $X = C$, and the total integrated modulus of continuity, where $X = L$, are defined by

$$\omega(\delta, f)_X = \sup\left\{\|f(x \oplus u, y \oplus v) - f(x,y)\|_X : u^2 + v^2 \le \delta^2\right\}.$$

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{M,N}(f,x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m,n) w_m(x) w_n(y).$$

The logarithmic means of cubical partial sums of double Walsh-Fourier series are defined as follows

$$t_n(f,x,y) = \frac{1}{l_n} \sum_{i=1}^{n-1} \frac{S_{i,i}(f,x,y)}{n-i},$$

where

$$l_n = \sum_{k=1}^{n-1} \frac{1}{k}.$$

In [4] we investigated the role of the modulus of continuity and convergence in norm. Namely, let either $X = C$, or $X = L$, $f \in X(I^2)$, and

$$\omega(1/2^n, f)_X = o(1/n^2).$$

Then we have

$$\|t_n(f) - f\|_X \to 0.$$

In this paper [4] we investigate the sharpness of these results.

Theorem 5. *There exists a function $f \in C(I^2)$ such that*

$$\omega(1/2^n, f)_C = O(1/n^2)$$

and $t_n(f, 0, 0)$ diverges. Besides, there exists a function $g \in L(I^2)$ such that

$$\omega(1/2^n, g)_L = O(1/n^2)$$

and $t_n(g)$ does not converge to g in L-norm.

In [4] Gát and Goginava proved that the L-norm convergence holds for the space $L(\log^+ L)^2(I^2)$ and this cannot be improved.

Theorem 6. *Let $\delta : [0, +\infty) \to [0, +\infty)$ be measurable and $\delta(+\infty) = 0$. Then there exists a two-variable function $h \in L(\log^+ L)^2\delta(L)(I^2)$ such that $t_n(h)$ does not converge to h in $L(I^2)$-norm.*

In [5] Gát and Goginava investigated the two-dimensional logarithmic means of the Walsh-Fourier series which are defined by

$$t_{n,m}(f, x, y) = \frac{1}{l_n l_m} \sum_{k=1}^{n-1} \sum_{l=1}^{m-1} \frac{S_{k,l} f(x, y)}{(n-k)(m-l)}.$$

The situation is the same, not worse than that for the logarithmic means of the cubical partial sums, i.e.,

$$\frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_{k,k} f(x, y)}{n-k}.$$

This is quite surprising from the point of view of the two-dimensional Fejér and Marcinkiewicz means. As it is well-known, the two-dimensional Fejér means are defined by

$$\sigma_{n,m}(f) := \frac{1}{nm} \sum_{j=1}^{n} \sum_{k=1}^{m} S_{j,k} f,$$

and the two-dimensional Marcinkiewicz means by

$$M_n(f) := \frac{1}{n} \sum_{j=1}^{n} S_{j,j} f.$$

For the Marcinkiewicz means it is known [7, 15] that for all integrable bivariate functions $f \in L(I^2)$ we have a.e. the relation $M_n(f) \to f$. On the other hand, the maximal convergence space for the two-dimensional Fejér means in the sense of a.e. convergence is the $L\log^+ L(I^2)$ space. For this result see the paper of the author [2].

So, let $X = C$, or $X = L$. Let $f \in X(I^2)$. The following was proved in [5].

Theorem 7. *$\omega(1/2^n, f)_X = o(1/n^2)$ implies that $\|t_{n,m}(f) - f\|_X \to 0$ as $n, m \to \infty$.*

Existence of a counterexample was shown in [5].

Theorem 8 ([5]). *There exists an $f \in X(I^2)$ such that the modulus of continuity is $O(1/n^2)$ and $t_{n,m}(f)$ does not converge to f in norm (in the Pringsheim sense).*

Moreover, we also proved ([5]) that the maximal convergence space for the norm convergence is $L(\log^+ L)^2$ again. That is, let $\delta : [0, +\infty) \to [0, +\infty)$ be a measurable function with the property $\delta(+\infty) = 0$. Then there exists a two-variable function $h \in L(\log^+ L)^2\delta(L)(I^2)$ such that $t_{n,m}(h)$ does not converge to h in $L(I^2)$-norm.

Finally, we would like to write a couple of words concerning a kind of generalization of the results above. Let $m := (m_k, k \in \mathbb{N})$ ($\mathbb{N} := \{0, 1, \dots\}$) be a sequence of integers and $m_k \geq 2$ ($k \in \mathbb{N}$). Let Z_{m_k} denote the discrete cyclic group of order m_k. The group

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}$$

is called a Vilenkin group. Let $M_0 := 1$, $M_{n+1} := m_n M_n$ ($n \in \mathbb{N}$) be the so-called generalized powers. Then each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, ..., m_i - 1\},\ i \in \mathbb{N}),$$

where only a finite number of n_i's differ from zero. It is well-known in the case of the so-called unbounded Vilenkin groups (in this situation the generating sequence m is an unbounded one) that there exists an integrable function $f \in L(G_m)$ such that the Fejér means $\sigma_{M_n}(f)$ do not converge to f in the Lebesgue norm. But, it is more interesting that despite the fact that the behavior of the Nörlund logarithmic means is worse than the behavior of the Fejér means in general, I. Blahota and the author can give (the proof will be published elsewhere) a class of unbounded generating sequence m such that the norm convergence $t_{M_n}(f) \to f$ holds for all $f \in L(G_m)$. That is why it would be also interesting to investigate the approximation properties of the Nörlund logarithmic means on Vilenkin groups both in one and two-dimensional cases.

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