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On Rational B-Spline Functions

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We study the degree of approximation by rational B-spline functions. The relationship with the degree of approximation by variation-diminishing Schoenberg splines is discussed, both in the non-uniform and in the uniform case. Furthermore it is proved that rational Bernstein operators reproduce linear functions if and only if all weights are equal. Starting from this result, more precise error estimates are given. Several examples will illustrate the impact of the weight choice on the rate of approximation.

1. Introduction and Some Basics

It was in Computer Aided Geometric Design (CAGD) that so-called NURBS ("non-uniform rational B-splines") were introduced. Farin cites in his book [4] the thesis of Vesprille [18] and articles by Tiller [17] and Piegl & Tiller [12] as early papers on the subject. The standard source on this method is now the book by Piegl & Tiller [13]. Further monographs on the subject are those by Fiorot & Jeannin [6] and by Farin [3]. NURBS are today in use in commercially available software libraries such as SISL from SINTEF in Oslo (see, e.g., [16]).

The abbreviation NURBS is an unfortunate acronym. The term is misleading since it suggests that one is exclusively dealing with non-uniform knot spacing which is not true. We thus prefer the term rational B-spline function. They constitute a generalization of Schoenberg's variation-diminishing splines. Adapted to the context of approximation (of functions) theory which we discuss here, this generalization is as follows.

Definition 1. Let $\Delta_n: \ 0 = x_0 < x_1 < \ldots < x_n = 1$, $n \in \mathbb{N}$, be a finite partition of the interval $I = [0,1]$, $k \in \mathbb{N}$. We extend this partition by

$$x_{-k} = \ldots = x_{-1} = x_0 = 0, \quad x_n = x_{n+1} = \ldots = x_{n+k} = 1.$$

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Define "nodes" (Greville abscissae = evaluation parameters) by

$$\xi_{j,k} := \frac{x_{j+1} + \cdots + x_{j+k}}{k}, \quad -k \le j \le n-1.$$

To each Greville abscissa associate a weight $w_{j,k} > 0$. Putting

$$N_{j,k}(x) := (x_{j+k+1} - x_j)[x_j, x_{j+1}, \ldots, x_{j+k+1}](\cdot - x)_+^k,$$

for $f \in \mathbb{R}^{[0,1]}$ (the space of real-valued functions on $[0,1]$) we define

$$\begin{aligned} R_{\Delta_n,k}(f;x) &:= \frac{\sum_{j=-k}^{n-1} w_{j,k}\, f(\xi_{j,k})\, N_{j,k}(x)}{\sum_{j=-k}^{n-1} w_{j,k}\, N_{j,k}(x)} \\ &=: \sum_{j=-k}^{n-1} f(\xi_{j,k})\, R_{j,k}(x), \quad 0 \le x < 1, \\ R_{\Delta_n,k}(f;1) &:= \lim_{\substack{x \to 1 \\ x < 1}} R_{\Delta_n,k}(f;x). \end{aligned}$$

$R_{\Delta_n,k}$ is the rational B-spline operator and $R_{\Delta_n,k}(f;\cdot)$ is a rational B-spline function.

Proposition 1 (Some properties of $R_{\Delta_n,k}$).

(i) $R_{\Delta_n,k}$ *is a positive linear operator reproducing constant functions.*

(ii) Both the numerator and the denominator are splines of degree k *and in* $C^{k-1}[0,1]$.

(iii) One has

$$R_{\Delta_n,k}(f;0) = f(0) \quad \textit{and} \quad R_{\Delta_n,k}(f;1) = f(1).$$

(iv) $R_{\Delta_n,k}$ *is discretely defined, i.e., it depends only on the* $n+k$ *values* $f(\xi_{j,k})$, $-k \le j \le n-1$ *(and on the weights* $w_{j,k}$ *associated with* $\xi_{j,k}$*).*

Example 1. (i) Suppose that $w_{j,k} = w > 0$ for $-k \le j \le n-1$. Then

$$\begin{aligned} R_{\Delta_n,k}(f;x) &= \frac{w \sum_{j=-k}^{n-1} f(\xi_{j,k}) \cdot N_{j,k}(x)}{w \sum_{j=-k}^{n-1} N_{j,k}(x)} \\ &= \sum_{j=-k}^{n-1} f(\xi_{j,k}) \cdot N_{j,k}(x) \\ &= S_{\Delta_n,k}(f;x), \quad x \in [0,1]. \end{aligned}$$

The latter is the famous (polynomial) variation-diminishing Schoenberg spline. It was introduced by Schoenberg and Greville in 1965 (see [15]).

(ii) Suppose that $w_{j,k} = w > 0,\ k = 1,\ n \in \mathbb{N}$.

Then the "knots" are given as

$$x_{-1} = x_0 < x_1 < \ldots < x_n = x_{n+1}$$

and the "nodes" are

$$\xi_{j,1} = x_{j+1}, \quad -1 \le j \le n-1.$$

The fundamental functions are now $N_{j,1},\ -1 \le j \le n-1$, and the operator $S_{\Delta_n,1}$ describes piecewise linear interpolation at the points

$$0 = x_0 < x_1 < \ldots < x_n = 1.$$

(iii) Suppose that $w_{j,k} = w > 0,\ n = 1,\ k \in \mathbb{N}$.

Then the "knots" are given as

$$x_{-k} = \ldots = x_0 = 0, \quad x_1 = \ldots = x_{1+k} = 1,$$

so there are no knots in $(0,1)$.

For the "nodes" one has

$$\xi_{-k,k} = 0,\ \xi_{-k+1,k} = \frac{1}{k},\ \ldots,\ \xi_{0,k} = 1 \text{ (equidistant)}.$$

For the fundamental functions one gets from the Mansfield identity:

$$N_{j,k}(x) = \binom{k}{j+k} x^{j+k}(1-x)^{-j}, \quad -k \le j \le 0 = n-1.$$

Hence

$$\begin{aligned}\sum_{j=-k}^{0} f(\xi_{j,k})\, N_{j,k}(x) &= \sum_{j=-k}^{0} f\left(\frac{j+k}{j}\right) \binom{k}{j+k} x^{j+k}(1-x)^{-j} \\ &= \sum_{j=0}^{k} f\left(\frac{j}{k}\right) \binom{k}{j} x^{j}(1-x)^{k-j} \\ &= B_k(f;x), \quad 0 \le x \le 1.\end{aligned}$$

The latter is the Bernstein polynomial of degree k.

(iv) Suppose that the weights are not identical, but again $n = 1,\ k \in \mathbb{N}$.

Writing $p_{k,j}(x) := \binom{k}{j} x^j (1-x)^{k-j}$ we arrive at

$$R_{\Delta_1,k}(f;x) = \frac{\displaystyle\sum_{j=0}^{k} w_{j,k}\, f\left(\frac{j}{k}\right)\, p_{k,j}(x)}{\displaystyle\sum_{j=0}^{k} w_{j,k}\, p_{k,j}(x)}.$$

This is a rational Bernstein function.

All five methods considered play a fundamental role in CAGD.

The approximation theoretical knowledge about the spline methods mentioned is in contrast to their importance in applications and to the many experimental results available. Therefore, in the present note we start our discussion on rational B-spline functions from the viewpoint of quantitative approximation theory.

2. Degree of Approximation by Rational B-spline Functions

In the present section we study the degree of approximation by rational B-spline functions defined on arbitrary knot sequences without multiplicities. To that end we use the following theorem given by the first author several years ago.

Theorem 1. *For a positive linear operator $L : C[0,1] \to C[0,1]$, reproducing constant functions, the following inequality holds:*

$$|L(f;x) - f(x)| \le \max\left\{1, \frac{1}{h} L(|e_1 - x|; x)\right\} \tilde{\omega}_1(f;h)$$

for all $f \in C[0,1]$, $x \in [0,1]$ and $h > 0$.

Here $\tilde{\omega}_1(f;\cdot)$ denotes the least concave majorant of the (classical) first order modulus of continuity.

The proof of Theorem 1 can be obtained as a combination of Theorem 3.3 in [9] and Corollary 2.3 in [8]. Applying the above theorem leads to

Proposition 2. *Let $R_{\Delta_n,k}$ be given as above. Define*

$$w^{\min}_{\Delta_n,k} := \min\{w_{j,k} \,:\, -k \le j \le n-1\} > 0,$$
$$w^{\max}_{\Delta_n,k} := \max\{w_{j,k} \,:\, -k \le j \le n-1\} > 0,$$

and the "weight ratio" by

$$\rho_{\Delta_n,k} := \frac{w^{\max}_{\Delta_n,k}}{w^{\min}_{\Delta_n,k}} \geq 1.$$

Then, for $f \in C[0,1]$, there holds

$$\|R_{\Delta_n,k} f - f\| \leq \rho_{\Delta_n,k} \cdot \tilde{\omega}_1\Big(f; \sqrt{\min\Big\{\frac{1}{2k},\ \frac{(k+1)\,\|\Delta_n\|^2}{12}\Big\}}\Big).$$

Proof. All that remains is to give an estimate for $R_{\Delta_n,k}(|e_1 - x|; x)$. We have

$$\begin{aligned}
R_{\Delta_n,k}(|e_1 - x|; x) &= \frac{\sum_{j=-k}^{n-1} w_{j,k}\, |\xi_{j,k} - x|\, N_{j,k}(x)}{\sum_{j=-k}^{n-1} w_{j,k}\, N_{j,k}(x)} \\
&\leq \frac{\sum_{j=-k}^{n-1} w^{\max}_{\Delta_n,k}\, |\xi_{j,k} - x|\, N_{j,k}(x)}{\sum_{j=-k}^{n-1} w^{\min}_{\Delta_n,k}\, N_{j,k}(x)} \\
&= \rho_{\Delta_n,k} \sum_{j=-k}^{n-1} |\xi_{j,k} - x|\, N_{j,k}(x) \\
&= \rho_{\Delta_n,k}\, S_{\Delta_n,k}(|e_1 - x|; x) \\
&\leq \rho_{\Delta_n,k} \sqrt{S_{\Delta_n,k}((e_1 - x)^2; x)}.
\end{aligned}$$

For the latter quantity Marsden [10] proved the uniform estimate

$$S_{\Delta_n,k}\big((e_1 - x)^2; x\big) \leq \min\Big\{\frac{1}{2k},\ \frac{(k+1)\|\Delta_n\|^2}{12}\Big\}, \quad 0 \leq x \leq 1.$$

Hence, we conclude from Theorem 1 that

$$\begin{aligned}
|R_{\Delta_n,k}(f; x) - f(x)| \leq\ &\max\Big\{1, \frac{1}{h}\, \rho_{\Delta_n,k} \sqrt{\min\Big\{\frac{1}{2k},\ \frac{(k+1)\,\|\Delta_n\|^2}{12}\Big\}}\Big\} \\
&\times \tilde{\omega}_1(f; h), \quad \forall h > 0
\end{aligned}$$

and putting $h = \sqrt{\min\Big\{\frac{1}{2k},\ \frac{(k+1)\,\|\Delta_n\|^2}{12}\Big\}}$ leads to

$$|R_{\Delta_n,k}(f; x) - f(x)| \leq \rho_{\Delta_n,k} \cdot \tilde{\omega}_1\Big(f; \sqrt{\min\Big\{\frac{1}{2k},\ \frac{(k+1)\,\|\Delta_n\|^2}{12}\Big\}}\Big).$$

The right hand side is independent of x, and thus we arrive at our claim. □

Corollary 1. *(i) If all weights equal $w > 0$, then for Schoenberg's variation-diminishing spline operator we get*

$$\|S_{\Delta_n,k} f - f\| \le \tilde{\omega}_1\Big(f; \sqrt{\min\Big\{\frac{1}{2k}, \frac{(k+1)\,\|\Delta_n\|^2}{12}\Big\}}\Big).$$

Similar inequalities were given by Marsden in [10].

(ii) For the Bernstein operators the above reduces to

$$\|B_k f - f\| \le \tilde{\omega}_1\Big(f; \frac{1}{\sqrt{2k}}\Big).$$

The latter inequality is a classical one similar to the given by Popoviciu [14].

Here we consider URBS – uniform rational B-splines. In this case much better information is available than in the general case. Denoting $S_{\Delta_n,k}$ in this case by $S_{n,k}$, the following was proved in [1, Theorem 2].

Theorem 2. *For $n \ge 1,\ k \ge 1,\ x \in [0,1]$, we have*

$$S_{n,k}\big((e_1 - x)^2; x\big) \le \frac{\min\{2x(1-x), \frac{k}{n}\}}{n+k-1}. \qquad (1)$$

We noted in [2, Remark 3.6] that the upper bound of (1) matches Marsden's uniform order in all cases $k, n \ge 1$ and is hence a pointwise refinement. Proceeding as in the general case we now have

Theorem 3.

$$|R_{n,k}(f;x) - f(x)| \le \rho_{\Delta_n,k} \cdot \tilde{\omega}_1\Big(f; \sqrt{\frac{\min\{2x(1-x), \frac{k}{n}\}}{n+k-1}}\Big).$$

A special case of URBS functions is given by rational Bernstein functions $R_{1,k}f$. They are obtained from Definition 1 for $n = 1$, so that in this case we have

$$|R_{1,k}(f;x) - f(x)| \le \rho_{\Delta_1,k} \cdot \tilde{\omega}_1\Big(f; \sqrt{\frac{2x(1-x)}{k}}\Big). \qquad (2)$$

The best constant is obtained if the "weight ratio" $\rho_{\Delta_1,k} = 1$, that is, if all weights are equal. This is the case for the polynomial Bernstein operator. We conjecture that this classical method is best possible among all rational Bernstein functions in a sense to be specified. This is supported by

Proposition 3. *The rational Bernstein operator reproduces linear functions if and only if all weights are equal.*

Proof. Since $R_{1,k}e_0 = e_0$ and $R_{1,k}$ is linear, it suffices to consider the function $e_1(x) = x$.

Writing w_j instead of $w_{j,k}$ for simplicity, we have

$$\begin{aligned} R_{1,k}(e_1;x) - x &= \frac{\sum_{j=0}^{k} w_j \, \frac{j}{k} \, p_{k,j}(x)}{\sum_{j=0}^{k} w_j \, p_{k,j}(x)} - x \cdot \frac{\sum_{j=0}^{k} w_j \, p_{k,j}(x)}{\sum_{j=0}^{k} w_j \, p_{k,j}(x)} \\ &= \frac{1}{N} \Big[\sum_{j=0}^{k} w_j \, \frac{j}{k} \, p_{k,j}(x) - x \cdot \sum_{j=0}^{k} w_j \, p_{k,j}(x) \Big] \\ &= \frac{T_1 - T_2}{N}. \end{aligned}$$

We raise the degree of T_1 from k to $k+1$ and get

$$T_1 = \sum_{j=0}^{k+1} \Big[\frac{j}{k+1} \, w_{j-1} \, \frac{j-1}{k} + \Big(1 - \frac{j}{k+1}\Big) \, w_j \, \frac{j}{k} \Big] \, p_{k+1,j}(x).$$

Here we put $w_{-1} := w_0$ and $w_{k+1} := w_k$ to be formally correct.

T_2 can be written as

$$\begin{aligned} T_2 &= \sum_{j=0}^{k} w_j \binom{k}{j} x^{j+1}(1-x)^{k-j} \\ &= \sum_{j=0}^{k} w_j \, \frac{j+1}{k+1} \binom{k+1}{j+1} x^{j+1}(1-x)^{k-j} \\ &= \sum_{j=1}^{k+1} w_{j-1} \, \frac{j}{k+1} \binom{k+1}{j} x^{j}(1-x)^{k+1-j} \\ &= \sum_{j=0}^{k+1} w_{j-1} \, \frac{j}{k+1} \binom{k+1}{j} x^{j}(1-x)^{k+1-j} \\ &= \sum_{j=0}^{k+1} w_{j-1} \, \frac{j}{k+1} \, p_{k+1,j}(x). \end{aligned}$$

Combining the representations of T_1 and T_2 we obtain

$$\begin{aligned} R_{1,k}(e_1;x)-x &= \frac{1}{N}\sum_{j=0}^{k+1}\Big[\Big(\frac{j-1}{k}-1\Big)w_{j-1}\,\frac{j}{k+1}+\Big(1-\frac{j}{k+1}\Big)w_j\,\frac{j}{k}\Big]p_{k+1,j}(x) \\ &= \frac{1}{N}\sum_{j=0}^{k+1}\frac{j}{k+1}\,\frac{k+1-j}{k}(w_j-w_{j-1})\,p_{k+1,j}(x) \\ &= x(1-x)\,\frac{1}{N}\sum_{j=0}^{k-1}(w_{j+1}-w_j)\,p_{k-1,j}(x). \end{aligned}$$

Hence, $R_{1,k}(e_1;x)-x=0$ for all $x\in[0,1]$ if and only if $w_{j+1}-w_j=0$ for $0\le j\le k-1$, i.e., $w_0=w_1=\ldots=w_k$. □

For rational Bernstein–Bézier curves the situation is somewhat different; see [5].

3. Degree of Approximation by Rational Bernstein Functions

For rational Bernstein functions $R_{1,k}(f;\,\cdot\,)$ inequality (2) from Section 2 can be brought into a more adequate form involving first and second order moduli.

We recall first that the proof of Proposition 3 was based upon the representation

$$R_{1,k}(e_1;x)-x=x(1-x)\,\frac{1}{N}\sum_{j=0}^{k-1}(w_{j+1}-w_j)\,p_{k-1,j}(x).$$

It would be desirable to have a similar representation for the more general $R_{\Delta_n,k}$, or at least for $R_{n,k},\ n\ge 2$.

In case of the positive quantities $R_{\Delta_n,k}((e_1-x)^2;x)$ we can proceed as in the proof of Proposition 2 to arrive at

$$R_{\Delta_n,k}\big((e_1-x)^2;x\big)\le\rho_{\Delta_n,k}\cdot S_{\Delta_n,k}\big((e_1-x)^2;x\big).$$

In particular, in the Bernstein case this means

$$\begin{aligned} R_{1,k}\big((e_1-x)^2;x\big) &= \frac{1}{N}\sum_{j=0}^{k}w_j\,\Big(\frac{j}{k}-x\Big)^2\,p_{k,j}(x) \\ &\le \rho_{1,k}\,B_k\big((e_1-x)^2;x\big) \\ &= \rho_{1,k}\,\frac{x(1-x)}{k}. \end{aligned}$$

Thus we can apply a Gonska-type theorem due to Păltănea [11, p. 31] to arrive at

Proposition 4. *For $f \in C[0,1]$, $x \in [0,1]$, $0 < h \le \frac{1}{2}$, there holds*

$$\begin{aligned}|R_{1,k}(f;x) - f(x)| \le \frac{x(1-x)}{N} \Big| \sum_{j=0}^{k-1} (w_{j+1} - w_j)\, p_{k-1,j}(x) \Big| \, h^{-1}\, \omega_1(f;h) \\ + \Big(1 + \frac{1}{2} h^{-2} \rho_{1,k} \frac{x(1-x)}{k}\Big) \omega_2(f;h).\end{aligned}$$

In particular, for $h = \sqrt{\frac{x(1-x)}{k}}$, *this implies*

$$\begin{aligned}|R_{1,k}(f;x) - f(x)| &\le \frac{\sqrt{k}}{N} \sqrt{x(1-x)} \Big| \sum_{j=0}^{k-1} (w_{j+1} - w_j)\, p_{k-1,j}(x) \Big| \\ &\quad \times \omega_1\Big(f; \sqrt{\frac{x(1-x)}{k}}\Big) \\ &\quad + \Big(1 + \frac{1}{2} \rho_{1,k}\Big) \omega_2\Big(f; \sqrt{\frac{x(1-x)}{k}}\Big) \\ &\le \sqrt{k} \sqrt{x(1-x)} \frac{\max_j |w_{j+1} - w_j|}{\min_j w_j} \omega_1\Big(f; \sqrt{\frac{x(1-x)}{k}}\Big) \\ &\quad + \Big(1 + \frac{1}{2} \rho_{1,k}\Big) \omega_2\Big(f; \sqrt{\frac{x(1-x)}{k}}\Big).\end{aligned}$$

Corollary 2. *If*

$$\frac{\max\limits_j |w_{j+1} - w_j|}{\min\limits_j w_j} \le c\, \frac{1}{k}, \quad c \ge 0,\ k = 1, 2, \ldots,$$

then

$$\begin{aligned}|R_{1,k}(f;x) - f(x)| \le c \sqrt{\frac{x(1-x)}{k}}\, \omega_1\Big(f; \sqrt{\frac{x(1-x)}{k}}\Big) \\ + \Big(1 + \frac{1}{2} \rho_{1,k}\Big) \omega_2\Big(f; \sqrt{\frac{x(1-x)}{k}}\Big).\end{aligned}$$

Example 2. (i) If, with $c \ge 0$, $w_{j,k} := w_j = 1 + \frac{c \cdot j}{k}$, $0 \le j \le k$, then – with the same c – the assumptions of the corollary are satisfied. Moreover, $\rho_{1,k} = c + 1$.

(ii) In the Bernstein polynomial case we have $c = 0$, so $\rho_{1,k} = 1$. Hence here the latter estimate reads

$$|B_k(f;x) - f(x)| \le \frac{3}{2}\, \omega_2\Big(f; \sqrt{\frac{x(1-x)}{k}}\Big).$$

Due to the presence of $x(1-x)$ in the representation of $R_{1,k}(e_1-x;x)$ and the upper bound of $R_{1,k}\big((e_1-x)^2;x\big)$ it is also possible to give an inequality in terms of Ditzian–Totik moduli. Here we used another theorem of Păltănea [11, p. 64], who generalized and optimized an earlier result in [7].

Proposition 5. *For $f \in C[0,1]$, $x \in [0,1]$, $0 < h \leq \frac{1}{2}$, one has*

$$|R_{1,k}(f;x) - f(x)| \leq \frac{\sqrt{x(1-x)}}{N}\,\frac{1}{2h}\,\Big|\sum_{j=0}^{k-1}(w_{j+1}-w_j)\,p_{k-1,j}(x)\Big| \\ \times\, \omega_1^{\varphi}(f;2h) + \Big[1 + \frac{3}{2}\,h^{-2}\,\frac{\rho_{1,k}}{k}\Big]\,\omega_2^{\varphi}(f;h).$$

In particular, for $h = \dfrac{1}{\sqrt{k}}$, $k \geq 2$, this implies

$$\begin{aligned}|R_{1,k}(f;x) - f(x)| &\leq \frac{\sqrt{k}}{2N}\,\sqrt{x(1-x)}\,\Big|\sum_{j=0}^{k-1}(w_{j+1}-w_j)\,p_{k-1,j}(x)\Big| \\ &\quad \times\, \omega_1^{\varphi}\Big(f;\frac{2}{\sqrt{k}}\Big) + \Big(1+\frac{3}{2}\,\rho_{1,k}\Big)\,\omega_2^{\varphi}\Big(f;\frac{1}{\sqrt{k}}\Big) \\ &\leq \frac{\sqrt{k}}{2}\,\sqrt{x(1-x)}\,\frac{\max_j |w_{j+1}-w_j|}{\min_j w_j}\,\omega_1^{\varphi}\Big(f;\frac{2}{\sqrt{k}}\Big) \\ &\quad + \Big(1+\frac{3}{2}\,\rho_{1,k}\Big)\,\omega_2^{\varphi}\Big(f;\frac{1}{\sqrt{k}}\Big).\end{aligned}$$

Remark 1. Observations analogous to the ones in Corollary 2 and Example 2 can also be made in the case of Ditzian–Totik moduli. In particular, for the polynomial Bernstein operators we arrive at

$$\|B_k f - f\|_\infty \leq \frac{5}{2}\,\omega_2^{\varphi}\Big(f;\frac{1}{\sqrt{k}}\Big).$$

4. Some Illustrations

In this section we give some examples illustrating the impact of the weights on the behaviour of rational Bernstein functions.

Example 3. Here we show that with inappropriate choices of the weights not even for the function $e_1(x) = x$ uniform convergence can be expected.

Indeed, for $0 < w = w_j$, $0 \le j \le k-1$, and $w_k > w$, to be determined later, we have

$$
\begin{aligned}
R_{1,k}(e_1;x) - x &= \frac{1}{\sum_{j=0}^{k} w_j\, p_{k,j}(x)}\, x(1-x) \sum_{j=0}^{k-1} (w_{j+1} - w_j)\, p_{k-1,j}(x) \\
&= \frac{1}{w \sum_{j=0}^{k-1} p_{k,j}(x) + w_k\, p_{k,k}(x)}\, x(1-x)(w_k - w)\, p_{k-1,k-1}(x) \\
&= \frac{1}{w(1 - p_{k,k}(x)) + w_k\, p_{k,k}(x)}\, x(1-x)(w_k - w)\, p_{k-1,k-1}(x) \\
&= \frac{1}{w + (w_k - w)\, p_{k,k}(x)}\, x(1-x)\,(w_k - w)\, p_{k-1,k-1}(x) \\
&= x(1-x)\, \frac{(w_k - w)\, p_{k-1,k-1}(x)}{w + (w_k - w)\, p_{k,k}(x)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
R_{1,k}(e_1;\frac{1}{2}) - \frac{1}{2} &= \frac{1}{4}\, \frac{(w_k - w)\,(\frac{1}{2})^{k-1}}{w + (w_k - w)\,(\frac{1}{2})^{k}} \\
&\ge \frac{1}{4}\, \frac{(w_k - w)\,(\frac{1}{2})^{k-1} + w - w}{w + (w_k - w)\,(\frac{1}{2})^{k-1}} \\
&= \frac{1}{4} \Big(1 - \frac{w}{w + (w_k - w)\,(\frac{1}{2})^{k-1}}\Big).
\end{aligned}
$$

Now choose w_k such that $w_k - w = 2^{k-1}$ and arrive at

$$
R_{1,k}(e_1;\frac{1}{2}) - \frac{1}{2} \ge \frac{1}{4} \Big(1 - \frac{w}{w+1}\Big) = \frac{1}{4}\, \frac{1}{w+1} \ne 0, \quad \forall k.
$$

Thus

$$
R_{1,k}(e_1;\frac{1}{2}) \not\to \frac{1}{2} \quad \text{for} \quad k \to \infty.
$$

Example 4. While the last example showed that a "wrong" choice of the weights can lead to divergence, the next illustration indicates that the approximation might be better if the weights are adjusted to the function.

This can be expected from the trivial relationship

$$
\inf\{\|R_{1,k} f - f\|_\infty \,:\, (w_0, \dots, w_k) \in \mathbb{R}_+^{k+1}\} \le \|B_k f - f\|_\infty.
$$

Here we consider the function

$$
f(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} \le x \le 1. \end{cases}
$$

Hence

$$B_2(f;x) = 2x(1-x),$$

and with $w_0 = w_2 = 1$ and $w_1 > 0$ we have

$$R_{1,2}(f;x) = \frac{2w_1\, x(1-x)}{(1-x)^2 + 2w_1\, x(1-x) + x^2}.$$

It can be seen by inspection that the approximation of f is better for $w_1 = 2$ or $w_1 = 3$ (for example), than it is for $w_1 = 1$ (the Bernstein polynomial of f).

Example 5. In the last example we illustrated the fact that choosing non-equal weights can lead to better approximations. Here we show that these can be even best possible.

We look again at $R_{1,2}$ associated to the weight sequence $(w_0, w_1, w_2) = (1, w_1, 1)$ and consider the function

$$g_{w_1}(x) = \frac{w_1\, x(1-x) + x^2}{(1-x)^2 + 2w_1\, x(1-x) + x^2}.$$

For $(w_0, w_1, w_2) = (1,1,1)$ we have $g_1(x) = x$, so in this case $R_{1,2}(g_1, x) = g_1(x) = x$.

But even for all $w_1 > 0$ it is true that $g_{w_1}(0) = 0$, $g_{w_1}(\frac{1}{2}) = \frac{1}{2}$, $g_{w_1}(1) = 1$, so that also in this case $R_{1,2}(g_{w_1}, x) = g_{w_1}(x)$.

Hence with $(w_0, w_1, w_2) = (1, w_1, 1)$, $R_{1,2}$ has e_0 and g_{w_1} as eigenfunctions with respect to the eigenvalue 1.

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