

On Recovering of a Bivariate Polynomial from Its Radon Projections

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For a given set of angles $\theta_0 < \dots < \theta_n$ in $[0, \pi)$, we give explicitly a set of $k + 1$ parallel X -ray beams in each direction θ_k , $k = 0, \dots, n$, which can be used to reconstruct a bivariate polynomial in Π_n^2 uniquely from a set of $\binom{n+2}{2}$ Radon projections in the above directions. Also we give an outline of algorithm which can be used to reconstruct the polynomial. Some numerical results are given too.

1. Introduction

In many practical problems the information of a bivariate function comes from the values of its integrals along a finite set of line segments, for example in tomography and electronic microscopy. This motivates many authors to study the problem of approximating bivariate functions using this type of information. See, for example, [4], [5], [7], [9], [10], [12], and the bibliography therein.

In [10], Marr proved that a bivariate polynomial of total degree at most n can be reconstructed by using its integrals along a finite number of chords joining equidistant points on the unit circumference.

A particular case of a general multivariate result, given by Hakopian in [6] (see also [4]), shows that a bivariate polynomial of total degree $\leq n$ can be recovered uniquely from its integrals over the set of $\binom{n+2}{2}$ chords joining any given $n + 2$ distinct points on the boundary of the unit disk. Another result, based on integrals over $(n + 1)(n + 2)/2$ chords taken in $n + 1$ distinct directions was given in [1]. Recently the problem of recovering a bivariate polynomial of total degree $\leq n$ from its Radon projections in $2\lfloor \frac{n+1}{2} \rfloor + 1$ equally spaced directions with $\lfloor \frac{n}{2} \rfloor + 1$ line segments in each direction was studied in [3]. It was shown that the problem has a unique solution if these line segments are associated with the zeros of the Chebyshev polynomial of a certain degree. Here we show that the zeros of the Chebyshev polynomial of second kind will

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ensure the uniqueness of the solution of the interpolation problem considered in [1]. First we need some preliminaries.

We shall denote by Π_n^2 the set of all real algebraic polynomials in two variables of total degree $\leq n$. If $P \in \Pi_n^2$, then

$$P(x, y) = \sum_{i+j \leq n} \alpha_{ij} x^i y^j, \quad \alpha_{ij} \in \mathbb{R}.$$

It is known that the dimension of Π_n^2 is $\binom{n+2}{2}$. Let $\mathbf{B} := \{x := (x, y) : \|x\| \leq 1\}$ be the unit disk in the plane \mathbb{R}^2 , where $\|x\|^2 = x^2 + y^2$. For a given real valued function f defined in \mathbb{R}^2 we shall assume that the integrals of f are known along all line segments on the unit disk \mathbf{B} . Given $t \in [-1, 1]$ and an angle $\theta \in [0, \pi)$, measured counterclockwise from the positive x axis, the equation

$$\ell(x, y) := x \cos \theta + y \sin \theta - t = 0$$

defines the line ℓ which passes through the point $(t \cos \theta, t \sin \theta)$ and is perpendicular to the vector $\langle \cos \theta, \sin \theta \rangle$. Let $I(\theta, t)$ be the set of points of intersection of the line ℓ and the unit disk \mathbf{B} , i.e.,

$$I(\theta, t) := \ell \cap \mathbf{B}, \quad \theta \in [0, \pi), \quad t \in [-1, 1].$$

It is easy to see that the points (x, y) on $I(\theta, t)$ are given by

$$x = t \cos \theta - s \sin \theta, \quad y = t \sin \theta + s \cos \theta$$

where $|s| \leq \sqrt{1 - t^2}$.

For a given $t \in [-1, 1]$ and $0 \leq \theta < \pi$, the *Radon projection* of a function f along the line segment $I(\theta, t)$ is given by

$$\mathcal{R}_\theta(f; t) := \int_{I(\theta, t)} f(x) dx = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t \cos \theta - s \sin \theta, t \sin \theta + s \cos \theta) ds.$$

It is also called an *X-ray*. Since $I(\theta, t) \equiv I(\theta + \pi, -t)$, we have

$$\mathcal{R}_\theta(f; t) \equiv \mathcal{R}_{\theta+\pi}(f; -t).$$

Thus, we assume that $0 \leq \theta < \pi$. It is known that the set of Radon projections

$$\{\mathcal{R}_\theta(f; t) : -1 \leq t \leq 1, 0 \leq \theta < \pi\}$$

determines f uniquely (see [11], [8]). A more recent result in [12] shows that any $f \in L^1(\mathbb{R}^2)$ with compact support in \mathbf{B} is uniquely determined by any infinite set of *X-rays*.

Since the function $f \equiv 0$ has all its projections equal to zero, it follows that the only function which has the zero Radon transform is the zero function.

In fact (see [2]), if $P \in \Pi_n^2$ then for each fixed θ there exists a univariate polynomial p of degree at most n such that

$$\mathcal{R}_\theta(P; t) = \sqrt{1 - t^2} p(t), \quad -1 \leq t \leq 1.$$

Moreover, every algebraic polynomial $P \in \Pi_n^2$ is uniquely determined by only a finite number of projections [9]:

Theorem A. *Let $\theta_0 < \dots < \theta_n$ be any given angles in $[0, \pi)$. Then the projections*

$$\mathcal{R}_{\theta_k}(P; t), \quad -1 \leq t \leq 1, \quad k = 0, \dots, n,$$

determine P uniquely.

The problem of recovering a bivariate polynomial $P \in \Pi_n^2$ by using a finite number of X -rays, matching the dimension of Π_n^2 , was considered in [1]. The X -rays were taken along a finite set of chords $J = \{I(\theta, t)\}$ where the cardinality of J is equal to the dimension of Π_n^2 . The set J was partitioned into $n + 1$ subgroups, such that the k th subgroup consists of $k + 1$ parallel line segments in the unit ball B .

The problem was described by a set of given angles $\theta_0 < \dots < \theta_n$ in $[0, \pi)$ and a triangular matrix $T = \{t_{ki}\}$ of points $t_{kk} < \dots < t_{kn}$, $k = 0, \dots, n$, associated with these angles.

Here we consider the problem of characterizing a location of the nodes $\{t_{kj}\}$ for which the interpolation of the data $\{\mathcal{R}_{\theta_k}(\cdot; t_{kj})\}$ by polynomials of degree at most n is poised (i.e., unisolvent).

Let

$$U_m(t) := \frac{\sin(m + 1)\theta}{\sin \theta}, \quad t = \cos \theta, \tag{1}$$

be the Chebyshev polynomial of the second kind of degree m . Consider the matrices

$$\mathbf{U}_k := \begin{pmatrix} U_k(t_{kk}) & U_{k+1}(t_{kk}) & \dots & U_n(t_{kk}) \\ U_k(t_{k,k+1}) & U_{k+1}(t_{k,k+1}) & \dots & U_n(t_{k,k+1}) \\ \dots & \dots & \dots & \dots \\ U_k(t_{kn}) & U_{k+1}(t_{kn}) & \dots & U_n(t_{kn}) \end{pmatrix}.$$

The following result was proved in [1].

Theorem B. *For given angles $0 \leq \theta_0 < \dots < \theta_n < \pi$ and associated points $T = \{t_{ki}\}_{k=0, i=k}^n$, the interpolation problem*

$$\int_{I(\theta_k, t_{ki})} P(x) dx = \gamma_{ki}, \quad k = 0, \dots, n, \quad i = k, \dots, n, \quad P \in \Pi_n^2, \tag{2}$$

is poised if and only if

$$\det \mathbf{U}_k \neq 0 \quad \text{for } k = 1, \dots, n.$$

The matrices U_k are not invertible for all choices of $T = \{t_{ki}\}_{k=0, i=k}^n$. Indeed, for $k = n$ we have $\det U_k = U_n(t_{nn}) = 0$ if t_{nn} is a zero of U_n .

However for each fixed k the determinant of U_k can be considered as a polynomial in the span $\{U_k, \dots, U_n\}$, thus it is different than zero for almost all choices of $t_{ki} \in \mathbb{R}$. But given a set of points $\{t_{ki}\}$ it is often difficult to determine if it gives a unique solution of the problem (2).

2. Main Result

For a given positive integer n let $\alpha = \frac{\pi}{n+2}$, and let $\eta_i := \cos \frac{i\pi}{n+2} = \cos i\alpha$, $i = 1, \dots, n + 1$, be the zeros of the polynomial $U_{n+1}(x)$. We have:

Theorem 1. *Suppose that $t_{ki} = \eta_{i+1}$ for $i = k, \dots, n$, $k = 0, \dots, n$. Then*

$$\det U_k \neq 0 \quad \text{for } k = 1, \dots, n$$

and consequently the interpolation problem (2) is poised.

Proof. In view of (1), by direct evaluation, it follows that

$$U_{n-j}(t_{k,n-l}) = (-1)^{n-l-j} \frac{\sin(j+1)(l+1)\alpha}{\sin(n-l+1)\alpha},$$

for $l = 0, \dots, n - k$, $j = 0, \dots, n$. From the rows of $\det U_k$ we factor out $1/\sin(n-l+1)\alpha$ for each $l = 0, \dots, n - k$, then what is left is a symmetric determinant. If n is even we factor out (-1) from the even rows of the resulting determinant to get that all elements in each column have the same sign, thus we factor out (-1) from its even columns. This will result in a factor of $\pm 1 =: \varepsilon$. (If n is odd we factor out (-1) from the odd rows and we get the same result).

Reversing the order of the rows and the order of the columns of U_k will not affect the value of its determinant. Put $m := n - k + 1$, and let

$$U_k^* := \begin{pmatrix} \sin \alpha & \sin 2\alpha & \dots & \sin m\alpha \\ \sin 2\alpha & \sin 4\alpha & \dots & \sin 2m\alpha \\ \dots & \dots & \dots & \dots \\ \sin m\alpha & \sin 2m\alpha & \dots & \sin m^2\alpha \end{pmatrix},$$

then we have

$$\det U_k = \frac{\varepsilon}{\prod_{j=1}^m \sin(n-j+2)\alpha} \det U_k^*.$$

We shall show that

$$\det U_k^* = 2^{m(m-1)/2} \prod_{j=1}^m \sin(j\alpha) \det V_k,$$

where

$$V_k := \begin{pmatrix} 1 & \cos \alpha & \cdots & \cos^{m-1} \alpha \\ 1 & \cos 2\alpha & \cdots & \cos^{m-1} 2\alpha \\ \dots & \dots & \dots & \dots \\ 1 & \cos m\alpha & \cdots & \cos^{m-1} m\alpha \end{pmatrix}.$$

Indeed, by the formula: $e^{i\phi} = \cos \phi + i \sin \phi$, it follows that

$$\cos n\phi + i \sin n\phi = (\cos \phi + i \sin \phi)^n = \sum_{\mu=0}^n \binom{n}{\mu} i^\mu \sin^\mu \phi \cos^{n-\mu} \phi. \quad (3)$$

Comparing the imaginary parts in both sides of (3) we get:

$$\begin{aligned} \sin 2m\phi &= \sin \phi \sum_{j=1}^m \beta_j \cos^{2j-1} \phi, \\ \sin(2m+1)\phi &= \sin \phi \sum_{j=0}^m \alpha_j \cos^{2j} \phi, \end{aligned}$$

where α_i, β_i are constants. Hence for the general element in U_k^* we have

$$u_{l,2j}^* = \sin l\alpha \sum_{\nu=1}^j a_\nu \cos^{2\nu-1} l\alpha, \quad (4)$$

$$u_{l,2j-1}^* = \sin l\alpha \sum_{\nu=0}^j b_\nu \cos^{2\nu} l\alpha, \quad (5)$$

for some constants a_ν, b_ν . Let us mention that the leading coefficients in (4), (5) are given by $a_j = 2^{2j-1}, b_j = 2^{2j}$, this follows from the fact that a_j, b_j are, respectively, the leading coefficients of the Chebyshev polynomials U_{2j-1}, U_{2j} .

Now for each $r = 1, \dots, m$ we factor out $\sin r\alpha$ from the r th row of $\det U_k^*$, thus we have

$$\det U_k^* = \prod_{r=1}^m \sin(r\alpha) \det U'_k,$$

where U'_k is the matrix with general element $u'_{i,j}$ given by

$$u'_{l,2j} = \sum_{\nu=1}^j a_\nu \cos^{2\nu-1} l\alpha, \quad (6)$$

$$u'_{l,2j-1} = \sum_{\nu=0}^j b_\nu \cos^{2\nu} l\alpha. \quad (7)$$

Thus, for each $k, u'_{l,k}$ is a polynomial of degree $k - 1$ in $\cos l\alpha$, which is even or odd depending on the parity of k .

For a fixed j we introduce the notation $C_{l,j} := [u'_{l,j}]$, $l = 1, \dots, m$, to denote the j th column of U'_k . In view of (6), (7) we have $C'_{l,1} := C_{l,1} = [1]$, and $C_{l,2} = [a_1 \cos l\alpha]$. Factoring out a_1 from $C_{l,2}$, the second column of U'_k becomes: $C'_{l,2} = [\cos l\alpha]$. Since $C_{l,3} = [b_2 \cos^2 l\alpha + b_1]$, we subtract $b_1 \times C'_{l,1}$ from $C_{l,3}$, then we factor out the coefficient b_2 , we find that the third column of U'_k becomes $C'_{l,3} = [\cos^2 l\alpha]$.

Continuing in this way, by subtracting from each odd column the multiples of the preceding odd columns and subtracting from each even column the multiples of the preceding even columns then factoring out the leading coefficients a_j, b_j we reduce all the columns of U'_k to the form $C'_{l,\mu} = [\cos^{\mu-1} l\alpha]$, $\mu = 1, \dots, m$, $l = 1, \dots, m$.

Since adding multiples of columns of a determinant to another column of the same determinant will not affect its value, we obtain

$$\begin{aligned} \det U'_k &= \prod_{j=1}^{[m/2]} a_j \prod_{j=0}^{[(m-1)/2]} b_j \det V_k \\ &= 2^{1+2+\dots+(m-1)} \det V_k = 2^{m(m-1)/2} \det V_k. \end{aligned}$$

Thus, from the above steps we finally get that

$$\det U_k = \varepsilon 2^{m(m-1)/2} \prod_{j=1}^m \frac{\sin j\alpha}{\sin(n-j+2)\alpha} \det V_k.$$

It is evident that $\det V_k$ is a Vandermonde determinant. Thus

$$\det V_k = \prod_{1 \leq l < j \leq m} (\cos j\alpha - \cos l\alpha) = \prod_{1 \leq l < j \leq m} 2 \sin \frac{(j+l)\alpha}{2} \sin \frac{(j-l)\alpha}{2}.$$

Since $j+l < 2m = 2(n-k+1) < 2(n+2)$, then $0 < \frac{(j+l)\alpha}{2}, \frac{(j-l)\alpha}{2} < \pi$, hence $\det V_k \neq 0$. This ends the proof. \square

3. An Algorithm and Numerical Examples

Now we present an algorithm that can be used to reconstruct the interpolating polynomial.

Every polynomial $P \in \Pi_n^2$ has the form

$$P(x, y) = \sum_{p+q \leq n} a_{pq} x^p y^q. \quad (8)$$

Step 1. Compute the Radon projections of the monomials $x^p y^q$:

$$\int_{I(\theta_k, t_{ki})} x^p y^q dx dy = \int_{-\sqrt{1-\eta_{i+1}}}^{\sqrt{1-\eta_{i+1}}} (\eta_{i+1} \cos \theta_k - s \sin \theta_k)^p (\eta_{i+1} \sin \theta_k + s \cos \theta_k)^q ds.$$

Step 2. Solve the following linear system for the coefficients $\{a_{pq}\}$:

$$\sum_{p+q \leq n} a_{pq} \int_{I(\theta_k, t_{ki})} x^p y^q dx dy = \gamma_{ki}, \quad \text{for } i = k, \dots, n, k = 0, \dots, n.$$

Step 3. Substitute the values of the coefficients $\{a_{pq}\}$ in (8).

We apply the above algorithm for two functions that are usually used for testing new methods for recovering bivariate functions: namely, to the so called “Mexican hat”:

$$z = \frac{\sin 3\pi \sqrt{x^2 + y^2 + 10^{-18}}}{3\pi \sqrt{x^2 + y^2 + 10^{-18}}}, \quad x^2 + y^2 \leq 1$$

and to the cone: $z = \sqrt{x^2 + y^2}, x^2 + y^2 \leq 1$.

We choose $\theta_i = (i + 1)\pi/(n + 2), i = 0, \dots, n$. The numerical results are illustrated in the figures below.

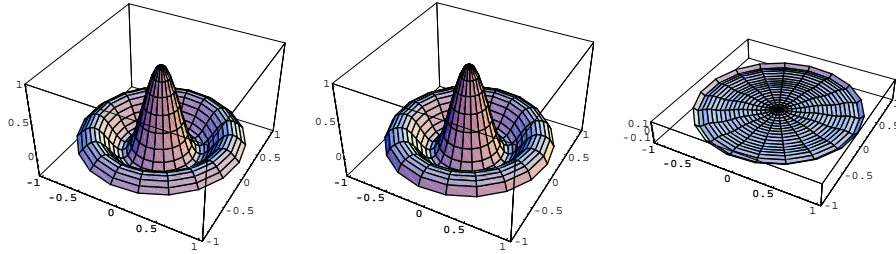


Figure 1. The Mexican hat, its interpolating polynomial of degree 10 and the error.

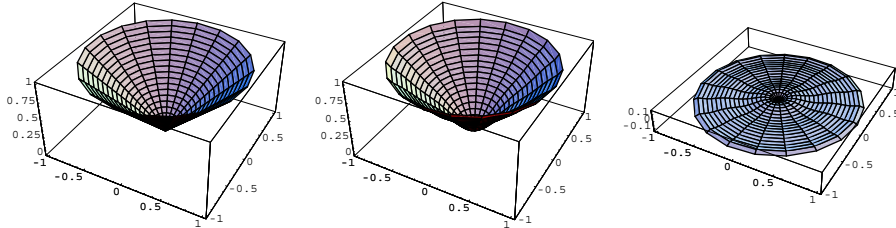


Figure 2. The cone, its interpolating polynomial of degree 10 and the error.

The uniform and the L_2 norms of the error are given in the following tables.

n	Mexican hat	Cone
8	0.136998	0.0702261
10	0.0122434	0.0582879

n	Mexican hat	Cone
8	0.0449385	0.0225118
10	0.0057766	0.0169353

References

- [1] B. BOJANOV AND I. K. GEORGIEVA, Interpolation by bivariate polynomials based on Radon projections, *Studia Math.* **162** (2004), 141–160.
- [2] B. BOJANOV AND G. PETROVA, Uniqueness of the Gaussian cubature for a ball, *J. Approx. Theory* **104** (2000), 21–44.
- [3] B. BOJANOV AND Y. XU, Reconstruction of a bivariate polynomials from its Radon projections, *SIAM J. Math. Anal.* **37** (2005), 238–250.
- [4] A. S. CAVARETTA, C. A. MICHELLI AND A. SHARMA, Multivariate interpolation and the Radon transform, Part I, *Math. Z.* **174** (1980), 263–279; Part II, in “Quantitative Approximation” (Ron DeVore and K. Scherer, Eds.), pp. 49–62, Acad. Press, New York, 1980.
- [5] M. E. DAVISON AND F. A. GRUNBAUM, Tomographic reconstruction with arbitrary directions, *Comm. Pure Appl. Math.* **34** (1981), 77–120.
- [6] H. HAKOPIAN, Multivariate divided differences and multivariate interpolation of Lagrange and Hermite type, *J. Approx. Theory* **34** (1982), 286–305.
- [7] C. HAMAKER AND D. C. SOLMON, The angles between the null spaces of X -rays, *J. Math. Anal. Appl.* **62** (1978), 1–23.
- [8] F. JOHN, Abhängigkeiten zwischen den Flächenintegralen einer stetigen Funktion, *Math. Anal.* **111** (1935), 541–559.
- [9] B. LOGAN AND L. SHEPP, Optimal reconstruction of a function from its projections, *Duke Math. J.* **42** (1975), no. 4, 645–659.
- [10] R. MARR, On the reconstruction of a function on a circular domain from a sampling of its line integrals, *J. Math. Anal. Appl.* **45** (1974), 357–374.
- [11] J. RADON, Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten, *Ber. Verh. Sächs. Akad.* **69** (1917), 262–277.
- [12] D. C. SOLMON, The X -ray transform, *J. Math. Anal. Appl.* **56** (1976), no. 1, 61–83.

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