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Leja Points for Cantor-Type Sets

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We show that the sequence (x_n) of points used in [3] for construction of basis in the space of Whitney functions on Cantor-type sets, locally satisfies the Leja condition.

1. Introduction

Given a compact set K we take the points $a_1, a_2 \in K$ with $|a_1 - a_2| = \text{diam} K$ and then inductively define a_n, $n = 3, 4, \dots$, as a point (in general it is not unique) that provides the maximum modulus of the polynomial $(x - a_1) \cdots (x - a_{n-1})$ on the set K. In this way we get *Leja points*, which are important in potential theory since they approximate the equilibrium distribution on K (see, e.g. [4]).

In [3] a method was suggested to construct a Schauder basis in spaces of functions given on a Cantor-type set K. The partial sums $S_N(f)$ of the basic expansion of a function f are the polynomials that interpolate f locally at certain points from a fixed sequence (x_n). In the case of polar Cantor-type sets $S_N(f)$ are interpolating polynomials on the whole set K.

In [1] the same sequence was used for the construction of a continuous linear extension operator acting from the space of Whitney function on the Cantor-type set into the space of C^∞-functions defined on the whole space.

Here it is shown that although (x_n) is not the Leja sequence, the points x_n locally satisfy the Leja condition.

2. Uniformly Distributed Points

Here as in [1] we consider the Cantor set $K^{(\alpha)}$.

Let $(l_s)_{s=0}^\infty$ be a sequence such that $l_0 = 1$, $0 < 2l_{s+1} < l_s$, $s \in \mathbb{N}$. Let K be the Cantor set associated with the sequence (l_s), that is, $K = \bigcap_{s=0}^\infty E_s$, where $E_0 = I_{1,0} = [0, 1]$, E_s is a union of 2^s closed *basic* intervals $I_{j,s}$ of length

l_s and where E_{s+1} is obtained by deleting the open concentric subinterval of length $h_s = l_s - 2l_{s+1}$ from each $I_{j,s}$, $j = 1, 2, \dots, 2^s$. We call two intervals $I_{2j-1,s+1}, I_{2j,s+1}$ *adjacent* if they are derived from the same interval $I_{j,s}$.

Fix $\alpha > 1$ and l_1 with $4l_1 \le 1$, $4l_1^{\alpha-1} \le 1$. Then $l_s \ge 4l_{s+1}$ for any s. We will denote by $K^{(\alpha)}$ the Cantor set associated with the sequence (l_s), where $l_0 = 1$ and $l_{s+1} = l_s^\alpha = \cdots = l_1^{\alpha^s}$ for $s \ge 1$. For a fixed s we denote by K_s the set $K^{(\alpha)} \cap [0, l_s]$. The set $K^{(\alpha)}$ is polar if and only if $\alpha \ge 2$ (see, e.g. [2]).

Let x be an endpoint of some basic interval. Then there exists a minimal number k (the *type* of x) such that x is the endpoint of some $I_{j,m}$ for every $m \ge k$.

Given K_s, let us choose the points $x_1, x_2, \dots$ by including all endpoints of the basic subintervals of K_s, using the following *rule of increase of the type.* At first we take two points of type $\le s$: $x_1 = 0, x_2 = l_s$. These points are the endpoints of two basic intervals $I_{1,s+1}$ and $I_{2,s+1}$. Other endpoints of these intervals have type $s + 1$. We include these points in the sequence in the same order as the previous points. That is $x_3 = l_{s+1}, x_4 = l_s - l_{s+1}$. After the n-th step we form the points $(x_k)_1^{2^n}$ from all endpoints of type $\le s + n - 1$. We have 2^n basic intervals of length l_{s+n}. One of the endpoints of any interval $I_{j,s+n}$ is x_k. Then we choose the other endpoint of this interval to be x_{k+2^n}.

The points chosen in this way are uniformly distributed in the sense of the following definition. Let

$$N = 2^{r_m} + 2^{r_{m-1}} + \cdots + 2^{r_0} \quad \text{with} \quad 0 \le r_0 < r_1 < \cdots < r_m$$

and N ordered points $(x_k)_1^N \subset K_s$ be given. We divide them into $m + 1$ groups. The group X_m contains the first 2^{r_m} points, the next $2^{r_{m-1}}$ points are included into the group X_{m-1}, etc. We say that the ordered points $(x_k)_1^N$ are *uniformly distributed* on the set K_s if the following holds. Every basic interval $I_{j,s+r_m}$, $j = 1, 2, \dots, 2^{r_m}$, contains exactly one point from X_m. The points of X_m occupy concrete 2^{r_m} basic intervals $I_{j,s+r_m+1}$. Other 2^{r_m} basic intervals of length l_{s+r_m+1} are free of points from X_m. Then the remaining $2^{r_{m-1}} + \cdots + 2^{r_0}$ points can occupy only these free intervals, and moreover, $2^{r_{m-1}}$ points from X_{m-1} are uniformly distributed on $2^{r_{m-1}}$ intervals $I_{j,s+r_{m-1}}$ and the remaining $2^{r_{m-2}} + \cdots + 2^{r_0}$ points can occupy only the intervals $I_{j,s+r_{m-1}+1}$ that do not contain points from X_{m-1}, etc.

If the points $(x_k)_1^N$ are uniformly distributed on K_s, then the numbers of points x_k in two basic intervals $I_{i,q}$, $I_{j,q}$ of equal length are the same or differ by 1. To prove this let us fix the minimal r_q with $l_{s+r_q} \le l_q$. Then $I_{i,q}$ and $I_{j,q}$ contain the same number of points from X_k, $k = m, m-1, \dots, q$. On the other hand, since $l_{s+r_{q-1}} > l_q$, any interval of the length l_q can contain only one point from $\cup_0^{q-1} X_k$.

Given points $(x_k)_1^N$ let e_N denote the polynomial $(x - x_1) \cdots (x - x_N)$.

Lemma 1. *Suppose that N_s satisfies $l_s^{\alpha-1} N_s \le 1$. Then the first N, $N \le N_s$, Leja points are uniformly distributed on the set K_s.*

Proof. By direct computation one can check that for any K_s, $s \geq 0$, the first 9 Leja points are just the points $(x_k)_1^9$ chosen by the rule of increase of the type. Therefore we can assume $N_s \geq 10$. The proof is by induction on N. Suppose $(x_k)_1^N$ are uniformly distributed on K_s, with $N = 2^{r_m} + 2^{r_{m-1}} + \cdots + 2^{r_0} < N_s$. Consider at first the case $r_0 = 0$ and $N + 1 = 2^{r_m} + \cdots + 2^{r_p} + 2^p$ with $1 \leq p < r_p \leq r_m$. Here in the decomposition of N we have $r_k = k$ for $k = 0, 1, \ldots, p-1$. The last point x_N belongs to some interval of length l_{s+1}. Then another interval $I_{j_1,s+1}$ covers two intervals of length l_{s+2}, but it contains only one point from $X_1 = \{x_{N-2}, x_{N-1}\}$, since these points are uniformly distributed. By $I_{j_2,s+2}$ we denote the interval free of points from X_1. This interval contains only one point from X_2, therefore there exists $I_{j_3,s+3} \subset I_{j_2,s+2}$ such that $I_{j_3,s+3} \cap X_2 = \emptyset$. We continue in this fashion to obtain the interval $I_{j_p,s+p}$ that does not contain points of $\cup_{k=0}^{p-1} X_k$. Any interval $I_{j,s+p}$ contains $(2^{r_m} + \cdots + 2^{r_p})2^{-p}$ points from $\cup_{k=p}^m X_k$ and if $j \neq j_p$, then it contains exactly one point from $\cup_{k=0}^{p-1} X_k$. Let us show that there exists $x \in I_{j_p,s+p}$ such that

$$|e_N(x)| > |e_N(y)| \quad \text{for any} \quad y \in I_{j,s+p},\ j \neq j_p.$$

This will mean that the next Leja point x_{N+1} belongs to $I_{j_p,s+p}$ and the points $(x_k)_1^{N+1}$ are distributed uniformly on K_s.

Let $I_{\mu,s+p}$ be the adjacent to $I_{j_p,s+p}$ interval. There are $2^{r_p - p}$ intervals of length l_{s+r_p} covered by $I_{j_p,s+p}$. Each of them contains exactly one point from X_p. Let us take $J_p = I_{i,s+r_p}$ which is the most distant from $I_{\mu,s+p}$. One of the subintervals $I_{j,s+r_p+1}$ does not contain points from X_p. On this subinterval we take $J_{p+1} = I_{i,s+r_{p+1}}$, which is the most distant from $x \in J_p \cap X_p$. Continuing in this way we get a nested family of intervals $J_k = I_{i,s+r_k}$ such that J_k contains some point $x_i \in X_k$ and J_{k+1} is the most distant from x_i. Eventually we take x as the endpoint of J_m with $|x - x_m| \geq l_{s+r_m} - l_{s+r_m+1}$. Here $x_m \in J_m \cap X_m$.

Let $\pi_k(x) = \prod_{x_j \in X_k} |x - x_j|$, $k = 0, \ldots, m$. Then $|e_N(x)| = \prod_0^m \pi_k(x)$.

Since the points $(x_k)_1^N$ are uniformly distributed and due to the choice of x, we get $\pi_k(x) \geq h_{s+r_k} h_{s+r_k-1} h_{s+r_k-2}^2 \cdots h_s^{2^{r_k-1}}$ for $k = 0, \ldots, m$. Therefore,

$$\prod_{k=0}^{p-1} \pi_k(x) \geq h_{s+p-1} h_{s+p-2}^2 \cdots h_s^{2^{p-1}}.$$

On the other hand, for any $y \in I_{j,s+p}$, $j \neq j_p$ and for $k = p, p+1, \ldots, m$ we get $\pi_k(y) \leq l_{s+r_k} l_{s+r_k-1} l_{s+r_k-2}^2 \ldots l_s^{2^{r_k-1}}$. In order to get an upper bound of $\prod_{k=0}^{p-1} \pi_k(y)$, let us consider the worst case, when $I_{j,s+p}$ and $I_{j_p,s+p}$ are adjacent. Then

$$\prod_{k=0}^{p-1} \pi_k(y) \leq l_{s+p} l_{s+p-2}^2 \cdots l_s^{2^{p-1}}.$$

The sequence (l_k/h_k) decreases, as is easy to check. Therefore,

$$|e_N(y)|/|e_N(x)| = \prod_{k=0}^{m} \pi_k(y)/\pi_k(x) < l_{s+p}/h_{s+p-1}\,(l_s/h_s)^{N_s-2}.$$

By condition we get

$$l_s/h_s = (1-2l_s^{\alpha-1})^{-1} \le (1-\frac{2}{N_s})^{-1} = 1+\frac{2}{N_s-2}$$

and $(l_s/h_s)^{N_s-2} < e^2$. Also,

$$l_{s+p}/h_{s+p-1} < l_s^{\alpha-1}/(1-2l_s^{\alpha-1}) \le \frac{1}{N_s-2}.$$

Therefore,

$$|e_N(y)|/|e_N(x)| < \frac{e^2}{N_s-2} < 1, \quad \text{since} \quad N_s \ge 10.$$

Similar arguments apply to the case $r_0 = 0$, $N+1 = 2^{2r_m}$.

If $r_0 > 0$, then we get 2^{r_0+1} intervals $I_{j,s+r_0+1}$ such that 2^{r_0} of them contain a point from X_0 and the remaining 2^{r_0} intervals are free of points from X_0. Any interval $I_{j,s+r_0+1}$ contains the same number of uniformly distributed points from $\cup_{k=1}^{m} X_k$. Arguing as above, we see that the next Leja point x_{N+1} belongs to some of the free intervals $I_{j,s+r_0+1}$, therefore $(x_k)_1^{N+1}$ are uniformly distributed. □

Lemma 2. *Let s be such that $l_s^{\alpha-1} \le 1/6$ and $N = 2^n+\nu$ with $0 \le \nu < 2^n$. Suppose that the points $(x_k)_1^N$ of type $\le s+n$ are uniformly distributed on K_s. Then the polynomial e_N is monotone on any basic subinterval $I_{j,s+n}$, containing one point from $(x_k)_1^N$.*

Proof. We see that ν out of 2^n intervals of length l_{s+n} contain two points from $(x_k)_1^N$, whereas $2^n-\nu$ of them contain only one such point. Fix any interval $I_{j,s+n}$ of the second type. Let $I_{j,s+n} \ni x_m$ with some $m \le N$. Let

$$Q(x) = \prod_{k=1, k\ne m}^{N} (x-x_k).$$

Then, for $x \in I_{j,s+n}$, we get

$$e_N'(x) = Q(x)\Big[1+(x-x_m)\sum_{k=1,k\ne m}^{N}(x-x_k)^{-1}\Big]. \tag{1}$$

The adjacent to $I_{j,s+n}$ interval contains at most two selected points. Continuing in this manner, we see that

$$\sum_{k=1,k\neq m}^{N} (x-x_k)^{-1} < 2h_{s+n-1}^{-1} + 4h_{s+n-2}^{-1} + \cdots + 2^n h_s^{-1}.$$

Since here $l_k \geq 6^q l_{k+q}$ for $q \geq 1$, we get $h_k \geq \frac{2}{3}\, l_k$ and

$$\sum_{k=1}^{n} 2^k h_{s+n-k}^{-1} \leq \frac{9}{2} l_{s+n-1}^{-1}.$$

On the other hand, $|x - x_m| \leq l_{s+n}$. Therefore the expression in the square brackets in (1) is positive and e_N' does not change its sign on $I_{j,s+n}$. $\square$

Similarly one can show that e_N is monotone on any basic interval $I_{j,s+n+1}$, $j = 1, \ldots, 2^{n+1}$.

3. Distribution of First Leja Points

Let us specify the location of the points $(x_k)_1^N$, selected by the rule of increase of the type, where as before $N = 2^{r_m} + 2^{r_{m-1}} + \cdots + 2^{r_0}$. We define

$$d_k = \begin{cases} 0, & k = m \\ l_{s+r_{k+1}} - d_{k+1} & \text{for } k = m-1, m-2, \ldots, 0. \end{cases}$$

Therefore,

$$d_k = l_{s+r_{k+1}} - l_{s+r_{k+2}} + \cdots + (-1)^{m-k+1} l_{s+r_m}.$$

Then X_m consists of all endpoints of the intervals $I_{j,s+r_m-1} = [a_j^{(m)}, b_j^{(m)}]$, that is $X_m = \{a_j^{(m)}, b_j^{(m)}\}_{j=1}^{2^{r_m-1}}$. By induction one can show that for any k, $0 \leq k \leq m$, we have $X_k = \{a_j^{(k)} + d_k, b_j^{(k)} - d_k\}_{j=1}^{2^{r_k-1}}$, where $([a_j^{(k)}, b_j^{(k)}])_{j=1}^{2^{r_k-1}}$ is the family of all basic intervals of length l_{s+r_k-1} on K_s. The formula gives only the contents of X_k. Inside of X_k we put the points in the order: $a_1^{(k)} + d_k$, $b_{2^{r_k-1}}^{(k)} - d_k$, $b_{2^{r_k-2}}^{(k)} - d_k$, $a_{2^{r_k-2}}^{(k)} + d_k$, and so on.

Theorem 1. *Let $l_s^{\alpha-1} \leq 1/6$ and $l_s^{\alpha-1} N_s \leq 1$. Then the first N_s Leja points of the set K_s can be taken by the rule of increase of the type.*

Proof. Suppose the first 2^{r_m} Leja points are all endpoints of type $\leq s + r_m - 1$ on K_s. Every interval $I_{j,s+r_m}$ contains some $x \in X_m$ as its endpoint. We arrange increasingly the other endpoints of $I_{j,s+r_m}$, $j = 1, \ldots, 2^{r_m}$, and denote them by $Y = (y_n)$. Lemma 1 and Lemma 2 imply that the next 2^{r_m} Leja points have to be taken from Y.

Given $y \in Y$ let us set distances $|y - x_k|$, $x_k \in X_m$, increasingly and denote them by $(\rho_k(y))_{k=1}^{2^{r_m}}$. For example,

$$\rho_1(y_1) = l_{s+r_m},$$
$$\rho_2(y_1) = l_{s+r_m-1} - l_{s+r_m},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\rho_{2^{r_m}}(y_1) = l_s - l_{s+r_m}.$$

By symmetry, let us consider at first $y \in I_{1,s+1}$. Clearly, $\rho_k(y_1) \geq \rho_k(y_n)$ for $k \leq 2^{r_m}$, $n = 2, 3, \dots, 2^{r_m-1}$. Therefore, we can take the next Leja point as $y_1 = l_{s+r_m}$. Analogously, $x_{2^{r_m}+2} = l_s - l_{s+r_m}$. By Lemma 1, the point $x_{2^{r_m}+3}$ gets into $I_{j,s+2}$ with $j = 2$ or $j = 3$. Without loss of generality we can take $I_{2,s+2}$ and $x = l_{s+1} - l_{s+r_m}$. Let us show that $|e(x)| > |e(y)|$ for any $y \in Y \cap I_{2,s+2}$. Here e denotes the corresponding polynomial of degree $2^{r_m} + 2$. For a fixed y we have $x = y + d$ with $l_{s+r_m-1} - 2l_{s+r_m} \leq d \leq l_{s+2} - 2l_{s+r_m}$. We represent e as the product $P_L\, P\, P_R$, where

$$P(z) = \Pi(z - x_k) \quad \text{for} \quad x_k \in I_{2,s+2},$$

P_L corresponds to $2^{r_m-2} + 1$ roots of e on $I_{1,s+2}$ and P_R does so for the roots on the right, that is, on $I_{2,s+1}$. Since the point x is at the same distance or closer than y to the endpoints of $I_{2,s+2}$, we get $|P(x)| \geq |P(y)|$. Thus,

$$\frac{|e(y)|}{|e(x)|} \leq \prod_{x_k \in I_{1,s+2}} \Big(1 - \frac{d}{x - x_k}\Big) \prod_{x_k \in I_{2,s+1}} \Big(1 + \frac{d}{x_k - x}\Big).$$

We have $x - x_k \leq l_{s+1}$ in the first product and $x_k - x > h_s$ in the second one. Therefore,

$$\frac{|e(y)|}{|e(x)|} \leq R := \Big(1 - \frac{d}{l_{s+1}}\Big)^{2^{r_m-2}+1} \Big(1 + \frac{d}{h_s}\Big)^{2^{r_m-1}+1}$$

and

$$\ln R \leq (2^{r_m-1} + 1)\frac{d}{h_s} - (2^{r_m-2} + 1)\frac{d}{l_{s+1}} < 0,$$

as is easy to see. Therefore, $x_{2^{r_m}+3} = l_{s+1} - l_{s+r_m}$, and similarly, $x_{2^{r_m}+4} = l_s - l_{s+1} + l_{s+r_m}$. We continue in this fashion to put the next $2^{r_{m-1}}$ Leja points as the points from X_{m-1} taken in the corresponding order.

Let us show that the next $2^{r_{m-2}}$ Leja points can be chosen only as points from X_{m-2}. Set $M = 2^{r_m - r_{m-1} - 1}$. Every interval $I_{j,s+r_{m-1}}$ contains $2M$ points from X_m and one point from X_{m-1}.

Let us fix any basic interval $I_{j,s+r_{m-1}} = [a, b]$. It contains some $x_k \in X_{m-1}$. The interval $[a, b]$ is surrounded by two gaps. If the right gap has length

$h_{s+r_{m-1}-1}$, then the left gap is larger. In this case $x_k = a + d_{m-1}$, Otherwise, $x_k = b - d_{m-1}$, Without loss of generality let $x_k = a + d_{m-1}$. Let us show that the other point $x = b - d_{m-1}$ will realize the maximum modulus of the corresponding polynomial e of degree $2^{r_m} + 2^{r_{m-1}}$ on $[a, b] \cap K_s$. We want to prove that $|e(x)| > |e(y)|$ for any $y \in Y \cap [a, b]$, $y \neq x$. Since the polynomial e is monotone on $[x, b]$ and $e(b) = 0$, it suffices to consider $y < x$. By Lemma 1 we can restrict ourselves to study only those y on the right subinterval of length $l_{s+r_{m-1}+1}$. Thus, $x = y + d$ with

$$l_{s+r_m-1} - 2l_{s+r_m} \leq d \leq l_{s+r_{m-1}+1} - 2l_{s+r_m}.$$

As before, we factorize e into the product $P_L\, P\, P_R$. Here P_L corresponds to the roots of e on the left. We have there at least the interval $I_{2j-1,s+r_{m-1}+1}$, containing $M+1$ points from $X_m \cup X_{m-1}$. The second term P is the polynomial with M roots on $I_{2j,s+r_{m-1}+1}$ and P_R stands for all other roots on the right from $[a, b]$. As above, $|P(x)| \geq |P(y)|$. For the zeros from $I_{2j-1,s+r_{m-1}+1}$ we have $x - x_j \leq l_{s+r_{m-1}}$ and

$$\frac{|P_L(y)|}{|P_L(x)|} = \prod \Big(1 - \frac{d}{x - x_j}\Big) \leq \Big(1 - \frac{d}{l_{s+r_{m-1}}}\Big)^{M+1}.$$

Here we consider only zeros on $I_{2j-1,s+r_{m-1}+1}$ and neglect the other possible left zeros.

On the other hand, on the right we have $2M + 1$ points x_j with $x_j - x > h_{s+r_{m-1}-1}$ and at worst $4M + 2$ zeros of e with $x_j - x > h_{s+r_{m-1}-2}$, and so on. Therefore,

$$\frac{|P_R(y)|}{|P_R(x)|} = \prod \Big(1+\frac{d}{x_j - x}\Big) < \Big(1+\frac{d}{h_{s+r_{m-1}-1}}\Big)^{2M+1} \Big(1+\frac{d}{h_{s+r_{m-1}-2}}\Big)^{4M+2} \cdots .$$

Therefore,

$$\ln \frac{|e(y)|}{|e(x)|} \leq -\frac{(M+1)d}{l_{s+r_{m-1}}} + \frac{(2M+1)d}{h_{s+r_{m-1}-1}} + \frac{2(2M+1)d}{h_{s+r_{m-1}-2}} + \cdots \frac{2^{r_{m-1}-1}(2M+1)d}{h_s}.$$

By condition, $h_q \geq 4l_{q+1} > 4h_{q+1}$. Therefore,

$$\sum_{k=s}^{n} 2^{k-n} h_k^{-1} < 2h_n^{-1}$$

and

$$\ln \frac{|e(y)|}{|e(x)|} \leq -\frac{(M+1)d}{l_{s+r_{m-1}}} + \frac{2(2M+1)d}{h_{s+r_{m-1}-1}}.$$

The expression on the right side is negative with enough to spare. Even if we distribute some points from X_{m-2} and fix the interval $I_{j,s+r_{m-1}}$, which does not contain these points, then the new polynomial e will attain its maximum modulus on this interval at the corresponding point from X_{m-2}. Thus for $2^{r_{m-1}}$ intervals $I_{j,s+r_{m-1}} = [a,b]$ we have $2^{r_{m-1}+1}$ points of type $a + d_{m-1}$, $b - d_{m-1}$. A half of them forms X_{m-1} and the remaining $2^{r_{m-1}}$ points z_k are the only possible candidates for the next $2^{r_{m-1}}$ Leja points. We arrange them in increasing order. Then

$$\begin{aligned} z_1 &= d_{m-2}, \\ z_2 &= l_{s+r_{m-1}-1} - d_{m-2}, \\ &\dots\dots\dots\dots\dots\dots \\ z_{2^{r_{m-1}}} &= l_s - d_{m-2}. \end{aligned}$$

The configuration of the previous selected points is the same for any interval $I_{j,s+r_{m-1}}$. Therefore, $\rho_k(z_1) = \rho_k(z_n)$ for $k \le 2M + 1$ and for all n. But for the next k the first and the last points z have advantage over the other z_n. Let us take z_1 as the next Leja point $x_{2^{r_m}+2^{r_{m-1}}+1}$ and, by symmetry, $x_{2^{r_m}+2^{r_{m-1}}+2} = l_s - d_{m-2}$. Arguing as above, we see that the next points from X_{m-2}, taken according the rule of increase of the type, satisfy the Leja condition.

The same conclusion can be drawn for points from $X_{m-3}, \dots, X_0$. Suppose the points from $X_m \cup \dots \cup X_{m-k}$ are chosen. Fix any interval $I_{j,s+r_{m-k}} = [A,B]$. It covers two adjacent intervals $[A,B_1]$ and $[A_2,B]$ of length $l_{s+r_{m-k}+1}$. The interval $[A,B]$ contains one point $x_k \in X_{m-k}$. Let $x_k \in [A,B_1]$, therefore $x = A + d_{m-k}$. The interval $[A,B]$ covers $M_1 = 2^{r_{m-k+1}-r_{m-k}}$ intervals $[a_q,b_q]$ of length $l_{s+r_{m-k+1}}$. Every interval $[a_q,b_q]$ contains one point from X_{m-k+1} and this point is of the type $a_q + d_{m-k+1}$ or $b_q - d_{m-k+1}$. By induction, the next Leja point can be chosen only from the remaining M_1 points. And what is more, the first on $[A,B]$ point $x \in X_{m-k}$ has been taken by this time as $b_1 - d_{m-k+1} = a_1 + d_{m-k}$. By Lemma 1 we can consider only $M_1/2$ corresponding points on $[A_2,B]$. The point $a_{M_1} + d_{m-k+1} = b_{M_1} - d_{m-k} = B - d_{m-k}$ has advantage, since it is located closer than all its rivals to a large gap. Therefore, the only possible candidate for a Leja point on $I_{j,s+r_{m-k}}$ is the point symmetric to the point x_k. We denote all such points by w_q:

$$\begin{aligned} w_1 &:= d_{m-k-1}, \\ w_2 &:= l_{s+r_{m-k}-1} - d_{m-k-1}, \\ &\dots\dots\dots\dots\dots\dots\dots \\ w_{2^{r_{m-k}}} &= l_s - d_{m-k-1}. \end{aligned}$$

We will choose from them the next Leja points. Comparing distances $\rho_k(w_q)$, we see that the next $2^{r_{m-k-1}}$ Leja points are just the points of X_{m-k-1}, taken by the rule of increase of the type. $\square$

Remark. For arbitrary large α and s the sequence (x_k), chosen by the rule of increase of the type, is not the Leja sequence on K_s.

Proof. Suppose $N = 2^n$ points x_k are chosen according the above procedure. They are all endpoints of type $\leq s+n-1$ on K_s. Also we take $x_{N+1} = l_{s+n}$ and $x_{N+2} = l_s - l_{s+n}$. Then

$$e_{N+2}(x) = e_N(x)\,(x - x_{N+1})(x - x_{N+2}).$$

Fix $y = l_{s+2} - l_{s+n}$. Let us show that

$$|e_{N+2}(y)| > |e_{N+2}(x_{N+3})|, \quad \text{where} \quad x_{N+3} = l_{s+1} - l_{s+n}.$$

As before, let

$$|e_N(x)| = \prod_{k=1}^{N} \rho_k(x) \quad \text{with} \quad \rho_k(x) = |x - x_{j_k}| \uparrow.$$

Then $\rho_k(x_{N+3}) = \rho_k(y)$ for $k = 1, 2, \dots, \frac{N}{4}$. If $\frac{N}{4} < k \leq \frac{N}{2}$, then $\rho_k(x_{N+3}) = \rho_k(y) + l_{s+2} - 2l_{s+n}$ and $\rho_k(x_{N+3}) = \rho_k(y) - l_{s+1} + l_{s+2}$ for $\frac{N}{2} < k \leq N$. Therefore,

$$\frac{|e(y)|}{|e(x_{N+3})|} = \prod_{\frac{N}{4}<k\leq\frac{N}{2}} \frac{\rho_k(y)}{\rho_k(x_{N+3})} \prod_{\frac{N}{2}<k\leq N} \frac{\rho_k(y)}{\rho_k(x_{N+3})} \frac{l_{s+2} - 2l_{s+n}}{l_{s+1} - 2l_{s+n}} \frac{l_s - l_{s+2}}{l_s - l_{s+1}}. \quad (2)$$

In the first product,

$$\frac{\rho_k(y)}{\rho_k(y) + l_{s+2} - 2l_{s+n}} > 1 - \frac{l_{s+2}}{\rho_k(y) + l_{s+2}} > 1 - \frac{l_{s+2}}{l_{s+1} - l_{s+2}} \geq 1 - \frac{4}{3}\frac{l_{s+2}}{l_{s+1}}.$$

For the second product,

$$\frac{\rho_k(y)}{\rho_k(y) - l_{s+1} + l_{s+2}} > 1 + \frac{l_{s+1} - l_{s+2}}{l_s}.$$

The last two terms in (2) can be estimated from below by $\frac{1}{2}\,l_{s+1}^{\alpha-1}$. Therefore,

$$\frac{|e(y)|}{|e(x_{N+3})|} > \frac{1}{2}\,l_{s+1}^{\alpha-1}\Big(1 - \frac{4}{3}\,l_{s+1}^{\alpha-1}\Big)^{N/4}\Big(1 + \frac{l_{s+1} - l_{s+2}}{l_s}\Big)^{N/2}. \quad (3)$$

Since

$$\Big(1 - \frac{4}{3}\,l_{s+1}^{\alpha-1}\Big)\Big(1 + \frac{l_{s+1} - l_{s+2}}{l_s}\Big)^2 > 1,$$

the expression on the right-hand side in (3) is greater than 1 for N large enough. Therefore, the point x_{N+3} cannot be taken as the next Leja point. □

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What Is the Size of the Lebesgue Constant for Newton Interpolation?*

Let X be an infinite triangular array of nodes in $[-1,1]$. Let $\Lambda_n(X)$ denote the $n-$th Lebesgue constant, that is, the uniform norm of the Lagrange interpolating operator defined by the $(n+1)-$st row of X. It is well-known that the sequence $(\Lambda_n(X))$ has at least logarithmic growth and that the Chebyshev array T is close to the optimal choice. Now suppose that the array X is monotone, that is, any row of X consists of the previous row plus one more value. Is it possible to reach a polynomial growth of the sequence $(\Lambda_n(X))$ for some array X? Affirmative answer allows to construct interpolational topological bases for, example, in the space $C^\infty[-1,1]$. The good choice for X gives the nested family of zeros of Chebyshev polynomials from some subsequence, for example (T_{3^n}), or the Leja sequence, or another sequence approximating the equilibrium distribution.

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