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# Sharp Inequalities for Hermite Polynomials

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We establish the following inequality for the orthonormal Hermite polynomials $\mathbf{H}_k(x)$ of degree $k \geq 1$ on the real line,

$$\frac{1}{4} < \max_x \left\{ k^{1/6} e^{-x^2} (\mathbf{H}_k(x))^2 \right\} < \frac{3}{2}\,.$$

The result is a corollary of new pointwise estimates of the Hermite polynomials in the oscillatory region.

## 1. Introduction

All basic properties of the Hermite polynomials used in this paper can be found in [8]. We will use bold letters for the orthonormal case versus regular characters for Hermite polynomials in the standard normalization.

We set

$$\mathcal{M}_k = \max_x \left\{ e^{-x^2} (\mathbf{H}_k(x))^2 \right\}.$$

The aim of this paper is to improve and simplify the result of [3], where two explicit yet rather weak constants $C_1, C_2$ were given such that $C_1 < k^{1/6}\mathcal{M}_k < C_2$. The bounds of [3] were based on the pointwise estimate of Hermite polynomials in the oscillatory region obtained in [2]. Here we give a simple proof of the following theorem which will be used instead.

**Theorem 1.** *Let $k \geq 1$ and*

$$|x| < \sqrt{2k - \left(\frac{k}{2}\right)^{1/3}}. \tag{1}$$

*Then*

$$(\mathbf{H}_k(x))^2 e^{-x^2} < \frac{4(2k - x^2)}{\pi\left(2\,(2k - x^2)^{3/2} - \sqrt{2k}\right)}. \tag{2}$$

*Moreover,*

$$(\mathbf{H}_k(x))^2 e^{-x^2} > \frac{8k-4x^2-2}{\pi\left(2\left(2k-x^2\right)^{3/2}+\sqrt{2k}\right)}, \tag{3}$$

*in* $k+1$ *points interlacing with the zeros of* $\mathbf{H}_k(x)$.

The inequalities (2) and (3) are quite sharp. Namely, for $0<\epsilon<1$, and $x=\sqrt{2(1-\epsilon)k}$, their ratio is $1+O\left(\frac{\epsilon^{-3/2}}{k}\right)$; whereas for $x=\sqrt{2k-\left(\frac{k}{2}\right)^{1/3}(1-\epsilon)^{-1}}$, it is $O(\epsilon^{-1})$.

Using the pointwise estimates of Theorem 1 we deduce the following inequalities on the Hermite polynomials which are rather close to the asymptotical value

$$\lim_{k\to\infty} k^{1/6}\mathcal{M}_k = \frac{3^{2/3}\sqrt{2}}{\pi^2}\max A^2(x) \approx 0.406, \tag{4}$$

where $A(x)$ is the Airy function, but hold independently on the degree.

**Theorem 2.**

$$\frac{1}{4} < k^{1/6}\mathcal{M}_k < \frac{3}{2},$$

*provided* $k \geq 1$.

The method used in this paper is quite general and can be applied in some other cases. Similar but less precise results uniform in all the parameters involved for Jacobi and Laguerre polynomials were established in [1, 4, 5]. One should consult [6, 7] for general inequalities for exponential weights uniform in the degree.

## 2. Proofs

A real polynomial is called hyperbolic if it has only real zeros. In particular all orthogonal polynomials are hyperbolic.

It will be convenient to use variables $m=\sqrt{2k}$, $y=y(x)=\sqrt{m^2-x^2}$ along with $k$ and $x$. For a fixed $k$ we set $f(x)=\mathbf{H}_k(x)$, $t=f'/f$.

Let

$$g = a_n \prod_i^k (x-x_i)$$

be a hyperbolic polynomial. We define

$$V(g,x,c) = U(g-c\,g',x),$$

where

$$U(g,x) = V(g,x,0) = (g')^2 - gg'' = g^2 \sum_{i=1}^{k} \frac{1}{(x-x_i)^2} \geq 0,$$

is the well-known Laguerre inequality.

Since $g - cg'$ is also a hyperbolic polynomial for any real value of $c$, we have

$$V(g,x,c) \geq 0.$$

The next two lemmas were established in [3].

**Lemma 1.** *Assume that the absolute maximum of the function* $(H_k(x))^2 e^{-x^2}$ *is attained for* $x = \omega$. *Then*

$$|\omega| < (\mu^4 - \frac{1}{3})^{3/2} \mu^{-3},$$

*where* $\mu = \left(k + \sqrt{k^2 + \frac{1}{27}}\right)^{1/6}$.

As an easy corollary we get

$$y^2(\omega) \geq m^{2/3} - \frac{1}{3m^{2/3}}. \tag{5}$$

Indeed, $k = \frac{27\mu^{12}-1}{54\mu^6}$, and

$$m^2 - \omega^2 - m^{2/3} + \frac{1}{3m^{2/3}} = \frac{\left((27\mu^{12}-1)^{1/3}+1\right)\left(3\mu^4 - (27\mu^{12}-1)^{1/3}\right)}{3\mu^2(27\mu^{12}-1)^{1/3}} \geq 0.$$

The following lemma gives rather precise bounds for the extreme zeros of $H_k$. The upper bound here is a classical result [8], whereas the lower one was obtained recently in [3].

**Lemma 2.** *For* $k \geq 2$ *the largest zero* $x_k$ *of* $H_k$ *satisfies*

$$\sqrt{2k} - \frac{9}{4}(2k)^{-1/6} < x_k < \sqrt{2k+1} - 6^{-1/3}(2k+1)^{-1/6} i_1,$$

*where* $i_1$ *is the least zero of Airy's function* $A(x)$, $6^{-1/3} i_1 = 1.8557\ldots$ .

To bound $\mathcal{M}_k$ we introduce the following functions

$$W_k(x) = \frac{(c^{-1}+x)V(f,x,c) + (c^{-1}-x)V(f,x,-c)}{2c},$$

where $c = (2k+1)^{-1/2}$.

Using the differential equation

$$f'' = 2xf' - 2kf,$$

we calculate

$$W_k(x) = \frac{\left(2xy^2 f - (2y^2-1)f'\right)^2}{2y^2-1} + \frac{4y^6 - m^2}{2y^2-1} f^2.$$

Thus, we have

$$f^2(x) \le \frac{2y^2-1}{4y^6-m^2} W_k(x), \tag{6}$$

provided $4y^6 - m^2 > 0$, that is for $x$ satisfying (1).

Let

$$z_1 = 2x - \frac{6xy}{2y^3-m}, \qquad z_2 = 2x + \frac{6xy}{2y^3+m},$$

then one finds

$$\frac{d}{dx} W_k(x) - z_1 W_k(x) = \frac{6x(m\sqrt{m+y}\, f - \sqrt{m-y}\, f')^2}{2y^3-m} \ge 0, \tag{7}$$
$$\frac{d}{dx} W_k(x) - z_2 W_k(x) = -\frac{6x(m\sqrt{m-y}\, f - \sqrt{m+y}\, f')^2}{2y^3+m} \le 0.$$

Suppose now that $x$ satisfies (1), thus $2y^3 > m$ and $W_k(x) > 0$. Integrating $\frac{W_k'}{W_k}$ we obtain by (7),

$$\int_0^x z_1\, dx \le \int_0^x \frac{W_k'}{W_k}\, dx \le \int_0^x z_2\, dx,$$

hence

$$x^2 + \ln\frac{2y^3-m}{2m^3-m} \le \ln\frac{W_k(x)}{W_k(0)} \le x^2 + \ln\frac{2y^3+m}{2m^3+m}.$$

Thus, we obtain

$$\frac{2y^3-m}{2m^3-m} W_k(0) \le W_k(x) e^{-x^2} \le \frac{2y^3+m}{2m^3+m} W_k(0). \tag{8}$$

To bound $W_k(0)$ we need the following elementary inequality.

**Lemma 3.**

$$\frac{1}{\sqrt{n+\frac{1}{3}}} < \sqrt{\pi}\binom{2n}{n} 4^{-n} < \frac{1}{\sqrt{n+\frac{1}{4}}}, \tag{9}$$

***Proof.*** It is easy to check that the function

$$F_c(n) = \sqrt{\pi(n+c)}\binom{2n}{n}4^{-n}$$

increases in $n$ for $c = \frac{1}{4}$, and decreases for $c = \frac{1}{3}$. Now the result follows by calculating

$$\lim_{n\to\infty} F_c(n) = 1$$

for a fixed value of $c$ by the Stirling formula. □

**Lemma 4.**

$$\frac{12m^3 - 2m}{3\pi} < W_k(0) < \frac{8m^4+1}{2\pi m}. \tag{10}$$

***Proof.*** Using

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}, \qquad H'_{2n+1}(0) = (-1)^n \frac{(2n+2)!}{(n+1)!},$$

and $\|H_k\|^2 = \sqrt{\pi}\, 2^k\, k!$, we find

$$W_k(0) = \begin{cases} \frac{2k(4k+1)}{\sqrt{\pi}} \binom{k}{k/2} 2^{-k}, & k \text{ even} \\ \frac{(4k-1)(k+1)}{\sqrt{\pi}} \binom{k+1}{(k+1)/2} 2^{-k}, & k \text{ odd.} \end{cases}$$

Now the result follows from (9) by straightforward calculations. We omit the details. □

***Proof of Theorem* 1.** One readily checks that all extremal points of $f^2 e^{-x^2}$ which are in the interval given by (5) satisfy (1).

Comparing (6), (8) and (10), we obtain

$$\pi(2y^3 - m)\, f^2 e^{-x^2} < \frac{(8m^4+1)(2y^2-1)}{2m^2(2m^2+1)} < 4y^2.$$

This proves (2). The proof of (3) is similar on observing that

$$f^2(x) = \frac{2y^2-1}{4y^6 - m^2} W_k(x),$$

for all $x$ satisfying the equation

$$2xy^2 f - (2y^2-1)f' = 0.$$

The last equation has a root between any two consecutive zeros of $f$ since the function $\frac{2xy^2}{2y^2-1}$ obviously intersects all the branches of $t = f'/f$ and all the roots satisfy (1). It is left to show that (1) holds for the leftmost and the rightmost intersection points. Indeed, calculating the derivative of $f^2e^{-x^2}$ we obtain that all its maxima occur for $t = x$. A quick inspection of the graphs of the functions $t(x)$, $\frac{2xy^2}{2y^2-1}$ and $x$ reveals that the extreme solutions of the equation $t = \frac{2xy^2}{2y^2-1}$ are embraced by these of $t = x$, by $x < \frac{2xy^2}{2y^2-1}$. Since the solutions of $t = x$ are restricted by the interval (1) the result follows. □

***Proof of Theorem* 2.** For $k = 1, 2$ the result follows by straightforward calculations. Assume now that $k \geq 3$. Since the right hand side of (2) decreases in $y$, we obtain by (2) and (5),

$$k^{1/6}\mathcal{M}_k < \frac{6\,(2k)^{1/3}\left(3(2k)^{2/3}-1\right)}{2^{1/6}\pi\left(\sqrt{3}\,\left(3\,(2k)^{2/3}-1\right)^{3/2}-9k\right)}. \tag{11}$$

It is easy to check that the last function decreases in $k$ for $k \geq 1$. Plugging in $k = 3$ we obtain

$$k^{1/6}\mathcal{M}_k < 1.4467... < \frac{3}{2}.$$

For the lower bound we observe that the function $\frac{4y^2-2}{\pi(2y^3+m)}$ decreases in $y$, provided $2y^3 - 3y - m \geq 0$. One can check that for $k \geq 2$ any

$$|x| \leq \sqrt{2k} - \frac{9}{4}\,(2k)^{-1/6}$$

satisfies this condition. Therefore, by Lemma 2 and (3), we obtain with

$$x = \sqrt{2k} - \frac{9}{4}\,(2k)^{-1/6}$$

that

$$k^{1/6}\mathcal{M}_k > \frac{8\kappa\,(72\kappa^2 - 8\kappa - 81)}{2^{1/6}\pi(32\kappa^3 + 27(8\kappa^2-9)^{3/2})} := \nu(\kappa),$$

where $\kappa = (2k)^{1/3}$. The sign of the derivative $\frac{d\nu}{d\kappa}$ coincides with the sign of

$$27(64\kappa^3 - 648\kappa^2 + 144\kappa + 729)\sqrt{8\kappa^2-9} + 64\kappa^3(4\kappa+81).$$

Investigation of the last function in the assumption $k \geq 3$ shows that $\nu(\kappa)$ attains the only minimum corresponding to $k = 273$ which is $0.252... > \frac{1}{4}$. This completes the proof. □

It is worth noticing that one could use the asymptotic value given by (4) as a lower bound in Theorem 2 provided the following conjecture is true.

**Conjecture 1.** *The function $k^{1/6}\mathcal{M}_k$ is decreasing for $k \geq 2$.*

Concerning the upper bound $\frac{3}{2}$ we notice that it can be easily improved by calculating a few more exact values of $k^{1/6}\mathcal{M}_k$ for $k \leq k_0$, and then substituting $k = k_0$ into (11). Yet, such an improvement is bounded by the limiting value of the RHS of (11) for $k \to \infty$ which is $\frac{2^{11/6}}{\pi} \approx \frac{8}{7}$.

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