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About a Result of Tiberiu Popoviciu

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Using a result obtained by Tiberiu Popoviciu on functionals with degree of exactness n, where n is a natural number, we establish new forms for the remainder of some quadrature formulas.

1. Introduction

Let A be a linear functional defined on a subspace S, $S \subset C[a,b]$. We suppose that $\pi \subset S$, where π is the set of all real algebraic polynomial. Denote by e_i, $i = 0, 1, \ldots$, the function defined by

$$e_i(x) =: x^i, \quad x \in [a,b].$$

The functional A has degree of exactness -1 if $A(e_0) \neq 0$. The functional A has degree of exactness n, where n is a natural number, if:

$$A(e_i) = 0, \quad i = 0, 1, \ldots, n, \qquad A(e_{n+1}) \neq 0.$$

We say that the functional A has degree of exactness $+\infty$ if $A(e_i) = 0$, $i = 0, 1, 2, \ldots$. In what follows we will consider linear functionals of the form

$$A(f) = \sum_{i=1}^{p} \sum_{i=0}^{k_i - 1} a_{i,j}\, f^{(i)}(z_i) \tag{1}$$

where $p, k_1, k_2, \ldots, k_p$ are natural numbers, z_i, $i = 1, 2, \ldots, p$, are p distinct points in the interval $[a,b]$, $a_{i,j}$, $j = 0, 1, \ldots, k_i - 1$, $i = 1, 2, \ldots, p$, are constants independent of the function f.

Let us set

$$m := k_1 + \cdots + k_p.$$

In [3] Popoviciu proved the following results:

Theorem 1 (T.Popoviciu). *The degree of exactness of the functional* (1) *is at most* $m - 2$.

If the functional A has degree of exactness n, $n \leq m-2$, then

$$A(f) = \sum_{i=n+1}^{m-1} a_i[x_1, x_2, \ldots, x_{i+1}; f] \tag{2}$$

where

$$x_{k_1+k_2+\cdots+k_{i-1}+r} = z_i,$$

$r = 1, 2, \ldots, k_i$, $i = 1, 2, \ldots, p$ $(k_0 = 0)$, and $[x_1, x_2, \ldots, x_m; f]$ is the divided difference of the function f at the points $x_1, x_2, \ldots, x_m$ and the coefficients a_i, $i = n+1, \ldots, m-1$, do not dependent on the function f.

If the functional A has degree of exactness $m-2$, then from (2) we obtain

$$A(f) = a_{m-1}[\underbrace{z_1, \ldots, z_1}_{k_1}, \underbrace{z_2, \ldots, z_2}_{k_2}, \ldots, \underbrace{z_p, \ldots, z_p}_{k_p}; f] \tag{3}$$

where $a_{m-1} = A(e_{m-1})$.

In the following we denote by $[z_1^{(k_1)}, \ldots, z_p^{(k_p)}; f]$ the divided difference $[\underbrace{z_1, \ldots, z_1}_{k_1}, \ldots, \underbrace{z_p, \ldots, z_p}_{k_p}; f]$. Then equality (3) can be rewritten as

$$A(f) = A(e_{m-1})[z_1^{(k_1)}, \ldots, z_p^{(k_p)}; f].$$

In this paper, using the results from Theorem 1 we establish new forms for the remainder of some classical quadrature formulas.

2. Gauss Quadrature Formula

Let us consider the Gauss quadrature formula

$$\int_{-1}^{1} f(x)\,dx = \sum_{k=1}^{n} A_k f(x_k) + R(f) \tag{4}$$

where x_k, $k = 1, 2, \ldots, n$, are the zeros of the Legendre polynomial P_n.

In this case we have

$$R(e_i) = 0, \quad i = 0, 1, \ldots, 2n-1.$$

Theorem 2. *If $f : [-1,1] \longrightarrow \mathbb{R}$ is absolutely continuous on $[-1,1]$, then the Gauss quadrature formula (4) can be written in the form*

$$\int_{-1}^{1} f(t)\,dt = \sum_{k=1}^{n} A_k f(x_k) + k[-1, x_1^{(2)}, \ldots, x_n^{(2)}, 1; F], \tag{5}$$

where

$$F(x) = \int_{-1}^{x} f(t)\,dt$$

and

$$K = \frac{2^{2n+1}}{\binom{2n}{n}^2}.$$

Proof. Let us consider the linear functional

$$A(f) = F(1) - F(-1) - \sum_{k=1}^{n} A_k F'(x_k). \tag{6}$$

We have

$$A(e_i) = 1 - (-1)^i - i \sum_{k=1}^{n} A_k x_k^{i-1}. \tag{7}$$

But

$$\sum_{k=1}^{n} A_k x_k^{i-1} = \frac{1-(-1)^i}{i}, \quad i = 1, \dots, 2n, \tag{8}$$

and

$$A(e_0) = 0. \tag{9}$$

From (7), (8) and (9) we deduce that the functional A has degree of exactness $2n$.

The number m is equal to $2n+2$. By Theorem 1 we get

$$A(F) = A(e_{2n+1})[-1, x_1^{(2)}, \dots, x_n^{(2)}, 1; F].$$

On the other hand, we have

$$A(e_{2n+1}) = A(P) \tag{10}$$

where $P(x) = (x+1)(x-x_1)^2 \dots (x-x_n)^2$. From (6) and (10) we get

$$A(e_{2n+1}) = P(1) = \frac{2^{2n+1}}{\binom{2n}{n}^2}.$$

Now, for $F(x) = \int\limits_{-1}^{x} f(t)\,dt$ we obtain equality (5). □

Remark 1. Theorem 2 was proved by Popoviciu [3] and by Lupas and Constantinescu [1] in a different way.

3. Lobatto's Quadrature Formula

If $x_i,\ i = 1,\dots,n$, are the zeros of the Jacobi polynomial $P_n^{(1,1)}$ of degree n, then the interpolatory quadrature formula

$$\int_{-1}^{1} f(x)\,dx = \frac{2}{(n+1)(n+2)}[f(-1)+f(1)] + \sum_{i=1}^{n} A_i f(x_i) + R(f) \tag{11}$$

is called Lobotto's quadrature formula [2]. It is well-known that

$$R(e_i) = 0, \quad i = 0,1,\dots,2n+1. \tag{12}$$

A representation of the remainder of the quadrature formula (11) is given in the following theorem.

Theorem 3. *If $f : [-1,1] \longrightarrow \mathbb{R}$ is absolutely continuous on $[-1,1]$ then the remainder $R(f)$ of Lobatto's quadrature (11) can be written as*

$$F(f) = K[-1,-1,x_1^{(2)},\dots,x_n^{(2)},1,1;F]$$

where

$$F(x) = \int_{-1}^{x} f(t)\,dt$$

and

$$K = -\frac{2^{2n+3}\,n!\,[(n+1)!]^2(n+2)!}{[(2n+2)!]^2}.$$

Proof. We consider the following linear functional

$$A(F) = F(1) - F(-1) - \alpha\big(F'(1)+F'(-1)\big) - \sum_{i=1}^{n} A_i F'(x_i) \tag{13}$$

where $\alpha = \frac{2}{(n+1)(n+2)}$. We have

$$A(e_0) = 0, \tag{14}$$

and from (12) we obtain

$$A(e_i) = 0, \quad i = 1,2,\dots,2n+2. \tag{15}$$

From (14) and (15) we conclude that the degree of exactness of the functional A is $2n+2$ and the number m from Theorem 1 is $2n+4$. Applying the result from the theorem of T. Popoviciu we have the following representation:

$$A(F) = A(e_{n+3})[-1,-1,x_1^{(2)},\dots,x_n^{(2)},1,1;F]. \tag{16}$$

On the other hand, we have

$$A(e_{n+3}) = A(Q)$$

where $Q = (x-1)(x+1)^2(x-x_1)^2 \dots (x-x_n)^2$. From (13) we get

$$A(Q) = -\alpha\, 2^2(1-x_1)^2 \dots (1-x_n)^2. \tag{17}$$

Since

$$(1-x_1)^2 \dots (1-x_n)^2 = \frac{2^{2n}\, n!\, [(n+1)!]^2}{[(2n+2)!]^2}, \tag{18}$$

from (18), (17) and (16) we obtain

$$A(F) = -\frac{2^{2n+3}\, n!\, [(n+1)!]^2 (n+2)!}{[(2n+2)!]^2}. \tag{19}$$

Finally we choose $F(x) = \int_{-1}^{x} f(t)\, dt$ in (19) and the proof of the theorem is completed. □

4. Turan's Quadrature Formula

We consider the interpolatory quadrature formula

$$\int_{-1}^{1} f(x)\, dx = \sum_{k=0}^{2s} \sum_{i=0}^{n} A_{k,i} f^{(k)}(x_i) + R(f) \tag{20}$$

where $f \in C^{2s}_{[-1,1]}$.

The linear functional R in (20) has degree of exactness at most $(2s+2)n$ (see, for example, [2]).

If $x_i,\ i = 1, 2, \dots, n$, are the zeros of the s-orthogonal polynomial $P_{s,n}$ ([2]), then R has degree of exactness $n(2s+2)$. In this case the quadrature formula is called Turan's quadrature formula.

Theorem 4. *Let $f \in C^{2s}[-1,1]$. Turan's quadrature formula can be written in the following form:*

$$\int_{-1}^{1} f(x)\, dx = \sum_{k=0}^{2s} \sum_{i=1}^{n} A_{k,i} f^{(k)}(x_i) + K[-1, x_1^{(2s+2)}, \dots, x_n^{(2s+2)}, 1; F]$$

where

$$F(x) = \int_{-1}^{x} f(t)\, dt$$

and

$$K = 2(1-x_1)^{2s+2} \dots (1-x_n)^{2s+2}.$$

Proof. We consider the linear functional A defined by

$$A(F) = F(1) - F(-1) - \sum_{k=0}^{2s} \sum_{i=0}^{n} A_{k,i} F^{(k+1)}(x_i). \tag{21}$$

We have

$$A(e_0) = 0, \qquad A(e_i) = 0, \quad i = 1, 2, \ldots, (2s+2)n.$$

It follows that the degree of exactness of the functional A defined by (21) is $(2s+2)n$. We note that

$$m = (2s+2)n + 2.$$

From Theorem 1 we have

$$A(F) = A(e_{(2s+2)n+1})[-1, x_1^{(2s+2)}, \ldots, x_n^{(2s+2)}, 1; F]. \tag{22}$$

On the other hand,

$$A(e_{(2s+2)n+1}) = A(T) \tag{23}$$

where

$$T(x) = (x+1)(x-x_1)^{(2s+2)} \ldots (x-x_n)^{(2s+2)}.$$

From (21) we get

$$A(T) = 2(1-x_1)^{(2s+2)} \ldots (1-x_n)^{(2s+2)}. \tag{24}$$

We choose $F(x) = \int_{-1}^{x} f(t)\,dt$ in (22) and our assertion follows from (23) and (24). □

References

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