CONSTRUCTIVE THEORY OF FUNCTIONS, Varna 2005
(B. Bojanov, Ed.), pp. 187-193
Marin Drinov Academic Publishing House, Sofia, 2006

Weighted Markov Inequalities on Infinite Intervals *

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We present exact Markov inequalities for weighted polynomials of the form $u(x) = e^{-x^2} p_n(x)$ or $v(x) = e^{-x} p_n(x)$, where $p_n(x)$ is an algebraic polynomial of degree at most n. The inequalities include estimates for the L_p-norm of the derivative of a weighted polynomial in terms of its supremum norm.

1. Introduction

We shall denote by π_n the set of all real algebraic polynomials p_n of degree at most n. Given an interval $I \subseteq \mathbb{R}$, we shall use the notations

$$\|f\|_{C(I)} := \sup_{x \in I} |f(x)|,$$
$$\|f\|_{L_p(I)} := \Big(\int_I |f(x)|^p dx \Big)^{1/p}, \quad 1 \le p < \infty,$$

for the supremum norm and the L_p norm on I, respectively. A polynomial f from π_n is oscillating in (a, b) if f has n simple zeros in (a, b). Let $\mathcal{P}_n$ be the subset of π_n which consists of the oscillating polynomials in $(-1, 1)$.

Every $f \in \mathcal{P}_n$ has $n+1$ extremal points in $[-1, 1]$. Let $h_j(f)$, $j = 0, \dots, n$, be the absolute values of the local extrema of f including the endpoints. According to a result of Davis [10] (see also [15] and [11]) they determine the polynomial uniquely (up to multiplication by -1). The following inheritance theorem was proved by Bojanov and Rahman [8].

Theorem A. *If f and g are polynomials from $\mathcal{P}_n$ and*

$$h_j(f) \le h_j(g), \quad j = 0, \dots, n,$$

*Research was supported by the Sofia University Science Foundation under Contract 121/2005.

then for every $k = 1, \dots, n$,

$$h_j(f^{(k)}) \le h_j(g^{(k)}), \quad j = 0, \dots, n-k. \tag{1}$$

Moreover, all the inequalities in (1) *are strict, unless $h_j(f) = h_j(g)$ for $j = 0, \dots, n$.*

A consequence of Theorem A is that the supremum norm of the k-th derivative of a polynomial f from $\mathcal{P}_n$ is a strictly increasing function of $h_0(f), \dots, h_n(f)$, which in turn implies V. Markov's inequality in $\mathcal{P}_n$.

The next result from [8] shows that other important functionals in $\mathcal{P}_n$ also depend monotonically on $\{h_j(f)\}_0^n$. Let us denote by Φ the class of all continuously differentiable, strictly increasing convex functions on $[0, \infty)$.

Theorem B. *Let $\varphi \in \Phi$. Then for every $f \in \mathcal{P}_n$ and $k = 1, \dots, n$ the integral*

$$I(f) = \int_{-1}^{1} \varphi\big(|f^{(k)}(x)|\big)\, dx$$

is a strictly increasing function of $h_0(f), \dots, h_n(f)$.

In particular, if $\varphi(x) = x^p$ $(1 \le p < \infty)$ then $I(f) = \|f^{(k)}\|^p_{L_p[-1,1]}$. The following inequality is a corollary of Theorem B.

If $\varphi \in \Phi$ and $f \in \mathcal{P}_n$, then

$$\int_{-1}^{1} \varphi\big(|f^{(k)}(x)|\big)\, dx \le \int_{-1}^{1} \varphi\big(\|f\|_{C[-1,1]}\, |T_n^{(k)}(x)|\big)\, dx, \quad k = 1, \dots, n. \tag{2}$$

The equality is attained if and only if $f(x) = \pm\|f\|\, T_n(x)$, where $T_n(x) = \cos(n \arccos x)$.

In fact, Theorems A and B were proved in [8] in a more general setting, treating polynomials with multiple zeros. Then the inequality (2) remains valid, but the extremal polynomial is the generalized Chebyshev polynomial, defined in [1].

In the case $k = 1$ the above inequality was proved by Bojanov [3, 4] even in the class π_n, without any restrictions on the zeros. This is a consequence of the following

Theorem C. *Let $\varphi \in \Phi$. Then, for each natural number n, the quantity*

$$\sup\left\{ \int_{-1}^{1} \varphi\big(|f'(x)|\big)\, dx \;:\; f \in \pi_n, \|f\|_{C[-1,1]} \le 1 \right\}$$

is attained if and only if $f = \pm T_n$.

The method developed in [2], [3], [4], and [8] was used to solve various extremal problems, concerning estimation of a derivative of a function from a given class of functions. We refer to the papers [5], [12], [7], and the recent survey paper [6].

In [12] one of the authors obtained results of the type of Theorem A for oscillating weighted polynomials of the form $e^{-x^2}p_n(x)$ or $e^{-x}p_n(x)$.

In Section 2 we present the main results of the paper [13], where we proved theorems of the type of Theorem B for the same classes of oscillating weighted polynomials.

Section 3 contains a summary of our recent paper [14], where we established a theorem of the type of Theorem C for weighted polynomials of the form $e^{-x}p_n(x)$.

2. Markov Inequalities in Integral Norm for Oscillating Weighted Polynomials

Let $\mathcal{U}_n$ be the set of all weighted polynomials of the form $u(x) = e^{-x^2}p_n(x)$, where $p_n \in \pi_n$ is an oscillating polynomial in $(-\infty, \infty)$. We shall denote by Ψ the class of all functions $\psi \in C^1[0,\infty) \cap C^2(0,\infty)$, which are convex and strictly increasing on $[0,\infty)$ and satisfy $\psi(0) = 0$.

We first quote two results from [12].

Theorem D. *Given positive numbers $h_0, \dots, h_n$, there exists a unique $u \in \mathcal{U}_n$ and a unique set of points $t_0 < \cdots < t_n$ such that*

$$\begin{aligned} u(t_k) &= (-1)^{n-k}h_k, \quad k = 0, \dots, n, \\ u'(t_k) &= 0, \quad k = 0, \dots, n. \end{aligned} \tag{3}$$

Since every $u \in \mathcal{U}_n$ has exactly $n+1$ extremal points $t_0 < \cdots < t_n$, Theorem D shows that the parameters $h_j(u) := |u(t_j)|$, $j = 0, \dots, n$, determine u uniquely (up to multiplication by -1).

Theorem E. *Let u_1 and u_2 be polynomials from $\mathcal{U}_n$. Suppose that*

$$0 < h_j(u_1) \le h_j(u_2), \quad j = 0, \dots, n.$$

Then for every natural number k, the inequalities

$$0 < h_j(u_1^{(k)}) \le h_j(u_2^{(k)}), \quad j = 0, \dots, n+k, \tag{4}$$

hold. In particular,

$$\|u_1^{(k)}\| \le \|u_2^{(k)}\|. \tag{5}$$

The equality in (4) (for some j) and (5) is attained if and only if $h_i(u_1) = h_i(u_2)$ for all $i = 0, \dots, n$.

Consequently, the supremum norm of the k-th derivative of a polynomial from $\mathcal{U}_n$ is a strictly increasing function of $h_0, \dots, h_n$.

Given $\mathbf{h} = (h_0, \dots, h_n)$, where $h_j > 0$, $j = 0, \dots, n$, we shall denote by $u(\mathbf{h}; \cdot)$ the unique solution of (3).

Lemma 1. *Let $\psi \in C^1[0,\infty) \cap C^2(0,\infty)$ be a convex and increasing on $[0,\infty)$ function such that $\psi(0) = \psi'(0) = 0$. Then the integral*

$$I(h_0, \dots, h_n) := \int_{-\infty}^{\infty} \psi(|u'(\mathbf{h}; x)|)\, dx$$

is an increasing function of every argument h_j, $j = 0, \dots, n$, in the domain $h_0 > 0, \dots, h_n > 0$. Moreover, if ψ is strictly increasing, then $I(h_0, \dots, h_n)$ is strictly increasing too.

In the next lemma we continue the investigation of the integral $I(h_0, \dots, h_n)$, which was defined in Lemma 1.

Lemma 2. *Let $\psi \in \Psi$. Then $I(h_0, \dots, h_n)$ is a strictly increasing function of every argument h_j, $j = 0, \dots, n$, in the domain $h_0 > 0, \dots, h_n > 0$.*

Theorem 1. *Let $\psi \in \Psi$. Then for every $u \in \mathcal{U}_n$ and every natural number k the integral*

$$I_k(u) = \int_{-\infty}^{\infty} \psi(|u^{(k)}(x)|)\, dx$$

is a strictly increasing function of $h_0(u), \dots, h_n(u)$.

Let $u_{*,n}$ be the weighted polynomial from $\mathcal{U}_n$ with positive leading coefficient and satisfying $h_j(u_{*,n}) = 1$, $j = 0, \dots, n$. Setting $\psi(t) = t^p$, $1 \le p < \infty$ in Theorem 1, we get the following exact Markov-type inequality in $L_p(\mathbb{R})$ for polynomials from $\mathcal{U}_n$.

Corollary 1. *For every $u \in \mathcal{U}_n$, $k \in \mathbb{N}$, and $1 \le p < \infty$, the inequality*

$$\|u^{(k)}\|_{L_p(\mathbb{R})} \le \|u_{*,n}^{(k)}\|_{L_p(\mathbb{R})}\, \|u\|_{C(\mathbb{R})}$$

holds true. The equality is attained if and only if $u = c\,u_{*,n}$, $c = const.$

We obtain related results for the weight e^{-x} on $[0,\infty)$. Let us denote by $\mathcal{V}_n$ the set of all weighted polynomials of the form $v(x) = e^{-x} p_n(x)$, where $p_n(x)$ is an oscillating polynomial in $(0,\infty)$.

Every $u \in \mathcal{V}_n$ has exactly $n+1$ extremal points $0 =: t_0 < \cdots < t_n$ and the parameters $h_j(v) := |v(t_j)|$, $j = 0, \dots, n$, determine v uniquely (up to multiplication by -1).

Theorem 2. *Let $\psi \in \Psi$. Then for every $v \in \mathcal{V}_n$ and every natural number k the integral*

$$J_k(v) = \int_0^\infty \psi(|v^{(k)}(x)|)\,dx$$

is a strictly increasing function of $h_0(v), \dots, h_n(v)$.

Theorem 2 implies the exact Markov-type inequality in $L_p(\mathbb{R}_+)$ for polynomials from $\mathcal{V}_n$. Let $v_{*,n}$ be the weighted polynomial from $\mathcal{V}_n$, which has positive leading coefficient and satisfies $h_j(v_{*,n}) = 1,\ j = 0, \dots, n$.

Corollary 2. *For every $v \in \mathcal{V}_n$, $k \in \mathbb{N}$ and $1 \le p < \infty$ the inequality*

$$\|v^{(k)}\|_{L_p(\mathbb{R}_+)} \le \|v_{*,n}^{(k)}\|_{L_p(\mathbb{R}_+)}\, \|v\|_{C(\mathbb{R}_+)}$$

holds true. The equality is attained if and only if $v = c\,v_{*,n}$, $c = const.$

3. An Extension of the Markov Inequality for the Laguerre Weight

Let $V_n = \{e^{-x}p_n(x) : p_n \in \pi_n\}$. Recall that

$$\Psi = \{\psi \in C^1[0,\infty) \cap C^2(0,\infty) : \psi \text{ is convex and strictly increasing on } [0,\infty),\ \psi(0) = 0\}.$$

The Chebyshev polynomial from V_n will be denoted by $v_{*,n}$.

The main result in this section is the following

Theorem 3. *Let $\psi \in \Psi$. Then for every natural number n, the only solutions of the extremal problem*

$$\sup\left\{\int_0^\infty \psi(|v'(x)|)\,dx : v \in V_n,\ \|v\|_{C(\mathbb{R}_+)} \le 1\right\} \tag{6}$$

are $v = \pm v_{*,n}$.

Corollary 3. *For every $n \in \mathbb{N}$, $v \in V_n$, and $1 \le p < \infty$, the inequality*

$$\|v'\|_{L_p(\mathbb{R}_+)} \le \|v'_{*,n}\|_{L_p(\mathbb{R}_+)}\, \|v\|_{C(\mathbb{R}_+)} \tag{7}$$

holds true. The equality is attained if and only if $v = \pm v_{*,n}$.

Remark. In the case $p = \infty$ inequality (7) was proved by Carley, Li, and Mohapatra [9].

The proof of Theorem 3 is based on the next two lemmas. Let us denote by Ω_n the class of all $v \in V_n$ which satisfy the conditions:

(i) $\|v\|_{C(\mathbb{R}_+)} = 1$;

(ii) There exist an integer number m, $0 \le m \le n$, and points $0 =: x_0 < x_1 < \cdots < x_m$ such that $v(x_k) = (-1)^{n-k}$ for $k = 0, \dots, m$, where v is a strictly monotone function on each interval $[x_k, x_{k+1}]$, $k = 0, \dots, m$, $x_{m+1} := \infty$.

Lemma 3. *Let $\psi \in \Psi$. Then for every $v \in \Omega_n$ we have*

$$\int_0^\infty \psi(|v'(x)|)\,dx \le \int_0^\infty \psi(|v'_{*,n}(x)|)\,dx.$$

The equality is attained if and only if $v = v_{*,n}$.

Lemma 4. *Theorem 3 is true if $\psi \in \Psi$ and $\psi'(0) = 0$.*

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