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Eigenfunctions and Eigenvalues for Some Durrmeyer-Type Operators

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We consider a generalization of different variants of Durrmeyer-type modifications of Baskakov and Meyer-König and Zeller operators. We give an overview on the results in [11], [12] and a generalization of the Rodriguez-type representation for the eigenfunctions given in [12], which is based on a general result concerning the commutativity of the operators in question with appropriate differential operators.

1. Introduction and Definition of the Operators

Durrmeyer-type modifications of positive linear operators were first introduced for the classical Bernstein operators by Durrmeyer in [5]. Their eigenfunctions which turned out to be independent of n were given by Derriennic in [4].

In this paper we consider Durrmeyer-type modifications of two classical positive linear operators introduced by Baskakov in [2] and Meyer-König and Zeller in [13]. There are different variants for Durrmeyer-type modifications of the Meyer-König and Zeller operators in the literature [1], [3], [6] which can be regarded as special cases of our general definition.

In what follows the operators are always defined for functions f such that the series on the right hand side is convergent.

For $x \in \mathbb{R}$, $l \in \mathbb{N}_0$, we will use the notation of rising and falling factorials, i.e.,

$$x^{\overline{l}} = \prod_{\nu=0}^{l-1}(x+\nu), \quad x^{\underline{l}} = \prod_{\nu=0}^{l-1}(x-\nu) \quad \text{for } l \in \mathbb{N},\ x^{\overline{0}} = x^{\underline{0}} = 1 .$$

Definition 1. For fixed $\nu \in \mathbb{Z}$ and $\kappa \in \mathbb{R}$, let $n \in \mathbb{N}$ be such that $n+\nu \in \mathbb{N}$. Then we define

$$(M_n^{\nu,\kappa} f)(x) = \sum_{k=0,\ k>\kappa-1}^{\infty} m_{n,k}(x)(n+\nu)\int_0^1 m_{n+\nu,k-\kappa}(t)(1-t)^{-2} f(t)\, dt ,$$

where

$$m_{n,k}(x)=\frac{(k+1)^{\overline{n}}}{n!}\,x^k(1-x)^{n+1}\,,\quad x\in[0,1]\,.$$

The operators $M_n^{\nu,\kappa}$ are closely related to some Durrmeyer modifications of the Baskakov operators given by the following definition. This fact makes it easy to carry over the results from one operator to the other.

Definition 2. For fixed $\nu\in\mathbb{Z}$ and $\kappa\in\mathbb{R}$, let $n\in\mathbb{N}$ be such that $n+\nu\in\mathbb{N}$. Then we define

$$(B_{n+1}^{\nu,\kappa}\widetilde{f})(\sigma)=\sum_{k=0,\,k>\kappa-1}^{\infty} b_{n+1,k}(\sigma)(n+\nu)\int_0^{\infty} b_{n+\nu+1,k-\kappa}(\tau)\widetilde{f}(\tau)\,d\tau\,,$$

$$b_{n+1,k}(\sigma)=\frac{(k+1)^{\overline{n}}}{n!}\,\sigma^k(1+\sigma)^{-(n+1+k)}\,,\quad \sigma\in[0,\infty)\,.$$

For $\nu=\kappa=0$ this is the Baskakov-Durrmeyer operator considered in several papers by the author (see e.g. [8], [9]).

Remark 1. The operators $M_n^{\nu,\kappa}$ and $B_{n+1}^{\nu,\kappa}$ are related in the following way. Let

$$\sigma:[0,1)\longrightarrow[0,\infty)\,,\quad \sigma(x)=\frac{x}{1-x}\,,\quad f(\cdot)=\widetilde{f}(\sigma(\cdot)).$$

Then

$$(M_n^{\nu,\kappa}f)(x)=(B_{n+1}^{\nu,\kappa}\widetilde{f})(\sigma(x))\,.$$

For the classical Baskakov and Meyer-König and Zeller operators the analogous result was given by Totik [14].

Clearly we have

$$(1-\cdot)^{n+\nu-1}f(\cdot)\in L_\infty(0,1)\Longleftrightarrow(1+\cdot)^{-(n+\nu-1)}\widetilde{f}(\cdot)\in L_\infty(0,\infty)\,.$$

2. Eigenfunctions and Eigenvalues of the Operators

We first point out that the following results are only true in case $\kappa<1$. Furthermore, it is easy to see that in case $\kappa\in\mathbb{N}$ the monomial $e_\kappa(\cdot)=(\cdot)^\kappa$ is an eigenfunction of $M_n^{\nu,\kappa}$ (see [6, p.13] for $\kappa=2$) and $\widetilde{e}_\kappa(\cdot)=\left(\frac{\cdot}{1+\cdot}\right)^\kappa$ is an eigenfunction of $B_{n+1}^{\nu,\kappa}$. Unfortunately, this is the only result we are able to determine up to now in this case.

We now present a result concerning the eigenfunctions and eigenvalues of the operators $M_n^{\nu,\kappa}$ and $B_{n+1}^{\nu,\kappa}$ which we proved by direct calculations in [11].

Theorem 1. *Let $\kappa < 1$, $m \in \mathbb{N}_0$ with $m \le n+\nu-1$, $f_\mu(\cdot) = (1-\cdot)^{-\mu}$ and $\widetilde{f}_\mu(\cdot) = (1+\cdot)^\mu$, $\mu \in \mathbb{N}_0$.*

(i) For $m \ge \nu$ or $m \le \frac{\nu-1}{2}$ the operator $M_n^{\nu,\kappa}$ and the operator $B_{n+1}^{\nu,\kappa}$, respectively, has an eigenfunction of the form

$$g_m^{\nu,\kappa} = \sum_{\mu=0}^{m} a_{m,\mu}^{\nu,\kappa} f_\mu \quad \text{and} \quad \widetilde{g}_m^{\nu,\kappa} = \sum_{\mu=0}^{m} a_{m,\mu}^{\nu,\kappa} \widetilde{f}_\mu\,, \tag{1}$$

respectively, with corresponding eigenvalue $\lambda_{n,m}^{\nu} = \frac{(n+m)^{\underline{m}}}{(n+\nu-1)^{\underline{m}}}$.

(ii) Except for a constant factor the coefficients of (1) can be uniquely determined by

$$a_{m,\mu}^{\nu,\kappa} = (-1)^{m-\mu}\binom{m}{\mu}\frac{(m+\kappa-\nu)^{\underline{m-\mu}}}{(2m-\nu)^{\underline{m-\mu}}}\,. \tag{2}$$

(iii) For $\frac{\nu}{2} \le m \le \nu-1$ there exists no linear combination of the functions f_μ and $\widetilde{f}_\mu$, respectively, $\mu = 0,\ldots,m$, which is an eigenfunction of the operator $M_n^{\nu,\kappa}$ and the operator $B_{n+1}^{\nu,\kappa}$, respectively.

Remark 2. We want to remark that the eigenfunctions in Theorem 1 are independent of n. Furthermore the corresponding eigenvalues do not depend on the parameter $\kappa < 1$.

Evidently, the third proposition of Theorem 1 is only relevant in the case $\nu \ge 2$, as the inequality $\frac{\nu}{2} \le m \le \nu - 1$ has no solution $m \in \mathbb{N}_0$ if $\nu \le 1$.

We also point out that for $m \ge \nu-\kappa > 0$, $-\kappa \in \mathbb{N}_0$, the falling factorials $(m+\kappa-\nu)^{\underline{m-\mu}}$ are equal to zero for $\mu = 0,1,\ldots,\nu-\kappa-1$, and so are the corresponding coefficients $a_{m,\mu}^{\nu,\kappa}$. Therefore, the eigenfunctions reduce to

$$g_m^{\nu,\kappa} = \sum_{\mu=\nu-\kappa}^{m} a_{m,\mu}^{\nu,\kappa} f_\mu \quad \text{and} \quad \widetilde{g}_m^{\nu,\kappa} = \sum_{\mu=\nu-\kappa}^{m} a_{m,\mu}^{\nu,\kappa} \widetilde{f}_\mu$$

in this case.

It was proved in [11, Lemma 8] that the eigenfunctions can be rewritten in the form

$$g_m^{\nu,\kappa} = \sum_{j=0}^{m} \alpha_{m,j}^{\nu,\kappa} h_j \quad \text{and} \quad \widetilde{g}_m^{\nu,\kappa} = \sum_{j=0}^{m} \alpha_{m,j}^{\nu,\kappa} \widetilde{h}_j\,,$$

where

$$\alpha_{m,j}^{\nu,\kappa} = \binom{m}{j}\frac{(m-\kappa)^{\underline{m-j}}}{(2m-\nu)^{\underline{m-j}}}\,, \quad h_j(\cdot) = \left(\frac{\cdot}{1-\cdot}\right)^j\,, \quad \text{and} \quad \widetilde{h}_j(\cdot) = (\cdot)^j\,.$$

In order to derive a Rodriguez-type formula for these eigenfunctions we prove a general result concerning the commutativity of the operators with certain differential operators, which is a generalization of [12, Theorem 2.3].

Theorem 2. *Let $\kappa < 1$, $\widetilde{w}^{\nu,\kappa}(\sigma) = \sigma^\kappa(1+\sigma)^{\nu-\kappa}$, $\widetilde{\varphi}(\sigma) = \sqrt{\sigma(1+\sigma)}$, and $\widetilde{f}$ be a function such that*

$$(1+\cdot)^{-(n+\nu-1)}\widetilde{f}(\cdot)\,,\ (1+\cdot)^{-(n+\nu-1)}\widetilde{\mathcal{D}}^m\widetilde{f}(\cdot) \in L_\infty(0,\infty)\,.$$

Then we have

$$\widetilde{\mathcal{D}}^m(B_{n+1}^{\nu,\kappa}\widetilde{f}) = B_{n+1}^{\nu,\kappa}(\widetilde{\mathcal{D}}^m\widetilde{f}) \tag{3}$$

where

$$\widetilde{\mathcal{D}}^m\widetilde{f}(\sigma) = \widetilde{w}^{\nu,\kappa}(\sigma)\,\frac{d^m}{d\sigma^m}\Big\{\frac{1}{\widetilde{w}^{\nu,\kappa}(\sigma)}\,\widetilde{\varphi}(\sigma)^{2m}\,\frac{d^m}{d\sigma^m}\,\widetilde{f}(\sigma)\Big\}\,.$$

Let $w^{\nu,\kappa}(x) = x^\kappa(1-x)^{-\nu}$, $\varphi(x) = \frac{\sqrt{x}}{1-x}$ and f be a function such that

$$(1-\cdot)^{n+\nu-1}f(\cdot)\,,\ (1-\cdot)^{n+\nu-1}w^{\nu,\kappa}(\cdot)\mathcal{D}^m f(\cdot) \in L_\infty(0,1)\,.$$

Then we have

$$\mathcal{D}^m(M_n^{\nu,\kappa}f) = M_n^{\nu,\kappa}(\mathcal{D}^m f)\,, \tag{4}$$

where

$$\mathcal{D}^m f(x) = w^{\nu,\kappa}(x)D^m\Big\{\frac{1}{w^{\nu,\kappa}(x)}\,\varphi(x)^{2m}D^m f(x)\Big\}$$

with

$$(Df)(x) = \frac{f'(x)}{\sigma'(x)} = (1-x)^2 f'(x)\,, \quad D^m f = D^{m-1}(Df)\,.$$

Proof. We only have to prove (3). The second statement (4) then follows by the relation between the operators $B_{n+1}^{\nu,\kappa}$ and $M_n^{\nu,\kappa}$ (see Remark 1).

Throughout the proof we will use the following identities which can be easily verified:

$$b_{n+1,k}^{(m)} = \frac{(n+m)!}{n!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\,b_{n+1+m,k-j}\,, \tag{5}$$

$$\frac{1}{\widetilde{w}^{\nu,\kappa}(\sigma)}\,\widetilde{\varphi}(\sigma)^{2m}b_{n+1+m,k}(\sigma)\,b_{n+1+\nu-m,k-\kappa+m}(\tau) = b_{n+1+\nu-m,k-\kappa+m}(\sigma)\,\frac{1}{\widetilde{w}^{\nu,\kappa}(\tau)}\,\widetilde{\varphi}(\tau)^{2m}b_{n+1+m,k}(\tau)\,, \tag{6}$$

$$\widetilde{w}^{\nu,\kappa}(\sigma)\,b_{n+1+\nu,k-\kappa}(\sigma)\,b_{n+1,k}(\tau) = b_{n+1,k}(\sigma)\widetilde{w}^{\nu,\kappa}(\tau)\,b_{n+1+\nu,k-\kappa}(\tau)\,, \tag{7}$$

where $b_{n+1,k} := 0$ for $k < 0$.

Using (5), shifting the indices $k \to k + j$, again using (5) and after this m-times integration by parts gives:

$$\begin{aligned}(B_{n+1}^{\nu,\kappa}\widetilde{f})^{(m)}(\sigma) &= \frac{(n+m)!}{n!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{k=j}^{\infty} b_{n+1+m,k-j}(\sigma) \\ &\quad \times (n+\nu)\int_0^\infty b_{n+1+\nu,k-\kappa}(\tau)\widetilde{f}(\tau)\,d\tau \\ &= \frac{(n+m)!}{n!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{k=0}^{\infty} b_{n+1+m,k}(\sigma) \\ &\quad \times (n+\nu)\int_0^\infty b_{n+1+\nu,k-\kappa+j}(\tau)\widetilde{f}(\tau)\,d\tau \\ &= \frac{(n+m)!}{n!}\sum_{k=0}^{\infty} b_{n+1+m,k}(\sigma) \\ &\quad \times (n+\nu)\int_0^\infty \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j} b_{n+1+\nu,k-\kappa+j}(\tau)\widetilde{f}(\tau)\,d\tau \\ &= \frac{(n+m)!\,(n+\nu-m)!}{n!\,(n+\nu)!}\sum_{k=0}^{\infty} b_{n+1+m,k}(\sigma)(n+\nu) \\ &\quad \times (-1)^m\int_0^\infty b^{(m)}_{n+1+\nu-m,k-\kappa+m}(\tau)\widetilde{f}(\tau)\,d\tau \\ &= \frac{(n+m)!\,(n+\nu-m)!}{n!\,(n+\nu)!}\sum_{k=0}^{\infty} b_{n+1+m,k}(\sigma) \\ &\quad \times (n+\nu)\int_0^\infty b_{n+1+\nu-m,k-\kappa+m}(\tau)\widetilde{f}^{(m)}(\tau)\,d\tau\,.\end{aligned}$$

From this, by using (6), we derive

$$\begin{aligned}&\frac{1}{\widetilde{w}^{\nu,\kappa}(\sigma)\,\widetilde{\varphi}(\sigma)^{2m}}\,(B_{n+1}^{\nu,\kappa}\widetilde{f})^{(m)}(\sigma) = \frac{(n+m)!\,(n+\nu-m)!}{n!\,(n+\nu-1)!} \\ &\quad \times \sum_{k=0}^{\infty} b_{n+1+\nu-m,k-\kappa+m}(\sigma)\int_0^\infty \frac{1}{\widetilde{w}^{\nu,\kappa}(\tau)}\,\widetilde{\varphi}(\tau)^{2m} b_{n+1+m,k}(\tau)\widetilde{f}^{(m)}(\tau)\,d\tau\,.\end{aligned}$$

Applying (5), shifting the indices $k \to k+j-m$, again using (5) and m-times integration by parts we get for the m-th derivative

$$\begin{aligned}
&\frac{d^m}{d\sigma^m}\Big\{\frac{1}{\widetilde{w}^{\nu,\kappa}(\sigma)}\,\widetilde{\varphi}(\sigma)^{2m}(B_{n+1}^{\nu,\kappa}\widetilde{f}(\tau))^{(m)}(\sigma)\Big\}\\
&= \frac{(n+m)!}{n!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{k=0}^{\infty} b_{n+1+\nu,k-\kappa+m-j}(\sigma)\\
&\quad \times (n+\nu)\int_0^\infty \frac{1}{\widetilde{w}^{\nu,\kappa}(\tau)}\,\widetilde{\varphi}(\tau)^{2m} b_{n+1+m,k}(\tau)\widetilde{f}^{(m)}(\tau)\,d\tau\\
&= \frac{(n+m)!}{n!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\sum_{k=0}^{\infty} b_{n+1+\nu,k-\kappa}(\sigma)\\
&\quad \times (n+\nu)\int_0^\infty \frac{1}{\widetilde{w}^{\nu,\kappa}(\tau)}\,\widetilde{\varphi}(\tau)^{2m} b_{n+1+m,k-m+j}(\tau)\widetilde{f}^{(m)}(\tau)\,d\tau\\
&= \frac{(n+m)!}{n!}\sum_{k=0}^{\infty} b_{n+1+\nu,k-\kappa}(\sigma)(n+\nu)\\
&\quad \times \int_0^\infty \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j} b_{n+1+m,k-m+j}(\tau)\frac{1}{\widetilde{w}^{\nu,\kappa}(\tau)}\,\widetilde{\varphi}(\tau)^{2m}\widetilde{f}^{(m)}(\tau)\,d\tau\\
&= \sum_{k=0}^{\infty} b_{n+1+\nu,k-\kappa}(\sigma)(n+\nu)(-1)^m\int_0^\infty b_{n+1,k}^{(m)}(\tau)\frac{1}{\widetilde{w}^{\nu,\kappa}(\tau)}\,\widetilde{\varphi}(\tau)^{2m} f^{(m)}(\tau)\,d\tau\\
&= \sum_{k=0}^{\infty} b_{n+1+\nu,k-\kappa}(\sigma)(n+\nu)\int_0^\infty b_{n+1,k}(\tau)\Big[\frac{1}{\widetilde{w}^{\nu,\kappa}(\tau)}\,\widetilde{\varphi}(\tau)^{2m}\widetilde{f}^{(m)}(\tau)\Big]^{(m)}\,d\tau\,.
\end{aligned}$$

Multiplying by $\widetilde{w}^{\nu,\kappa}(\sigma)$ and using (7) we get

$$\begin{aligned}
\widetilde{\mathcal{D}}^m(B_{n+1}^{\nu,\kappa}\widetilde{f}) &= \sum_{k=0}^{\infty} b_{n+1,k}(\sigma)(n+\nu)\int_0^\infty b_{n+1+\nu,k-\kappa}(\tau)\widetilde{\mathcal{D}}^m\widetilde{f}(\tau)\,d\tau\\
&= B_{n+1}^{\nu,\kappa}(\widetilde{\mathcal{D}}^m\widetilde{f})\,.
\end{aligned}$$

□

As we want to derive the desired Rodriguez-type formulas for the eigenfunctions by applying Theorem 2 to special functions, we need to know the image of the operators $B_{n+1}^{\nu,\kappa}$ for the monomials. The following is a generalization of [12, Lemma 2.4].

Lemma 1. *Let $e_m(\cdot)=(\cdot)^m$, $m\in\mathbb{N}_0$, and $m\le n+\nu-1$. Then we have*

$$(B_{n+1}^{\nu,\kappa}e_m)(\sigma)=\frac{(n+\nu-m-1)!}{(n+\nu-1)!}\sum_{j=0}^{m}\binom{m}{j}\frac{(n+j)!}{n!}(m-\kappa)^{\underline{m-j}}\sigma^j\,.$$

Proof. As

$$\int_0^\infty b_{n+1+\nu,k-\kappa}(\tau)\,\tau^m\,d\tau=\frac{(n+\nu-m-1)!}{(n+\nu)!}\,(m+k-\kappa)^{\underline{m}},$$

we get

$$(B_{n+1}^{\nu,\kappa}e_m)(\sigma) = \frac{(n+\nu-m-1)!}{(n+\nu-1)!}\sum_{k=0}^{\infty} b_{n+1,k}(\sigma)(m+k-\kappa)^{\underline{m}}.$$

Using the representation

$$(m+k-\kappa)^{\underline{m}} = \sum_{j=0}^{m}\binom{m}{j}(m-\kappa)^{\underline{m-j}}k^{\underline{j}},$$

which can be derived by inserting $x = k$ into the Newton interpolation polynomial of $p_m(x) = (m+x-\kappa)^{\underline{m}}$ with the knots $x_i = i$, we get after interchanging the order of summation

$$\begin{aligned}(B_{n+1}^{\nu,\kappa}e_m)(\sigma) &= \frac{(n+\nu-m-1)!}{(n+\nu-1)!}\sum_{j=0}^{m}\binom{m}{j}(m-\kappa)^{\underline{m-j}}\sum_{k=j}^{\infty} b_{n+1,k}(\sigma)\,k^{\underline{j}}\\ &= \frac{(n+\nu-m-1)!}{(n+\nu-1)!\,n!}\sum_{j=0}^{m}\binom{m}{j}(m-\kappa)^{\underline{m-j}}(n+j)!\,\sigma^{j},\end{aligned}$$

where we used the identities

$$b_{n+1,k}(\sigma)\,k^{\underline{j}} = \frac{(n+j)!}{n!}\,b_{n+1+j,k-j}(\sigma)$$

and

$$\sum_{k=j}^{\infty} b_{n+1+j,k-j}(\sigma) = \sum_{k=0}^{\infty} b_{n+1+j,k}(\sigma) = 1$$

in the last equality. □

The desired Rodriguez-type formulas now follow as a corollary of Theorem 2 and Lemma 1.

Corollary 1. *Let $m \in \mathbb{N}_0$, $m \le n+\nu-1$, and $\lambda_{n,m}^{\nu} = \frac{(n+m)^{\underline{m}}}{(n+\nu-1)^{\underline{m}}}$. Then*

$$\widetilde{q}_m^{\nu,\kappa}(\sigma) = \widetilde{w}^{\nu,\kappa}(\sigma)\,\frac{d^m}{d\sigma^m}\Big(\frac{1}{\widetilde{w}^{\nu,\kappa}(\sigma)}\,\widetilde{\varphi}(\sigma)^{2m}\Big) \tag{8}$$

is an eigenfunction of $B_{n+1}^{\nu,\kappa}$ with corresponding eigenvalue $\lambda_{n,m}^{\nu}$ and

$$q_m^{\nu,\kappa}(x) = w^{\nu,\kappa}(x)D^m\Big(\frac{1}{w^{\nu,\kappa}(x)}\,\varphi(x)^{2m}\Big) \tag{9}$$

is an eigenfunction of $M_n^{\nu,\kappa}$ with corresponding eigenvalue $\lambda_{n,m}^{\nu}$.

Proof. Again we have to prove only the first statement (8) since the second one, (9), follows from Remark 1.

Applying Theorem 2 to the special functions $\widetilde{f}_m(\cdot) = \frac{1}{m!}(\cdot)^m$, i.e., making use of the equality $\widetilde{f}_m^{(m)}(\cdot) = 1$, and using the relation

$$(B_{n+1}^{\nu,\kappa}\widetilde{f}_m)^{(m)}(\sigma) = \frac{(n+m)^{\underline{m}}}{(n+\nu-1)^{\underline{m}}},$$

based on Lemma 1, gives

$$\begin{aligned}\Big\{B_{n+1}^{\nu,\kappa}\Big[\widetilde{w}^{\nu,\kappa}(\tau)\Big(\frac{1}{\widetilde{w}^{\nu,\kappa}(\tau)}\,\widetilde{\varphi}(\tau)^{2m}\Big)^{(m)}\Big]\Big\}(\sigma)& \\ &= \widetilde{w}^{\nu,\kappa}(\sigma)\,\frac{d^m}{d\sigma^m}\Big\{\frac{1}{\widetilde{w}^{\nu,\kappa}(\sigma)}\,\widetilde{\varphi}(\sigma)^{2m}(B_{n+1}^{\nu,\kappa}\widetilde{f}_m)^{(m)}(\sigma)\Big\} \\ &= \widetilde{w}^{\nu,\kappa}(\sigma)\,\frac{d^m}{d\sigma^m}\Big\{\frac{1}{\widetilde{w}^{\nu,\kappa}(\sigma)}\,\widetilde{\varphi}(\sigma)^{2m}\Big\}\frac{(n+m)^{\underline{m}}}{(n+\nu-1)^{\underline{m}}}.\end{aligned}$$

□

Comparing the results of Corollary 1 with those of Theorem 1 we see that there is no further restriction to m in our corollary. So, one could think now that one has derived more eigenfunctions in Corollary 1. But this is not true, as except for a constant factor an appropriate number of the functions $\widetilde{q}_m^{\nu,\kappa}$ coincide, which will be proved in the next lemma. Of course, an analogous result holds true for the eigenfunctions $q_m^{\nu,\kappa}$ of $M_n^{\nu,\kappa}$.

Lemma 2. *For* $0 \le m \le \lfloor \frac{\nu}{2} - 1 \rfloor$ *we have*

$$\widetilde{q}_m^{\nu,\kappa} = \frac{m!}{(\nu-1-m)!}\,\widetilde{q}_{\nu-1-m}^{\nu,\kappa} \quad \textit{and} \quad q_m^{\nu,\kappa} = \frac{m!}{(\nu-1-m)!}\,q_{\nu-1-m}^{\nu,\kappa},$$

respectively.

Proof. For $0 \le m \le \lfloor \frac{\nu}{2} - 1 \rfloor$ we derive, by applying Leibniz formula and the binomial formula,

$$\begin{aligned}\frac{d^{\nu-1-2m}}{d\sigma^{\nu-1-2m}}\Big\{\frac{1}{\widetilde{w}^{\nu,\kappa}(\sigma)}\,\widetilde{\varphi}(\sigma)^{2(\nu-1-m)}\Big\}& \\ &= \sum_{j=0}^{\nu-1-2m}\binom{\nu-1-2m}{j}\frac{(\nu-1-m)!}{(\nu-1-m-j)!}\,\sigma^{\nu-1-m-j} \\ &\quad\times(-1)^{\nu-1-2m-j}\frac{(\nu-1-m-j)!}{m!}\,(1+\sigma)^{m-\nu+j} \\ &= \frac{(\nu-1-m)!}{m!}\,\sigma^m(1+\sigma)^{m-\nu} \\ &\quad\times\sum_{j=0}^{\nu-1-2m}\binom{\nu-1-2m}{j}\sigma^{\nu-1-2m-j}(-1)^{\nu-1-2m-j}(1+\sigma)^j \\ &= \frac{(\nu-1-m)!}{m!}\,\sigma^m(1+\sigma)^{m-\nu}.\end{aligned}$$

Taking the m-th derivative and multiplying with $\widetilde{w}^{\nu,\kappa}(\sigma)$ leads to the proposition. □

In our last result we observe that the functions $\widetilde{q}_m^{\nu,\kappa}$ and $q_m^{\nu,\kappa}$ coincide with $\widetilde{g}_m^{\nu,\kappa}$ and $g_m^{\nu,\kappa}$, respectively, except for a constant factor.

Lemma 3. *For* $0 \le m \le \frac{\nu-1}{2}$ *or* $m \ge \nu$ *we have*

$$\widetilde{q}_m^{\nu,\kappa} = (2m-\nu)^{\underline{m}}\, \widetilde{g}_m^{\nu,\kappa}\,, \quad q_m^{\nu,\kappa} = (2m-\nu)^{\underline{m}}\, g_m^{\nu,\kappa}\,.$$

Proof. By Leibniz formula, applying the binomial formula to $\sigma^{m-j} = (1+\sigma-1)^{m-j}$, shifting the indices $\mu \to \mu - j$, and interchanging the order of summation, we get

$$\begin{aligned}
&\widetilde{q}_m^{\nu,\kappa}(\sigma) \\
&= \sigma^\kappa (1+\sigma)^{\nu-\kappa} \left\{ \sigma^{m-\kappa} (1+\sigma)^{m-\nu+\kappa} \right\}^{(m)} \\
&= \sum_{j=0}^{m} \binom{m}{j} (m-\kappa)^{\underline{j}}\, (m-\nu+\kappa)^{\underline{m-j}}\, \sigma^{m-j} (1+\sigma)^j \\
&= \sum_{j=0}^{m} \binom{m}{j} (m-\kappa)^{\underline{j}}\, (m-\nu+\kappa)^{\underline{m-j}} \sum_{\mu=0}^{m-j} \binom{m-j}{\mu} (-1)^{m-j-\mu} (1+\sigma)^{\mu+j} \\
&= \sum_{j=0}^{m} \binom{m}{j} (m-\kappa)^{\underline{j}}\, (m-\nu+\kappa)^{\underline{m-j}} \sum_{\mu=j}^{m} \binom{m-j}{\mu-j} (-1)^{m-\mu} (1+\sigma)^{\mu} \\
&= \sum_{\mu=0}^{m} (-1)^{m-\mu} (1+\sigma)^{\mu} \sum_{j=0}^{\mu} \binom{m}{j} \binom{m-j}{\mu-j} (m-\kappa)^{\underline{j}}\, (m-\nu+\kappa)^{\underline{m-j}}. \qquad (10)
\end{aligned}$$

Using the Vandermonde convolution (see e.g. [7, (3.1)]), we get for the inner sum of (10)

$$\begin{aligned}
&\sum_{j=0}^{\mu} \binom{m}{j} \binom{m-j}{\mu-j} (m-\kappa)^{\underline{j}}\, (m-\nu+\kappa)^{\underline{m-j}} \\
&\quad = \frac{m!\,(m-\nu+\kappa)^{\underline{m-\mu}}}{(m-\mu)!} \sum_{j=0}^{\mu} \frac{(m-\kappa)^{\underline{j}}\, (\mu-\nu+\kappa)^{\underline{\mu-j}}}{j!\,(\mu-j)!} \\
&\quad = \frac{m!\,(m-\nu+\kappa)^{\underline{m-\mu}}\, (m+\mu-\nu)^{\underline{\mu}}}{(m-\mu)!\,\mu!} \\
&\quad = \binom{m}{\mu} (m-\nu+\kappa)^{\underline{m-\mu}}\, (m+\mu-\nu)^{\underline{\mu}}.
\end{aligned}$$

Inserting this in (10) leads to

$$\widetilde{q}_m^{\nu,\kappa}(\sigma) = \sum_{\mu=0}^{m} a_{m,\mu}^{\nu,\kappa}\, \widetilde{f}_\mu(\sigma) (2m-\nu)^{\underline{m-\mu}}\, (m+\mu-\nu)^{\underline{\mu}}.$$

Observing that $(2m-\nu)^{\underline{m-\mu}}\, (m+\mu-\nu)^{\underline{\mu}} = (2m-\nu)^{\underline{m}}$ we have proved our proposition. □

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