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On the Markov Inequality for Tchebycheff Systems

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We describe conditions on a general Tchebycheff system of functions on a finite interval $[a, b]$ which secure that the generalized Tchebycheff polynomial is extremal function to the Markov type inequality

$$\|f'\|_{C[a,b]} \leq M\|f\|_{C[a,b]}.$$

We consider also some particular examples.

1. Introduction

For any positive integer n let π_n be the set of all real algebraic polynomials of degree not exceeding n. As usual, we denote by $T_n(x)$ the Tchebycheff polynomial of first kind

$$T_n(x) = \cos(n \arccos x), \quad |x| \leq 1.$$

As it is well-known, in the classical A. Markov inequality

$$\|f'\|_{C[-1,1]} \leq n^2\|f\|_{C[-1,1]}, \quad \forall f \in \pi_n, \tag{1}$$

the equality sign is attained only for $f = c\,T_n(x)$, $c =$const. This is equivalent to the statement that the supremum

$$M_n := \sup\left\{\|f'\|_{C[-1,1]} : f \in \pi_n \text{ and } \|f\|_{C[-1,1]} \leq 1\right\}$$

is attained only for $f = \pm T_n(x)$.

Let $[a, b]$ be a given interval. In the present note we consider a Tchebycheff system of functions on $[a, b]$,

$$U_n = \left\{u_0(x), u_1(x), \ldots, u_n(x)\right\}.$$

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Everywhere below we abbreviate $\|f\|_{C[a,b]}$ to $\|f\|$ and denote

$$\mathcal{U}_n := \operatorname{span} U_n,$$
$$\mathcal{U}_n^1 := \{u \in \mathcal{U}_n \,:\, \|u\| \le 1\}.$$

Also, supposing that $\mathcal{U}_n \subset C^1[a,b]$ we set

$$M(U_n) := \sup\{\|f'\| \,:\, f \in \mathcal{U}_n^1\}. \tag{2}$$

Let $u_*(U_n;[a,b];x)$, briefly u_*, be the *Tchebycheff polynomial for* U_n normalized with $||u_*|| = 1$ and $u_*(b) > 0$. That is, $u_* \in \mathcal{U}_n$ and there exist $n+1$ points $\eta_0 < \ldots < \eta_n$ in $[a,b]$, called *alternation points*, such that

$$u_*(\eta_i) = (-1)^{n-i}\|u_*\| = (-1)^{n-i}, \quad i = 0, \ldots, n.$$

It is known that such element in $\mathcal{U}_n$ exists and is unique (see, e.g. [6]).

Note that the alternation property is closely related to the supremum in (2). Indeed, if we consider the extremal problem of finding

$$\sup\{|f'(\xi)| \,:\, f \in \mathcal{U}_n^1\} \text{ with fixed } \xi \in [a,b],$$

then, according to Karlin and Studden [6], the supremum is attained for an element in $\mathcal{U}_n$ which has at least n alternation points. Actually, the number of alternation points of $u \in \mathcal{U}_n$ cannot exceed $n+1$ and it is $n+1$ only for $u = c\,u_*$, $c=$const. In a recent paper [3], Bojanov and the author have formulated some conditions that secure the extremality of u_* in (2). In the sequel, we recall the main result from [3]. It is assumed there that $\mathcal{U}_n \subset C^3[a,b]$, but this is not very restrictive condition because there exists a standard technique for smoothing Tchebycheff systems.

We shall say that U_n is an *Extended Tchebycheff system* (ET-system) of order k, if $U_n \subset C^k[a,b]$ and every non-zero linear combination $f(x) = \sum_{k=0}^{n} a_k u_k(x)$ has no more than n zeros in $[a,b]$, counted with the multiplicities up to order k.

We require the following five properties of U_n:

Property 1. $u_0(x) \equiv 1$.

Property 2. The functions $\{u_1'(x), \ldots, u_n'(x)\}$ *form an ET-system of order* 2 *on* $[a,b]$.

Property 3. If $w \in \mathcal{U}_n$ *has exactly* n *distinct zeros in* $[a,b]$*, then*

$$\{u_1'(x), \ldots, u_n'(x), w(x)\}$$

form an ET-system of order 2 *on* $[a,b]$.

Property 4. In the notations of Property 3,

$$\{u_1''(x), \ldots, u_n''(x), w'(x)\}$$

form an ET-system of order 2 *on* $[a,b]$.

To formulate the last condition we need to introduce a notion. We shall say that the function v has "opposite orientation" with respect to w if

$$v(a)w(a) < 0 \quad \text{and} \quad v(b)w(b) < 0.$$

The same notion will be used also when $w(a) = 0$ with $w(a)$ replaced by $-w'(a)$ and when $w(b) = 0$ with $w(b)$ replaced by $w'(b)$.

Property 5. Assume that the function $w_1 \in \mathcal{U}_n$ has exactly $n-1$ zeros in $[a,b]$ and the function

$$v(x) := w_1(x) + \lambda u'(x),$$

where $u \in \mathcal{U}_n$ and $\lambda \in \mathbb{R}$, has opposite orientation with respect to w_1. Then v has at most $n-1$ zeros, counting the multiplicities up to order 2.

Now we are prepared to formulate our main result in [3].

Theorem 1. *Assume that the system U_n consists of functions from $C^3[a,b]$ which satisfy Properties 1–5. Then, the Markov inequality holds, i.e.,*

$$\|u'\| \le \|u'_*\| \, \|u\| \quad \text{for every} \;\; u \in \mathcal{U}_n. \tag{3}$$

The equality is attained only for $u = C\,u_*$ with any constant C.

The proof is variational one, used in the case of algebraic polynomials (for higher derivatives), and developed gradually in the works of Markov [8], Bernstein [1], Duffin and Schaeffer [9], Tikhomirov [11], Shadrin [10], Bojanov [2] and other authors. Our approach in [3] follows closely that in [2].

2. Three Positive Examples

Exponential polynomials. For given numbers $\lambda_0 < \lambda_1 < \cdots < \lambda_n$, with $0 \in \{\lambda_j\}$, and a finite interval $[a,b]$ we consider

$$U_n := \left\{e^{\lambda_0 x}, e^{\lambda_1 x}, \dots, e^{\lambda_n x}\right\}.$$

It is easy to check that the system U_n satisfies Properties 1–5 and therefore the supremum (2) is attained for u_*.

Corollary 1. *Let $[a,b]$ be any given finite interval and $\lambda_0 < \lambda_1 < \cdots < \lambda_n$ with $0 \in \{\lambda_j\}$. Then the inequality*

$$\|f'\| \le \|u'_*(\{e^{\lambda_0 x}, e^{\lambda_1 x}, \dots, e^{\lambda_n x}\}; [a,b]; \cdot)\| \, \|f\|$$

holds for every exponential polynomial

$$f(x) = \sum_{k=0}^{n} a_k e^{\lambda_k x}.$$

The equality is attained only for $f = c\,u_*$, $c = const.$

Müntz polynomials. Consider the system of functions

$$U_n := \{1, x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_n}\}, \quad \text{where } \alpha_1 \geq 1,\ \alpha_{i+1} \geq \alpha_i + 1.$$

Using Descartes Rule it is easy to verify that U_n satisfy the assumptions of Theorem 1 on any fixed subinterval $[a, b]$ of $(0, \infty)$. Then, we obtain the following.

Corollary 2. *Assume that the sequence* $\{\alpha_i\}_0^n$ *is such that* $\alpha_0 = 0$ *and* $\alpha_{i+1} \geq \alpha_i + 1$ *for* $i = 0, \ldots, n-1$. *Let*

$$u_*(x) := u_*(\{1, t^{\alpha_1}, \ldots, t^{\alpha_n}\}; [a, b]; x)$$

be the generalized Tchebycheff polynomial for the system $\{1, t^{\alpha_1}, \ldots, t^{\alpha_n}\}$ *on the interval* $[a, b] \subset (0, \infty)$. *Then*

$$\|u'\| \leq \|u_*'\| \, \|u\|$$

for every $u \in \operatorname{span}\{1, t^{\alpha_1}, \ldots, t^{\alpha_n}\}$. *The equality is attained only for* $u = c\, u_*$, $c = const.$

More information about Markov's Inequality in the spaces of exponential and Müntz polynomials as well as rational functions with prescribed poles can be found in the book of Borwein and Erdely [4].

The space of splines. We denote by $S_r(\xi_1, \ldots, \xi_{n-r})$ the set of all splines of degree r with $n - r$ knots $\xi_1 < \cdots < \xi_{n-r}$. The system of functions

$$U_n := \{1, x, \ldots, x^r, (x - \xi_1)_+^r, \ldots, (x - \xi_{n-r})_+^r\},$$

is a basis of $S_r(\xi_1, \ldots, \xi_{n-r})$ and it is only a weak T-system on $[a, b]$ containing $\{\xi_i\}$. Then, in order to apply Theorem 1, we first approximate the spline functions using a standard smoothing technique based on the Gaussian transform (see, for example, [5]). For a given $\varepsilon > 0$, with each $u \in S_r(\xi_1, \ldots, \xi_{n-r})$ we associate the smooth function

$$G_\varepsilon[u](x) := \frac{1}{\sqrt{2\pi}\,\varepsilon} \int_{-\infty}^{\infty} e^{-(x-t)^2/2\varepsilon^2} u(t)\, dt.$$

For the system $G_\varepsilon[U_n]$ we verify Properties 1–5 and get that the corresponding Tchebycheff polynomial is extremal in the Markov inequality for this system. Then, by a careful limit passage when $\varepsilon \to 0$ we obtain

Corollary 3. *For every* $s \in \mathcal{U}_n := S_r(\xi_1, \ldots, \xi_{n-r})$,

$$\|s'\| \leq \|u_*'(\mathcal{U}_n; [a, b]; \cdot)\| \, \|s\|.$$

Moreover, (if $r > 1$*) the equality sign is attained if and only if* $s = c\, u_*$, $c = const.$

3. Two Negative Examples

The first example shows that the properties

(i) $u_0 \equiv 1$;

(ii) $\{u_1', \ldots, u_n'\}$ is an ET-system of any order on $[a, b]$;

(iii) $\{u_1'', \ldots, u_n''\}$ is an ET-system of any order on $[a, b]$,

do not imply that $u_*(U_n; [a, b]; .)$ is extremal polynomial in the Markov inequality for U_n.

Example 1. Let $[a, b] = [0, 1]$ and $0 < \xi_1 < \xi_2 < 1$ be fixed knots. Every spline from $S_1(\xi_1, \xi_2)$ is uniquely determined by the vector of its values at the points $0, \xi_1, \xi_2, 1$. Consider the system of functions $V_n = \{v_0, v_1, v_2\} \subset S_1(\xi_1, \xi_2)$, which correspond to the vectors $(1, 1, 1, 1)$, $(0, 1, \lambda, 0)$, $(1, 0, -\mu, 0)$.

Assuming that $0 < \mu < \lambda < 1$ it is easy to see that every linear combination of v_1 and v_2 has at most one local extremum. Then, $\{v_1', v_2'\}$ is a weak T-system and therefore V_n is a weak T-system too. With a sufficiently small $\varepsilon > 0$, let $U_n := G_\varepsilon[V_n]$.

Now, let us list some useful properties of the linear operator G_ε, (see [5]).

(a) $G_\varepsilon[1] \equiv 1$ (and thus $\|G_\varepsilon(u)\| \leq \|u\|$).

(b) $G_\varepsilon[u](x)$ is an infinitely differentiable function. Moreover, if $u^{(k)}(x)$ exists and is integrable, then

$$\frac{d^k}{dx^k} G_\varepsilon[u](x) = G_\varepsilon[u^{(k)}](x).$$

(c) For every $s \in S_r(\xi_1, \ldots, \xi_{n-r})$ we have

$$G_\varepsilon[s] \text{ tends uniformly to } s \text{ as } \varepsilon \to 0$$

on every finite interval.

(d) The kernel of G_ε is totally positive. This implies *the variation diminishing property*:

$$Z(G_\varepsilon[f]; \mathbb{R}) \leq S^-(f; \mathbb{R}),$$

where $Z(g; I)$ denotes the number of zeros of g, counting multiplicities, and $S^-(f; I)$ here is the number of sign changes of the function f on the interval I.

(e) G_ε transforms every weak T-system on $\mathbb{R}$ into an ET-system of any order.

From the properties of G_ε conditions (i) and (ii) follows immediately. Next, it is directly calculated that

$$\operatorname{span}\{u_1'', u_2''\} = \operatorname{span}\left\{ \frac{1}{\sqrt{2\pi}\,\varepsilon}\, e^{-(x-\xi_1)^2/2\varepsilon^2}, \frac{1}{\sqrt{2\pi}\,\varepsilon}\, e^{-(x-\xi_2)^2/2\varepsilon^2} \right\}.$$

So, $\{u_1'', u_2''\}$ is also an ET-system of any order.

But, if the distance $\Delta := \xi_2 - \xi_1$ is relatively smaller than ξ_1 and $1 - \xi_2$, then $v_* = v_0 - 2v_1$ has maximal slope at the interval (ξ_1, ξ_2) which equals $2(1-\lambda)/\Delta$. On the other hand, $\|v_2\| = 1$ and $v_2'(x) = \mu/\Delta$ on (ξ_1, ξ_2). Thus, choosing for example $\lambda = 3/4$ and $\mu = 1/2$, we get $\|v_2'\| > \|v_*'\|$. The similar inequality holds for u_2 and u_*, provided ε is sufficiently small.

Closely related to the considered problem is the following one:

Which properties of U_n imply that

$$\sup_{u \in \mathcal{U}_n^1} \|u'\| = \sup_{u \in \mathcal{U}_n^1} \max\{|u'(a)|, |u'(b)|\}. \tag{4}$$

Note that this property takes place for $\mathcal{U}_n = \pi_n$. Also, Properties 1 and 2 imply that the supremum in the right hand side is attained for $u_*(U_n, \cdot)$.

The next example shows that the Properties 1–5 all together do not imply (4).

Example 2. Consider the space $\mathcal{V}_n$ of parabolic splines with knots $\xi_1 < \dots < \xi_m$ in (a,b), $m = n - 2 \geq 5$. Then, the slightly smoothed space $\mathcal{U}_n = G_\varepsilon[\mathcal{V}_n]$ satisfy Properties 1–5 and, therefore, the both suprema in (4) are attained for u_*. But, fixing ξ_1 and ξ_m and choosing the middle knots sufficiently close, according to interlacing property between the zeros and knots of splines, we can make the middle extremal points of u_* very close to each other. Thus, we can obtain $\|u_*'\|$ as large as we want, while the values $|u_*'(a)|$ and $|u_*'(b)|$ remain bounded, so the equality in (4) does not hold.

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