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Polynomial Inequalities of Markov and Duffin-Schaeffer Type

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This article is a survey of some recent developments on Markov-type inequalities. Most of the results are on inequalities of the so-called Duffin and Schaeffer type. The latter may be viewed as comparison-type theorems of the following nature: inequalities between the absolute values of two polynomials of degree not exceeding n on a given set of $n+1$ points induce inequalities between some norms (uniform or L_2) of their derivatives.

1. Introduction

Throughout this paper, π_n will mean the class of all algebraic polynomials of degree not exceeding n, and π_n^r will consist of polynomials from π_n having only real coefficients. Unless otherwise specified, $\|\cdot\|$ will stand for the uniform norm in $[-1,1]$,

$$\|g\| := \max_{x\in[-1,1]} |g(x)|.$$

The classical inequality of the brothers Markov reads as follows:

Theorem A. *If $f \in \pi_n$ satisfies*

$$\|f\| \le 1, \tag{1.1}$$

then, for $k = 1,\dots,n$,

$$\|f^{(k)}\| \le \|T_n^{(k)}\|, \tag{1.2}$$

and the equality in (1.2) occurs if and only if $f = cT_n$, $|c| = 1$.

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Here and henceforth, T_n is the n-th Chebyshev polynomial of the first kind, which is defined by

$$T_n(x) = \cos(n \arccos x), \quad x \in [-1, 1].$$

Actually, Theorem A (as well as Theorem B below) was originally established for real-valued polynomials, i.e., for $f \in \pi_n^r$ only, but from argument due to S. Bernstein (see the proof of Theorem 2.5) its validity easily extends to $f \in \pi_n$.

The explicit formula for the constant in the right-hand side of (1.2) is

$$\|T_n^{(k)}\| = T_n^{(k)}(1) = \frac{n^2(n^2-1)\cdots(n^2-(k-1)^2)}{1 \cdot 3 \cdots (2k-1)}.$$

An interesting fact is that the impetus to this beautiful result was a question, asked by a man, who was not a mathematician. It was the famous Russian chemist D. I. Mendeleev, who raised in 1887 the following

Question: *What is the maximal magnitude of each of the coefficients of a quadratic* $p_0x^2 + p_1x + p_2$, *if its (uniform) deviation from zero in a given interval* $[a, b]$ *does not exceed* Δ*?*

Mendeleev's question was inspired by his attempts to establish empirically the dependence of the temperature of boiling of spirit solutions on their concentration. Mendeleev answered himself to this question, but, what is more important, his question motivated A. A. Markov [29] to obtain sharp upper bounds for the coefficients p_0, p_{n-1} and p_n of an algebraic polynomial

$$P(x) = p_0x^n + p_1x^{n-1} + \cdots + p_n, \tag{1.3}$$

if the uniform deviation of P in a given interval $[a, b]$ does not exceed Δ. Precisely, A. A. Markov found the exact upper bounds for the quantities $|P(c)|$ and $|P'(c)|$ ($c \in \mathbb{R}$), proving in this way the case $k = 1$ of Theorem A.

In 1892 V. A. Markov [30], the younger brother of A. A. Markov (being at that time a student at St Petersburg University, at age of 17), proved Theorem A for all k, $1 \le k \le n$. Regarding Mendeleev's question, V. A. Markov proved that if the polynomial in (1.3) satisfies $\|P\| \le 1$, then its coefficients satisfy

$$|p_0| \le 2^{n-1} \quad \text{(this is a result of P. L. Chebyshev (1854))},$$

$$|p_1| \le 2^{n-2}, \qquad |p_2| \le 2^{n-3}n, \qquad |p_3| \le 2^{n-4}(n-1),$$

and for $i > 1$,

$$|p_{2i}| \le 2^{n-2i-1}\frac{n(n-i-1)(n-i-2)\cdots(n-2i+1)}{i!},$$

$$|p_{2i-1}| \le 2^{n-2i-2}\frac{(n-1)(n-i-2)(n-i-3)\cdots(n-2i)}{i!}.$$

The above bounds for $\{p_i\}_0^n$ are sharp, and they are attained for $P = T_n$ or $P = T_{n-1}$ according as i is even or odd number. Equivalently, V. A. Markov found the least uniform deviation from zero in $[-1, 1]$ of the polynomials from π_n with unit coefficient of x^k $(1 \le k \le n)$, as well as the least deviating polynomials.

The original V. A. Markov's paper [30] runs over 110 pages, and it contains some brilliant ideas, which can find (and have found!) application to many other situations. V. Markov's paper is not an easy reading, and this motivated many authors to try to give new, simpler proofs. For exhaustive exposition on the story of the classical Markov inequality and its proofs given so far, we refer the reader to the nice survey of Shadrin [64].

Schur [61] studied Markov-type inequality for polynomials that satisfy additional restrictions. In particular, he proved the following result:

Theorem B. *If $f \in \pi_n$ satisfies* (1.1) *and $f(-1) = f(1) = 0$, then*

$$\|f'\| \le \|\overline{T}_n'\| = n \cot \frac{\pi}{2n}, \tag{1.4}$$

where $\overline{T}_n$ is the stretched Chebyshev polynomial

$$\overline{T}_n(x) := T_n\Big(x \cos \frac{\pi}{2n}\Big).$$

Moreover, the equality in (1.4) *occurs if and only if $f = c\overline{T}_n$, $|c| = 1$.*

One of the possible generalizations of Markov's inequality is to consider polynomials with *curved majorants*. Namely, instead of polynomials from π_n satisfying (1.1), one may consider the class of polynomials

$$\Omega_n(\varphi) := \big\{ f \in \pi_n : \ |f(x)| \le \varphi(x), \ x \in [-1, 1] \big\},$$

where $\varphi(x)$ is continuous and positive (or non-negative) function in $[-1, 1]$. The problem then is to find

$$M_{n,k,\varphi} := \sup_{f \in \Omega_n(\varphi)} \|f^{(k)}\|, \tag{1.5}$$

as well as the extremizers in (1.5), i.e., the polynomials from $\Omega_n(\varphi)$ for which the suppremum is attained. The inequality

$$\|f^{(k)}\| \le M_{n,k,\varphi}, \quad f \in \Omega_n(\varphi),$$

is called *Markov-type inequality for polynomials with a curved majorant φ.*

A natural candidate for extremizer in (1.5) is the so-called *snake-polynomial*, associated with φ. This is the unique polynomial $S_{n,\varphi} \in \Omega_n(\varphi) \cap \pi_n^r$ for which there exist $n+1$ points (*alternation points*) $1 \ge t_0 > t_1 > \cdots > t_n \ge -1$ such that

$$S_{n,\varphi}(t_i) = \begin{cases} \varphi(t_i), & i - \text{even} \\ -\varphi(t_i), & i - \text{odd}. \end{cases}$$

The uniqueness of $S_{n,\varphi}$ follows from standard zero-counting argument, while the existence was proved by Karlin [27, 28] even in a more general setting, e.g., for generalized polynomials in Chebyshev systems (T-systems), oscillating between two continuous functions $\varphi(x) \leq \psi(x)$, under the only additional assumption that there exists a polynomial in this T-system whose graph lies between those of φ and ψ. For further extensions the reader is referred to [52]. Notice that T_n is the snake polynomial corresponding to $\varphi \equiv 1$.

There are only few Markov-type inequalities for polynomials with curved majorants. The main contribution to this subject is due to Videnskii [68, 69, 70, 71], and to Q. Rahman and co-authors. In 1958 Videnskii [69] established such inequalities for the one-parametric family of majorants

$$\varphi(a;x) = \sqrt{1+(a^2-1)x^2}, \quad -1 \leq x \leq 1,$$

showing that the only (up to a constant multiplier of modulus one) extremizer is the snake-polynomial associated with $\varphi(a;x)$,

$$Q(x) = \frac{a+1}{2}\,T_n(x) + \frac{a-1}{2}\,T_{n-2}(x), \quad a \geq 0.$$

The particular case $a = 1$ reproduces Theorem A ($\varphi(x) \equiv 1$), while in the cases $a = 0$ and $a =$ "∞" one obtains Markov-type inequality for polynomials with a circular majorant $\varphi(x) = \sqrt{1-x^2}$ and with majorant $\varphi(x) = |x|$, respectively. Also, in [70] Videnskii proved Markov-type inequality for majorants of the form

$$\varphi(x) = \sqrt{(1+a^2x^2)(1+b^2x^2)},$$

while for

$$\varphi(x) = \sqrt{\prod_{\nu=1}^{m}(1+a_\nu^2x^2)}, \quad m > 2,$$

and

$$\varphi(x) = \sqrt{ax^2+bx+1}$$

he proved in [70] and [71], respectively, Markov-type inequalities only for $k = 1$, i.e., for the first derivative. To obtain these results, Videnskii utilized (and developed further) the method proposed by Schaeffer and Duffin in [59], where one of the simplest proofs of the Markov inequality was given.

Being unaware of Videnskii's results, Rahman [53] and Pierre and Rahman [48, 49] proved Markov-type inequality for polynomials with a circular majorant. In [49], Pierre and Rahman adopted V. Markov's method to obtain Markov-type inequality for majorants of the form $\varphi(x) = (1-x)^{m_1/2}(1+x)^{m_2/2}$, $m_1, m_2 \in \mathbb{N}$, for derivatives of order $k \geq (m_1+m_2)/2$. The missing case $k = 1$ for the particular choice $m_1 = m_2 = 2$ was settled by Pierre, Rahman and Schmeisser in [51].

In 1941 Duffin and Schaeffer [18] found the following remarkable extension of Theorem A:

Theorem C. *The conclusion of Theorem A remains true if* (1.1) *is replaced by the weaker assumption*

$$|f(\cos(\nu\pi/n))| \le 1 \quad \textit{for} \quad \nu = 0, \dots, n. \tag{1.6}$$

For polynomials with real coefficients Duffin and Schaeffer proved even more:

Theorem D. *If* $f \in \pi_n^r$ *satisfies* (1.6)*, then, for* $k = 1, \dots, n$,

$$|f^{(k)}(x+iy)| \le |T_n^{(k)}(1+iy)| \quad \textit{for every} \quad x \in [-1,1],\ y \in \mathbb{R}. \tag{1.7}$$

The equality in (1.7) *occurs if and only if* $f = \pm T_n$.

For the proof of this beautiful result of Duffin and Schaeffer, the reader may look also in [58], [60] and [56].

Duffin and Schaeffer showed also that Theorem C is no longer true if the set $\{\cos(\nu\pi/n)\}_0^n$ in (1.6) is replaced by any closed subset E of $[-1,1]$, such that $E \not\supset \{\cos(\nu\pi/n)\}_0^n$. The latter conclusion seems quite reasonable, as there are no points on the real line other than $\{\cos(\nu\pi/n)\}_0^n$, where $|T_n| = 1$, i.e., where the graph of the snake-polynomial T_n touches the graphs of $\pm\varphi(x)$, $\varphi(x) \equiv 1$ being the majorizing curve.

The above interpretation gives a good reason for expectation of the same phenomena for the case of curved majorants, namely, that restriction only on the set of alternation points secures Markov-type inequality. To the best of our knowledge, by 1992 there were only two results of such a nature. The first one is due to Rahman and Schmeisser [55], and settles the case of circular majorant, $\varphi(x) = \sqrt{1-x^2}$:

Theorem E. *Let* $Q(x) = (x^2-1)\,U_{n-2}(x)$. *If* $f \in \pi_n$ *satisfies the restriction*

$$|f(x)| \le \sqrt{1-x^2} \quad \textit{at the zeros of} \quad (x^2-1)\,T_{n-1}(x), \tag{1.8}$$

then, for $k = 2, \dots, n$,

$$\|f^{(k)}\| \le \|Q^{(k)}\|, \tag{1.9}$$

and

$$\|f'\| \le (n-1)\Big(\frac{2}{\pi}\log(n-1) + 3\Big).$$

The equality in (1.9) *occurs if and only if* $f = c\,Q$, $|c| = 1$.

Here and henceforth,

$$U_m(x) = \frac{\sin\left((m+1)\arccos x\right)}{\sqrt{1-x^2}}$$

is the m-th Chebyshev polynomial of the second kind.

The second result concerns the case of parabolic majorant $\varphi(x) = 1 - x^2$, and was established by Rahman and Watt [57]:

Theorem F. *Let* $Q(x) = (x^2-1)\,T_{n-2}(x)$. *If* $f(x) = (x^2-1)q(x)$, *where* $q \in \pi_{n-2}$ *satisfies*

$$|q(x)| \le 1 \quad \text{at the zeros of} \quad U_{n-3}(x), \tag{1.10}$$

then, for $k = 3, \dots, n$,

$$\|f^{(k)}\| \le \|Q^{(k)}\|.$$

The proof of both Theorems E and F relies on arguments similar to those used by Duffin and Schaeffer [18], and for those k postulated in the theorems, there is a complement for polynomials with real coefficients:

$$|f^{(k)}(x+iy)| \le |Q^{(k)}(1+iy)| \quad \text{for every} \quad x \in [-1,1], \; y \in \mathbb{R}.$$

Notice that in both Theorems E and F, Q is the snake-polynomial associated with the majorant φ, and the abscissae in the pointwise restrictions (1.8) and (1.10) are exactly the alternation points. We also observe a difference in the conclusions of those theorems, compared with the conclusions, if $|f(x)| \le \varphi(x)$, $x \in [-1,1]$, was assumed. Indeed, according to [53, 48, 49, 51], under the latter assumption the inequality $\|f^{(k)}\| \le \|Q^{(k)}\|$ holds also true for $k = 1$ in the case of circular majorant, and for $k = 2$ and $k = 1$, n - odd, in the case of parabolic majorant. The gap when (1.8) or (1.10) is assumed is not a proof-defect, it reflects the matter of things: in the missing cases of k, Q is not the extremal polynomial.

The lack of more results in the spirit of Theorems C, E, and F can be explained with the fact, that finding explicitly (for every n) the snake-polynomial associated with a given majorant $\varphi(x)$, is extremely difficult task; as it was shown by Bernstein [2], this can be done (to a certain extent), in the case $\varphi(x) = \sqrt{R_{2m}(x)}$, where R_{2m} is a non-negative in $[-1,1]$ polynomial of degree $2m$. On the other hand, although there are no counterexamples in literature, it is far not clear whether the snake-polynomial is always extremizer in Markov-type inequalities for polynomials with curved majorants.

Theorems C, E, and F are comparison-type theorems, they describe a typical situation: the assumption $|f| \le |Q|$ on a given set of $n+1$ points (in Theorem F, it is also actually assumed that $|f'| \le |Q'|$ at ± 1) induces the inequality $\|f^{(k)}\| \le \|Q^{(k)}\|$ for all k, or from some k onwards. This observation, as well as the above-mentioned difficulties with finding snake-polynomials motivated the author to propose in [37] the following definition:

Definition 1. Let Q be a polynomial of degree n, and $\Delta = \{t_\nu\}_{\nu=0}^{n}$, where $1 \geq t_0 > t_1 > \ldots > t_n \geq -1$. The pair (Q, Δ) is said to admit Duffin and Schaeffer-type (in short, DS-type) inequality, if for any $f \in \pi_n$, the assumption

$$|f| \leq |Q| \ \text{ at the points from } \ \Delta \tag{1.11}$$

implies for $k = 1, \ldots, n$,

$$\|f^{(k)}\| \leq \|Q^{(k)}\|. \tag{1.12}$$

If, in addition, Q satisfies $Q(-1) = Q(1) = 0$, then (Q, Δ) is said do admit Duffin-Schaeffer-Schur type (DSS-type) inequality.

This definition needs not to be restricted only to the $C[-1,1]$-norm, and then the restriction for the "check points" $\{t_\nu\}$ to lie in $[-1,1]$ can be dropped. Sometimes we shall say that a pair (Q, Δ) admits DS- or DSS-type inequality if (1.12) is established only for $k \geq 2$ or $k \geq 3$.

At this point, let us briefly discuss the problem of DS-type inequalities, and their relation to the Markov-type inequalities for polynomials with a curved majorant φ. First of all, with the DS-setting we avoid the necessity to search for the snake-polynomial associated with φ (as highly probable extremizer). Instead, we assign the role of majorant to a fixed polynomial Q. We choose also the points from Δ, which by (1.11) define a bounded set of polynomials $\Omega(Q, \Delta) \subset \pi_n$, with $Q \in \Omega(Q, \Delta)$. For Q and Δ suitably chosen, Q turns out to be the extremizer, i.e., $\|Q^{(k)}\| \geq \|f^{(k)}\|$ for every $f \in \Omega(Q, \Delta)$. Of course, the results quoted above provide some heuristic reasoning for the choice of Q and Δ: one may think that Q is the snake-polynomial and Δ is the set of its alternation points, associated with some (unknown) curve φ. This suggests that Q must be oscillating polynomial, i.e., having only real and simple zeros, and the zeros of Q must be separated by the points from Δ. On the other hand, if Q turns out to be extremizer in the DS-inequality for *every* choice of Δ with the latter property, then Q is the extremizer in *any* Markov-type inequality for polynomials with a curved majorant φ, provided that Q is the snake-polynomial, associated with φ. We shall show later that the Chebyshev polynomial T_n enjoys this property.

The rest of this paper is organized as follows: in the next section we present some pointwise inequalities for derivatives of polynomials, bounded on a discrete set of points. These results are employed in Section 3 for derivation of DS- and DSS-type inequalities in $C[-1,1]$-norm. In Section 4 we extend Theorem D to the class of ultraspherical polynomials. Section 5 contains certain DS-type inequalities in some weighted L_2-norms. In the last section we present some concluding remarks.

2. Pointwise Inequalities for Derivatives

The proof of a DS-type inequality requires pointwise estimation of the derivatives of a polynomial $f \in \pi_n$, satisfying the inequality

$$|f(t)| \le |Q(t)|, \quad t \in \Delta, \tag{2.1}$$

where Q is a given polynomial of degree n with only real zeros, which separate the points from Δ,

$$\Delta = \{t_\nu\}_{\nu=0}^{n}, \quad t_0 > t_1 > \cdots > t_n. \tag{2.2}$$

The sharp pointwise bound for polynomials satisfying (2.1) is readily deduced from the Lagrange interpolation formula:

$$\begin{aligned} |f^{(k)}(x)| &= \Big|\sum_{\nu=0}^{n} \frac{\omega_\nu^{(k)}(x)}{\omega_\nu(t_\nu)} f(t_\nu)\Big| \le \sum_{\nu=0}^{n} \Big|\frac{\omega_\nu^{(k)}(x)}{\omega_\nu(t_\nu)}\Big| |f(t_\nu)| \\ &\le \sum_{\nu=0}^{n} \Big|\frac{\omega_\nu^{(k)}(x)}{\omega_\nu(t_\nu)}\Big| |Q(t_\nu)| =: M_k(Q, \Delta; x). \end{aligned} \tag{2.3}$$

Here and elsewhere, $\omega(x) = (x - t_0)\cdots(x - t_n)$ is the monic polynomial with zeros defined by the check points, and $\omega_\nu(x) = \omega(x)/(x - t_\nu)$ for $\nu = 0, \dots, n$. Clearly, the bound $M_k(Q; x)$ is attainable, but to specify precisely those polynomials for which $|f^{(k)}(x)| = M_k(Q, \Delta; x)$, we need some results from V. Markov's paper [30]. We start with a definition:

Definition 2. Let p and q be polynomials having only real and simple zeros. The zeros of p and q are said to interlace if one can trace all the zeros of p and q, switching from a zero of p to a zero of q and vice versa, and moving only in one direction. If, in addition, no zero of p coincides with a zero of q, then the zeros of p and q are said to interlace strictly.

Clearly, interlacing is only possible if either p and q are of the same degree, or of degrees, which differ by one. In the latter case, if the zeros of p interlace strictly with the zeros of q and $\deg p < \deg q$, we shall also say that *the zeros of p separate the zeros of q*. The first V. Markov's lemma reads as follows:

Lemma 2.1. *If the zeros of two polynomials p and q interlace, $p \ne q$, then the zeros of p' and q' interlace strictly.*

Clearly, Lemma 2.1 extends to higher derivatives. It also implies that the zeros of the derivative of a polynomial p with only real and simple zeros are strictly monotone increasing functions of each zero of p.

There are many proofs of this simple, and, as a matter of fact, very useful lemma (see, e.g., [58, Lemma 2.7.1], [62], or [68] for a proof of the same property for more general systems of functions (this result is given also in [64, Lemma 4.8]), and [6]).

We shall describe below an important consequence of Lemma 2.1, which also goes back to V. Markov [30] (see also [62], [25], [64]). For any pair of indices (i,j), $1 \le i < j \le n$, the zeros of ω_i and ω_j interlace (not strictly), the zeros of ω_i being smaller than or equal to the corresponding zeros of ω_j. According to Lemma 2.1, for $k = 1,\dots,n-1$, the interlacing (and the same arrangement) is inherited by the zeros of $\omega_i^{(k)}$ and $\omega_j^{(k)}$. More precisely, if for $i = 1,\dots,n$, $\{\gamma_{\nu,i}\}_{\nu=1}^{n-k}$ are the zeros of $\omega_i^{(k)}$, labeled in ascending order, we have the following arrangement:

$$\gamma_{1,0} < \gamma_{1,1} < \cdots < \gamma_{1,n} < \gamma_{2,0} < \cdots < \gamma_{n-k,0} < \cdots < \gamma_{n-k,n}\,. \tag{2.4}$$

For $k \in \{1,2,\dots,n-1\}$, let $I_{n,k}(\omega) = I_{n,k}(\Delta)$ be defined by

$$I_{n,k}(\Delta) := (-\infty,\gamma_{1,0}] \cup_{\nu=1}^{n-k-1} [\gamma_{\nu,n},\gamma_{\nu+1,0}] \cup [\gamma_{n-k,n},\infty). \tag{2.5}$$

This definition naturally extends in the case $k = n$ by $I_{n,n}(\Delta) = \mathbb{R}$. The $n-k+1$ non-overlapping intervals, forming $I_{n,k}(\Delta)$, are called *Chebyshev intervals*, while the complementary intervals $\{(\gamma_{\nu,0},\gamma_{\nu,n})\}_{\nu=1}^{n-k}$ are referred to as *Zolotarev intervals*. For the convenience sake, we set

$$J_{n,k}(\Delta) := \cup_{\nu=1}^{n-k}(\gamma_{\nu,0},\gamma_{\nu,n}). \tag{2.6}$$

Lemma 2.2. *The set $I_{n,k}(\Delta)$ possesses the following properties:*

(i) A point $t \in \mathbb{R}$ belongs to $I_{n,k}(\Delta)$ if and only if $\omega_0^{(k)}(t)\,\omega_n^{(k)}(t) \ge 0$. Moreover, t is an interior point for $I_{n,k}(\Delta)$ if and only if $\omega_0^{(k)}(t)\,\omega_n^{(k)}(t) > 0$.

(ii) Let Q be a polynomial of degree n, whose zeros separate the points from Δ. If $f \in \pi_n$ satisfies (2.1), then

$$|f^{(k)}(x)| \le |Q^{(k)}(x)| = M_k(Q,\Delta;x) \quad \textit{for every} \quad x \in I_{n,k}(\Delta). \tag{2.7}$$

(iii) If x is an interior point for $I_{n,k}(\Delta)$ and $f \in \pi_n$ satisfies (2.1), then the inequality in (2.7) is strict unless $f = c\,Q$, $|c| = 1$.

Part (i) follows from the fact that the zeros of $\omega_0^{(k)}$ and $\omega_n^{(k)}$ are the endpoints of the Chebyshev intervals, and $\omega_0^{(k)}$ and $\omega_n^{(k)}$ have the same number of zeros located to the right from any fixed Chebyshev interval. Precisely, it follows from (2.4) that each interior point t for $I_{n,k}(\Delta)$ is characterized by the property

$$\operatorname{sign}\omega_0^{(k)}(t) = \operatorname{sign}\omega_1^{(k)}(t) = \cdots = \operatorname{sign}\omega_n^{(k)}(t). \tag{2.8}$$

This observation and the inequalities $Q(t_\nu)\,Q(t_{\nu+1}) < 0$, $\omega(t_\nu)\,\omega(t_{\nu+1}) < 0$ for $\nu = 0,\dots,n$ imply that, the substitution $f = Q$ in (2.3) turns all the inequalities therein into equalities, thus proving part (ii). Finally, part (iii) is settled by a careful inspection of the equality cases in the triangle inequality.

Lemma 2.2 shows that if the zeros of the oscillating polynomial Q separate the points from Δ, then Q is extremal on $I_{n,k}(\Delta)$, i.e., $|Q^{(k)}(x)| = M_k(Q,\Delta;x)$ for every $x \in I_{n,k}(\Delta)$. Moreover, at the interior points for $I_{n,k}(\Delta)$, the only extremal polynomials are of the form $cQ(x)$, where $c = e^{i\theta}$ and $\theta \in \mathbb{R}$. There is no uniqueness of the extremizer when x is a boundary points for $I_{n,k}(\Delta)$, as it is seen from (2.3) that in this case $f^{(k)}(x)$ does not depend on $f(t_0)$ or on $f(t_n)$.

The oscillating polynomial Q is not extremal on the supplementary set $J_{n,k}(\Delta)$ (notice that $J_{n,n}(\Delta) = \emptyset$). In order to describe the polynomials from π_n, which satisfy (2.1) and realize the sharp upper bound $M_k(Q,\Delta;x)$ for $x \in J_{n,k}(\Delta)$, $1 \le k \le n-1$, we consider the following n open subsets of $J_{n,k}(\Delta)$:

$$J^{\ell}_{n,k}(\Delta) = \cup_{i=1}^{n-k}(\gamma_{i,\ell},\gamma_{i,\ell+1}), \quad \ell = 0,\dots,n-1.$$

If $t \in J^{\ell}_{n,k}(\Delta)$ for some $\ell \in \{0,\dots,n-1\}$, then, instead of (2.8), the arrangement (2.4) yields

$$\begin{aligned}\operatorname{sign}\omega_0^{(k)}(t) &= \operatorname{sign}\omega_1^{(k)}(t) = \cdots = \operatorname{sign}\omega_\ell^{(k)}(t)\\ &= -\operatorname{sign}\omega_{\ell+1}^{(k)}(t) = \cdots = -\operatorname{sign}\omega_n^{(k)}(t).\end{aligned} \tag{2.9}$$

Looking back at (2.3) and taking into account that $Q(t_\nu)\,Q(t_{\nu+1}) < 0$ and $\omega(t_\nu)\,\omega(t_{\nu+1}) < 0$ for $\nu = 0,\dots,n$, we see that, for $t \in J^{\ell}_{n,k}(\Delta)$, the only polynomials $f \in \pi_n$ satisfying (2.1), for which $|f^{(k)}(t)| = M_k(Q,\Delta;t)$ are of the form $f = c\,P_\ell$, where $c = e^{i\theta}$, $\theta \in \mathbb{R}$, and $P_\ell \in \pi_n^r$ is determined by the condition

$$P_\ell(t_i) = \begin{cases} Q(t_i), & i = 0,\dots,\ell\\ -Q(t_i), & i = \ell+1,\dots,n.\end{cases}$$

The same polynomials remain extremal in the case when $t = \gamma_{i,\ell}$ or $t = \gamma_{i,\ell+1}$, but they are not the only extremizers, as evidently in this case their value at the point t_ℓ or $t_{\ell+1}$, respectively, can be chosen arbitrarily.

Since

$$I_{n,k}(\Delta) \cup_{\ell=0}^{n-1} \overline{J^{\ell}_{n,k}(\Delta)} = \mathbb{R},$$

the search for $f \in \pi_n$, which satisfy (2.1) and $|f^{(k)}(x)| = M_k(Q,\Delta;x)$ for some $x \in \mathbb{R}$ can be restricted to the set

$$\{Q, P_0, \dots, P_{n-1}\}.$$

In other words, one may consider only those polynomials $f \in \pi_n^r$, for which $|f(t_\nu)| = |Q(t_\nu)|$ for $\nu = 0,\dots,n$, and $f(t_\nu)f(t_{\nu+1}) < 0$ either for all ν or except for one ν.

Although the polynomials $\{P_\nu\}_{\nu=0}^{n-1}$ (together with the oscillating polynomial Q) are enough for evaluating $M_{n,k}(Q,\Delta;x)$ whenever $x \in \mathbb{R}$, they are not a convenient tool to work with, as there is no simple expression, relating Q and P_ν. The next theorem provides some more appropriate substitutes. For the simplicity sake, by the end of this section, unless explicitly otherwise stated, Q and Δ will be supposed to satisfy the following assumptions:

Assumption A: $Q(x) = \alpha(x-x_1)(x-x_2)\cdots(x-x_n)$, where $\alpha \in \mathbb{R}\setminus\{0\}$, and $1 > x_1 > x_2 > \cdots > x_n > -1$.

Assumption B: $\Delta = \{t_\nu\}_{\nu=0}^{n}$, $1 \ge t_0 > x_1 > t_1 > \cdots > x_n > t_n \ge -1$.

Theorem 2.1. *Let Q and Δ satisfy Assumptions A and B. If $f \in \pi_n$ and $\|f\| \le \|Q\|$ on Δ, then, for each $k \in \{1,\dots,n\}$ and for every $t \in \mathbb{R}$,*

$$\|f^{(k)}(t)\| \le \max\left\{\|Q^{(k)}(t)\|, \|Q_1^{(k)}(t)\|, \dots, \|Q_n^{(k)}(t)\|\right\}, \tag{2.10}$$

where

$$Q_\nu(t) = Q(t)\,\frac{1-x_\nu t}{t-x_\nu}, \quad \nu = 1,\dots,n.$$

Theorem 2.1 is due to Shadrin [63]. The proof below follows [37] (Shadrin's proof in [63] is more complicated).

Proof. We use the notation for $I_{n,k}(\Delta)$ and $J_{n,k}^{\ell}(\Delta)$ introduced above. All we need to prove is that, for every $t \in \mathbb{R}$, some of the quantities, appearing in the right-hand side of (2.10), is greater than or equal to $M_{n,k}(Q,\Delta;t)$. According to Lemma 2.2,

$$|Q^{(k)}(t)| = M_{n,k}(Q,\Delta;t) \ \text{ whenever } \ t \in I_{n,k}(\Delta). \tag{2.11}$$

Assume that $t \in J_{n,k}^{\ell}(\Delta)$ for some $\ell \in \{0,\dots,n-1\}$. From $\Delta \subset (-1,1)$ it follows that $|t| < 1$, and from

$$\left|\frac{1-uv}{u-v}\right| \ge 1 \ \text{ for } \ u,v \in [-1,1],\ u \ne v,$$

we infer

$$|Q_\ell(t_\nu)| \ge |Q(t_\nu)| \ \text{ for } \ \nu = 0,\dots,n.$$

Since the zeros of Q separate the points from Δ,

$$Q_\ell(t_\nu)\,Q_\ell(t_{\nu+1}) \begin{cases} < 0, & \text{for } \nu \ne \ell \\ > 0, & \text{for } \nu = \ell. \end{cases}$$

Also, (2.9) holds true for $t \in J^{\ell}_{n,k}(\Delta)$, therefore the sequences $\{Q_\ell(t_\nu)\}_{\nu=0}^n$ and $\{\omega_\nu^{(k)}(t)/\omega_\nu(t_\nu)\}_{\nu=0}^n$ have the same sign pattern. By Lagrange's interpolation formula,

$$\begin{aligned}|Q_\ell^{(k)}(t)| &= \Big|\sum_{\nu=0}^{n} \frac{\omega_\nu^{(k)}(t)}{\omega_\nu(t_\nu)} Q_\ell(t_\nu)\Big| = \sum_{\nu=0}^{n} \Big|\frac{\omega_\nu^{(k)}(t)}{\omega_\nu(t_\nu)}\Big| |Q_\ell(t_\nu)| \\ &\geq \sum_{\nu=0}^{n} \Big|\frac{\omega_\nu^{(k)}(t)}{\omega_\nu(t_\nu)}\Big| |Q(t_\nu)| = M_k(Q,\Delta;t).\end{aligned}$$

The inequality $|Q_\ell^{(k)}(t)| \geq M_k(Q,\Delta;t)$ remains also valid when $t \in \overline{J^{\ell}_{n,k}(\Delta)}$, i.e., when t is a boundary point for $J^{\ell}_{n,k}(\Delta)$. This observation together with (2.11) completes the proof of Theorem 2.1, as

$$\cup_{\ell=0}^{n-1} \overline{J^{\ell}_{n,k}(\Delta)} \cup I_{n,k}(\Delta) = \mathbb{R}. \qquad \square$$

The following theorem is an analogue of Theorem 2.1 for polynomials that vanish at the extreme check points (see [45]).

Theorem 2.2. *Let Q satisfy Assumption A with $x_n = -x_1$, and let the points $\{t_\nu\}_{\nu=1}^{n-1}$ separate the zeros of Q,*

$$x_1 > t_1 > x_2 > t_2 > \cdots > x_{n-1} > t_{n-1} > x_n.$$

If $f \in \pi_n$ satisfies the assumptions

$$\begin{aligned}&(i) \quad f(x_1) = f(x_n) = 0,\\ &(ii) \quad |f(t_\nu)| \leq |Q(t_\nu)| \ \text{ for } \ \nu = 1,\dots,n-1,\end{aligned}$$

then, for each $k \in \{1,\dots,n\}$ and for every $t \in \mathbb{R}$,

$$|f^{(k)}(t)| \leq \max_{1\leq\nu\leq n-1} |\widetilde{Q}_\nu^{(k)}(t)|,$$

where

$$\widetilde{Q}_\nu(t) := \frac{1}{x_1}\frac{x_1^2 - x_\nu x}{x - x_\nu} Q(x).$$

The proof of Theorem 2.2 is a slight modification of that of Theorem 2.1. Notice that, instead of the $n+1$ polynomials $Q, \{Q_\nu\}_1^n$, appearing in the right-hand side of (2.10), here we have only $n-1$ polynomials $\{\widetilde{Q}_\nu\}_1^{n-1}$ (actually, $n-2$, as $\widetilde{Q}_n = -\widetilde{Q}_1 = Q$). This reduction is a consequence of the assumption (i) and $x_n = -x_1$.

Theorems 2.1 and 2.2 can be useful in the derivation of DS- and DSS-type inequalities only for lower order derivatives (actually, so far they have been applied only to the case $k = 1$, see Theorems 3.10 and 3.12 in the next section).

The limited use of Theorems 2.1 and 2.2 is due to the fact that for larger k the expressions for $Q_\nu^{(k)}$ and $\widetilde{Q}_\nu^{(k)}$ become rather cumbersome. To avoid this effect, one may apply Lemmas 2.1 and 2.2 and iterate the arguments used in the proof of Theorem 2.1 to obtain another pointwise estimation. This idea is due to A. Shadrin, and we sketch it below.

Assumption B means that the zeros of Q interlace with the zeros of both ω_0 and ω_n; more precisely, for $i = 1, \dots, n$, the i-th zero of Q is between the i-th zeros of ω_0 and ω_n. According to Lemma 2.1, for any fixed $m \in \mathbb{N}$, $1 \le m < n$, the interlacing and the same arrangement prevails for the zeros of $Q^{(m)}$, $\omega_0^{(m)}$ and $\omega_n^{(m)}$. This means that the zeros of $Q^{(m)}$ separate the $n-m+1$ Chebyshev intervals forming $I_{n,m}(\Delta)$. One can collect a point from each Chebyshev interval, forming in this way a set Δ' of $n-m+1$ points. The points from Δ' are separated by the zeros of $Q^{(m)}$, and, according to Lemma 2.2, $|f^{(m)}| \le |Q^{(m)}|$ on Δ' provided that $f \in \pi_n$ satisfies $|f| \le |Q|$ on Δ. Now, application of Theorem 2.1 with f and Q replaced by $f^{(m)}$ and $Q^{(m)}$, Δ replaced by Δ', and k by $k-m$, yields the following theorem.

Theorem 2.3. *Let Q and Δ satisfy Assumptions A and B, and let $f \in \pi_n$ satisfy $|f| \le |Q|$ on Δ. Let $m \in \mathbb{N}$, $1 \le m < n$, and let $\tau_1 > \tau_2 > \cdots > \tau_{n-m}$ be the zeros of $Q^{(m)}$. Then, for each $k \in \{m+1, \dots, n\}$ and for every $t \in \mathbb{R}$,*

$$\|f^{(k)}(t)\| \le \max\left\{\|Q^{(k)}(t)\|, \|\hat{Q}_1^{(k-m)}(t)\|, \dots, \|\hat{Q}_{n-m}^{(k-m)}(t)\|\right\},$$

where

$$\hat{Q}_\nu(x) = Q^{(m)}(x)\,\frac{1-\tau_\nu x}{x-\tau_\nu}, \quad \nu = 1, \dots, n-m.$$

Theorem 2.3 has its analogue when $k = m$, in which case a more precise statement is possible.

Theorem 2.4. *Let Q, f and Δ satisfy the assumptions of Theorem 2.3. Then, for each $k \in \{1, \dots, n-1\}$ and for $r = 0, 1, \dots, n-k$,*

$$\max_{t\in[\tau_{r+1},\tau_r]} |f^{(k)}(t)| \le \max\left\{\|\hat{Q}_r\|_{C[\tau_{r+1},\tau_{r-1}]}, \|\hat{Q}_{r+1}\|_{C[\tau_{r+2},\tau_r]}\right\},$$

where $\tau_{-1} = \tau_0 = 1$, $\tau_{n-k+1} = \tau_{n-k+2} = -1$, $\tau_1 > \cdots > \tau_{n-k}$ are the zeros of $Q^{(k)}$, and

$$\hat{Q}_\nu(x) = Q^{(k)}(x)\,\frac{1-\tau_\nu x}{x-\tau_\nu}, \quad \nu = 0, \dots, n-k+1.$$

Theorem 2.4 was established in [46]. For its proof, we simply continue the arguments that led us to Theorem 2.3, but confined to the case $k = m$. Thus, we have a set Δ' consisting of $n-k+1$ points, say $\{y_\nu\}_{\nu=0}^{n-k}$, so that each of the $n-k+1$ Chebyshev intervals in $I_{n,k}(\Delta)$ has a representative in Δ'. The zeros of $Q^{(k)}$ are separated by the points from Δ', and we may assume that

$$y_0 > \tau_1 > y_1 > \cdots > y_{n-k-1} > \tau_{n-k} > y_{n-k}.$$

Let $\{\ell_\nu\}_{\nu=0}^{n-k}$ be the Lagrange fundamental polynomials associated with the interpolation nodes $\{y_\nu\}_{\nu=0}^{n-k}$. We consider two cases:

Case 1: $t \in (y_r, y_{r-1})$ for some $r \in \{1, \dots, n-k\}$. In this case

$$\operatorname{sign} \ell_\nu(t) = \begin{cases} (-1)^{r-1-\nu}, & \text{if } \nu \le r-1 \\ (-1)^{\nu-r}, & \text{if } \nu \ge r. \end{cases}$$

Since the sequence $\{\hat{Q}_r(y_\nu)\}_{\nu=0}^{n-k}$ has the same sign pattern as $\{\ell_\nu(t)\}_{\nu=0}^{n-k}$, and $|\hat{Q}_r(y_\nu)| \ge |Q^{(k)}(y_\nu)|$ for $\nu = 0, \dots, n-k$, we obtain by Lagrange's interpolation formula that

$$|\hat{Q}_r(t)| = \sum_{\nu=0}^{n-k} |\ell_\nu(t)|\, |\hat{Q}_r(y_\nu)| \ge \sum_{\nu=0}^{n-k} |\ell_\nu(t)|\, |Q^{(k)}(y_\nu)| \ge |f^{(k)}(t)|.$$

This result remains true also for $t = y_r, y_{r-1}$. Since $[y_r, y_{r-1}] \subset [\tau_{r+1}, \tau_{r-1}]$, we conclude that

$$\max_{t\in[y_r,y_{r-1}]} |f^{(k)}(t)| \le \|\hat{Q}_r\|_{C[\tau_{r+1},\tau_{r-1}]}. \tag{2.12}$$

Case 2: $t \in (y_0, \tau_0]$ or $t \in [\tau_{n-k+1}, y_{n-k})$. In this case $\{\ell_\nu(t)\}_{\nu=0}^{n-k}$ alternate in sign, and so do $\{\hat{Q}_0(y_\nu)\}_{\nu=0}^{n-k}$ and $\{\hat{Q}_{n-k+1}(y_\nu)\}_{\nu=0}^{n-k}$, since, by definition, $-\hat{Q}_0 = Q^{(k)} = \hat{Q}_{n-k+1}$. Using again Lagrange's formula, we see that (2.12) is true in this case, too (with $y_{-1} := 1$ and $y_{n-k+1} := -1$).

Since $[\tau_{r+1}, \tau_r] \subset [y_{r+1}, y_r] \cup [y_r, y_{r-1}]$, the claim of Theorem 2.4 follows from (2.12).

In Theorems 2.1 - 2.4, the number of polynomials majorizing $|f^{(k)}(x)|$, increases with n. This number can be reduced to two, provided that some special relation between Q and Δ is assumed. The next result was proved in [37].

Theorem 2.5. *Let Q satisfy Assumption A, and Δ satisfy Assumption B with $t_0 = 1$, $t_n = -1$. Set $\omega_*(x) = \prod_{\nu=1}^{n-1}(x - t_\nu)$. Assume that, for some $k \in \{1, \dots, n-1\}$, $Q^{(k)}$ can be represented in the form*

$$Q^{(k)}(x) = c_1 \omega_*^{(k-1)}(x) + c_2 x \omega_*^{(k)}(x),$$

with some constants $c_1, c_2 \in \mathbb{R}$ satisfying

$$\begin{aligned} c_1(c_1 - kc_2) > 0, &\quad \text{if } 1 \le k \le n-2, \\ (c_1 + c_2)(c_1 - kc_2) > 0, &\quad \text{if } k = n-1. \end{aligned} \tag{2.13}$$

If $f \in \pi_n$ and $|f| \le |Q|$ on Δ, then, for every $x \in [-1, 1]$,

$$|f^{(k)}(x)| \le \max\{|Q^{(k)}(x)|, |Z_{n,k}(x)|\}, \tag{2.14}$$

where

$$Z_{n,k}(x) := (c_1 - kc_2)\Big[\frac{x^2-1}{k}\, \omega_*^{(k)}(x) + x\omega_*^{(k-1)}(x)\Big] - c_2 \omega_*^{(k)}(x). \tag{2.15}$$

Proof. We shall prove first the case when $f \in \pi_n^r$, and later will show that this restriction is of no importance for the conclusion of the theorem. Let us consider the generic case $1 \le k \le n-2$. Denote by $\{\alpha_\nu^k\}_{\nu=1}^{n-k}$ and $\{\beta_\nu^k\}_{\nu=1}^{n-k}$ the zeros of $\omega_0^{(k)}$ and $\omega_n^{(k)}$, respectively, labeled in ascending order (here, $\omega_0(x) = (x+1)\omega_*(x)$ and $\omega_n(x) = (x-1)\omega_*(x)$). In view of the definition of the Chebyshev intervals, we have

$$I_{n,k}(\Delta) \cap [-1,1] = [-1, \alpha_1^k] \cup_{\nu=1}^{n-k-1} [\beta_\nu^k, \alpha_{\nu+1}^k] \cup [\beta_{n-k}^k, 1] =: \mathcal{I}_{n,k}(\Delta).$$

According to Lemma 2.2 (ii),

$$|f^{(k)}(x)| \le |Q^{(k)}(x)| \;\; \text{whenever} \;\; x \in \mathcal{I}_{n,k}(\omega). \tag{2.16}$$

Theorem 2.5 will be proved, if we succeed in showing that

$$|f^{(k)}(x)| \le |Z_{n,k}(x)| \;\; \text{for every} \;\; x \in J_{n,k}(\Delta) = \cup_{\nu=1}^{n-k}(\alpha_\nu^k, \beta_\nu^k). \tag{2.17}$$

As was already shown before, the zeros of $Q^{(k)}$ separate the Chebyshev intervals, i.e., each Zolotarev interval $(\alpha_\nu^k, \beta_\nu^k)$ $(\nu = 1, 2, \dots, n-k)$, contains exactly one zero of $Q^{(k)}$. On the other hand, the zeros of ω_* interlace with the zeros of both ω_0 and ω_n, and Lemma 2.1 tells us that the same is true for the zeros of their k-th derivatives. Therefore, if we denote by $\{\gamma_\nu^k\}_{\nu=1}^{n-k-1}$ the zeros of $\omega_*^{(k)}$, labeled in ascending order, we see that each Chebyshev interval contains in its interior exactly one zero of $Q^{(k)}$, i.e.,

$$\gamma_\nu^k \in (\beta_\nu^k, \alpha_{\nu+1}^k) \;\; \text{for} \;\; \nu = 1, \dots, n-k-1.$$

The representation of $Q^{(k)}$ yields

$$\operatorname{sign}\left\{Q^{(k)}(\gamma_{n-k-1}^k)\right\} = \operatorname{sign}\left\{c_1 \omega_*^{(k-1)}(\gamma_{n-k-1}^k)\right\} = -\operatorname{sign}\{c_1\}.$$

Since $Q^{(k)}$ has exactly one zero located to the right of γ_{n-k-1}^k (the one in $(\alpha_{n-k}^k, \beta_{n-k}^k)$), we conclude that the leading coefficient of Q and c_1 have the same sign. This observation, the assumption (2.13) and (2.15) imply that the leading coefficient of $Z_{n,k}$ has the same sign as c_1.

On using (2.15) and the representation for $Q^{(k)}$, we find the following relations between $Z_{n,k}$ and $Q^{(k)}$:

$$Z_{n,k}(x) - Q^{(k)}(x) = \frac{c_1 - kc_2}{k}(x - x_0)\omega_0^{(k)}(x),$$

$$Z_{n,k}(x) + Q^{(k)}(x) = \frac{c_1 - kc_2}{k}(x + x_0)\omega_n^{(k)}(x),$$

where

$$x_0 = \frac{c_1}{c_1 - kc_2}.$$

Hence,

$$Z_{n,k}(x) = \begin{cases} Q^{(k)}(x), & \text{for } x = x_0 \text{ and for } x = \alpha_\nu^k \\ -Q^{(k)}(x), & \text{for } x = -x_0 \text{ and for } x = \beta_\nu^k. \end{cases} \tag{2.18}$$

Since, as was mentioned above, each Zolotarev interval $(\alpha_\nu^k, \beta_\nu^k)$ contains exactly one zero of $Q^{(k)}$, the latter observation implies

$$\text{sign}\left\{Z_{n,k}(\beta_\nu^k)\right\} = -\text{sign}\left\{Z_{n,k}(\alpha_{\nu+1}^k)\right\} \text{ for } \nu = 1, \dots, n-k-1. \tag{2.19}$$

In particular, (2.18) yields

$$Z_{n,k}(\beta_{n-k}^k) - Q^{(k)}(\beta_{n-k}^k) = -2Q^{(k)}(\beta_{n-k}^k),$$

and since β_{n-k}^k is located to the right from the rightmost zero of $Q^{(k)}$, there holds

$$\text{sign}\left\{[Z_{n,k} - Q^{(k)}](\beta_{n-k}^k)\right\} = -\text{sign}\,\{c_1\}.$$

On the other hand, for x large enough,

$$\text{sign}\left\{[Z_{n,k} - Q^{(k)}](x)\right\} = \text{sign}\,\{c_1\},$$

therefore x_0 is the rightmost zero of $Z_{n,k} - Q^{(k)}$, and $x_0 > \beta_{n-k}^k$.

Similarly,

$$Z_{n,k}(\alpha_1^k) + Q^{(k)}(\alpha_1^k) = 2Q^{(k)}(\alpha_1^k),$$

and since α_1^k is located to the left from the leftmost zero of $Q^{(k)}$,

$$\text{sign}\left\{[Z_{n,k} + Q^{(k)}](\alpha_1^k)\right\} = (-1)^{n-k}\text{sign}\,\{c_1\}.$$

From

$$\text{sign}\left\{[Z_{n,k} + Q^{(k)}](x)\right\} = (-1)^{n-k+1}\text{sign}\,\{c_1\}$$

as $x \to -\infty$, we see that $Z_{n,k} + Q^{(k)}$ has a zero located to the left from α_1^k, i.e., $-x_0 < \alpha_1^k$.

We show next that each of the intervals $(-x_0, \alpha_1^k)$, (β_{n-k}^k, x_0) and $(\beta_\nu^k, \alpha_{\nu+1}^k)$ $(\nu = 1, \dots, n-k-1)$ contains exactly one zero of $Z_{n,k}$ (and as $Z_{n,k}$ is a polynomial of degree $n-k+1$, those are all its zeros). It suffices to show that $Z_{n,k}$ changes its sign in each of the above intervals. For $(\beta_\nu^k, \alpha_{\nu+1}^k)$ this is derived from (2.19), while for $(-x_0, \alpha_1^k)$ and (β_{n-k}^k, x_0) the conclusion follows from (2.18), more precisely, from

$$Z_{n,k}(-x_0) = -Q^{(k)}(-x_0), \qquad Z_{n,k}(\alpha_1^k) = Q^{(k)}(\alpha_1^k),$$
$$Z_{n,k}(\beta_{n-k}^k) = -Q^{(k)}(\beta_{n-k}^k), \qquad Z_{n,k}(x_0) = Q^{(k)}(x_0),$$

and the fact that $Q^{(k)}$ does not vanish outside $(\alpha_1^k, \beta_{n-k}^k)$.

We are ready to prove (2.17). We know from (2.18), that $|Z_{n,k}| = |Q^{(k)}|$ at the end-points of the Chebyshev intervals, and (2.16) implies

$$|f^{(k)}(\alpha_\nu^k)| \le |Z_{n,k}(\alpha_\nu^k)| \quad \text{for} \quad \nu = 1, \dots, n-k, \tag{2.20}$$

and

$$|f^{(k)}(\beta_\nu^k)| \le |Z_{n,k}(\beta_\nu^k)| \quad \text{for} \quad \nu = 1, \dots, n-k. \tag{2.21}$$

The inequalities (2.20) and (2.21) together with (2.19) yield

$$\left[Z_{n,k}(\beta_\nu^k) \pm f^{(k)}(\beta_\nu^k)\right]\left[Z_{n,k}(\alpha_{\nu+1}^k) \pm f^{(k)})(\alpha_{\nu+1}^k)\right] \le 0,$$

whence each polynomial $Z_{n,k} \pm f^{(k)}$ has at least one zero in each Chebyshev interval $[\beta_\nu^k, \alpha_{\nu+1}^k]$, $\nu = 1, \dots, n-k-1$. The same conclusion holds true for the intervals $[-x_0, \alpha_1^k]$ and $[\beta_{n-k}^k, x_0]$. The verification for $[\beta_{n-k}^k, x_0]$ is done as follows: since x_0 is the rightmost zero of $Z_{n,k} - Q^{(k)}$, there holds $|Z_{n,k}(x)| \ge |Q^{(k)}(x)|$ for every $x \ge x_0$. On the other hand, $|Q^{(k)}(x)| \ge |f^{(k)}(x)|$ for every $x \ge \beta_{n-k}^k$, in particular, for $x \ge x_0$. Therefore,

$$\operatorname{sign}\left\{[Z_{n,k} \pm f^{(k)}](x)\right\} = \operatorname{sign}\left\{Z_{n,k}(x)\right\} = \operatorname{sign} c_1 \quad \text{for} \quad x \ge x_0.$$

Since, in view of (2.21) and (2.18),

$$\operatorname{sign}\left\{[Z_{n,k} \pm f^{(k)}](\beta_{n-k}^k)\right\} = -\operatorname{sign}\left\{Q^{(k)}(\beta_{n-k}^k)\right\} = -\operatorname{sign} c_1,$$

the polynomial $Z_{n,k} \pm f^{(k)}$ changes its sign in $[\beta_{n-k}^k, x_0]$. Similar argument shows that $Z_{n,k} \pm f^{(k)}$ changes its sign in $[-x_0, \alpha_1^k]$.

We saw that $Z_{n,k} \pm f^{(k)}$ has $n-k+1$ zeros, all located in the Chebyshev intervals. Since $Z_{n,k} \pm f^{(k)}$ is a polynomial of degree $n-k+1$, it follows that $Z_{n,k} \pm f^{(k)}$ does not change its sign on the Zolotarev intervals $(\alpha_\nu^k, \beta_\nu^k)$ $(\nu = 1, \dots, n-k)$. According to (2.20) and (2.21), at the end-points of the Zolotarev intervals $|f^{(k)}| \le |Z_{n,k}|$. This completes the proof of (2.17).

The proof of the case $k = n-1$ goes along the same lines except for the fact that the leading coefficient of Q has the same sign as $c_1 + c_2$. From the second part of (2.13) we deduce that the same sign has the leading coefficient of $Z_{n,n-1}$, and accomplish the proof just as in the case considered above.

Theorem 2.5 is proved for the case $f \in \pi_n^r$. The argument extending its conclusion to arbitrary $f \in \pi_n$ is standard. Let $f \in \pi_n$ satisfy the assumptions of the theorem. If, for a fixed $\tau \in [-1,1]$, $f^{(k)}(\tau) = e^{i\theta}|f^{(k)}(\tau)|$, $\theta \in \mathbb{R}$, we consider $g(x) := \Re\{e^{-i\theta} f(x)\}$. Clearly, $g \in \pi_n^r$, and $|g| \le |Q|$ on Δ. By the case we just proved, $|f^{(k)}(\tau)| = |g^{(k)}(\tau)| \le \max\left\{|Q^{(k)}(\tau), |Z_{n,k}(\tau)|\right\}$. □

The same arguments are essentially used in the proof of the next theorem, which provides estimates for $|f^{(k)}|$ for all k.

Theorem 2.6 ([37]). *Let q be a polynomial of degree n having only simple zeros, all located in $[-1,1]$. Let $Q(x) = mxq'(x) + q(x)$, where $m \in \mathbb{R}$ and*

$$m \geq \max\Big\{\frac{q(-1)}{q'(-1)}, -\frac{q(1)}{q'(1)}\Big\}. \tag{2.22}$$

If $f \in \pi_n$ and $|f(x)| \leq |Q(x)|$ whenever $(x^2-1)q'(x) = 0$, then, for each $k \in \{1, \dots, n\}$ and for every $x \in [-1,1]$,

$$|f^{(k)}(x)| \leq \max\big\{|Q^{(k)}(x)|, |\mathcal{Z}_{n,k}(x)|\big\},$$

where

$$\mathcal{Z}_{n,k}(x) = \Big(\frac{x^2-1}{k} - m\Big)q^{(k+1)}(x) + xq^{(k)}(x).$$

The particular choice $m = 0$ in Theorem 2.6 comes down to a result of Shadrin [62], which he applied to produce the simplest hitherto known proof of V. Markov's inequality under the assumptions in Theorem C.

Corollary 2.1. *Let Q satisfy Assumption A. If $f \in \pi_n$ and $|f(x)| \leq |Q(x)|$ whenever $(x^2-1)Q'(x) = 0$, then, for each $k \in \{1, \dots, n\}$ and for every $x \in [-1,1]$,*

$$|f^{(k)}(x)| \leq \max\Big\{|Q^{(k)}(x)|, \Big|\frac{x^2-1}{k}\,Q^{(k+1)}(x) + xQ^{(k)}(x)\Big|\Big\}.$$

There are various generalizations of Theorems 2.5 and 2.6, e.g., replacement of the role of the extreme check points ± 1 by $\pm a$, "non-symmetric" variants of Theorems 2.5 and 2.6, etc. We shall use the following version of Theorem 2.5:

Theorem 2.5′. *Let ω_* be a polynomial of degree $n-1$ with simple zeros only, all located in the interval $(-a,a)$. Let Q be a polynomial of degree n, whose zeros separate the zeros of $\omega(x) := (x^2-a^2)\omega_*(x)$. Assume that, for some $k \in \{1, \dots, n-1\}$, $Q^{(k)}$ can be represented in the form*

$$Q^{(k)}(x) = c_1\omega_*^{(k-1)}(x) + c_2 x\omega_*^{(k)}(x)$$

where $c_1, c_2 \in \mathbb{R}$ satisfy (2.13). If $f \in \pi_n$ and $|f(x)| \leq |Q(x)|$ whenever $\omega(x) = 0$, then, for every $x \in [-a,a]$,

$$|f^{(k)}(x)| \leq \max\big\{|Q^{(k)}(x)|, |Z_{n,k}(x)|\big\},$$

where

$$Z_{n,k}(x) := \frac{c_1 - kc_2}{a}\Big[\frac{x^2-a^2}{k}\,\omega_*^{(k)}(x) + x\omega_*^{(k-1)}(x)\Big] - ac_2\omega_*^{(k)}(x).$$

3. DS- and DSS-type Inequalities in the Uniform Norm

All DS- and DSS-type inequalities, established in this section, are in $C[-1,1]$-norm (sometimes, their proof passes through some $C[-a,a]$-norm for suitably transformed polynomials). A property that is peculiar to the oscillating polynomial Q here, is

$$\|Q^{(k)}\| = Q^{(k)}(1)$$

for all k (or at least for those k for which DS- (DSS-) type inequality is established). Therefore, to obtain such inequalities from the pointwise estimates in the preceding section, one needs to show that the $C[-1,1]$-norm of all polynomials estimating $|f^{(k)}(x)|$ from above, does not exceed $Q^{(k)}(1)$. One should not bother for the highest derivatives, i.e., for $k = n$ and $k = n-1$ when Q is of degree n. In the former case always $\|f^{(n)}\| \le \|Q^{(n)}\|$, as we know that $I_{n,n}(\Delta) = \mathbb{R}$, while in the latter case $\|f^{(n-1)}\|$ is attained at some of the endpoints ± 1, as $f^{(n-1)}$ is a linear polynomial, and the endpoints always belong to $I_{n,n-1}(\Delta)$ (this observation is due to Shadrin [62]).

We are not able to carry out in full details the proof of all theorems in this section, as this would require much space and would involve a lot of technical details. Therefore, in some cases we only point out to the major steps in the proofs, and sketch the ideas, leading to their accomplishment.

Since, in the theorems below, Q is ultraspherical, or alien to some ultraspherical polynomial, we find it appropriate at this point to quote some important properties of the ultraspherical polynomials. Recall, that the ultraspherical polynomial $P_n^{(\lambda)}$ (also called Gegenbauer polynomial), is the n-th orthogonal polynomial with respect to the weight function $w(x) := (1-x^2)^{\lambda-1/2}$ $(\lambda > -1/2)$, and normalized by the condition $P_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n}$. For easy reference, we list the properties of $P_n^{(\lambda)}$ we shall need in a lemma. The reader may find these and many other properties in any textbook on orthogonal polynomials, e.g., in [66].

Lemma 3.1. *The ultraspherical polynomials possess the following properties:*

(i) $y = P_n^{(\lambda)}$ *satisfies the second order differential equation*

$$(1-x^2)y'' - (2\lambda+1)xy' + n(n+2\lambda)y = 0;$$

(ii) For $\lambda > 0$*, the local maxima of* $|P_n^{(\lambda)}|$ *increase as* $|x|$ *increases. Moreover,* $\|P_n^{(\lambda)}\| = |P_n^{(\lambda)}(\pm 1)|$*, and* $\|P_n^{(\lambda)}\|$ *is attained only at* ± 1*;*

(iii) $\dfrac{d}{dx}\{P_n^{(\lambda)}(x)\} = 2\lambda P_{n-1}^{(\lambda+1)}(x) \quad (\lambda \ne 0)$*;*

(iv) $\dfrac{d}{dx}\{P_{n+1}^{(\lambda)}(x)\} = x\dfrac{d}{dx}\{P_n^{(\lambda)}(x)\} + (n+2\lambda)P_n^{(\lambda)}(x) \quad (\lambda \ne 0)$*;*

(v) For any $\lambda > 0$,

$$P_n^{(\lambda)}(x) = \sum_{m=0}^{n} a(m,n,\lambda) T_m(x), \quad \textit{where} \;\; a(m,n,\lambda) \geq 0 \;\; (m = 0, 1, \dots, n).$$

Special cases of ultraspherical polynomials are the Legendre polynomials $P_n = P_n^{(1/2)}$ and the Chebyshev polynomial of the second kind $U_n = P_n^{(1)}$. The Chebyshev polynomial of the first kind T_n also belongs to this class (it corresponds to $\lambda = 0$), but it is normalized differently, namely, by $T_n(1) = 1$. This is the reason for properties (iii) and (iv) in Lemma 3.1 to be formulated for $\lambda \neq 0$, their substitutes look as follows:

(iii′) $T_n'(x) = nU_{n-1}(x) \;\; (= nP_{n-1}^{(1)}(x))$;

(iv′) $T_{n+1}'(x) = \dfrac{n+1}{n}\, xT_n'(x) + (n+1)T_n(x)$.

In property (v), only half of the coefficients in the expansion of $P_n^{(\lambda)}$ in the basis of the Chebyshev polynomials are positive, namely, those for which m is of the same parity with n; the remaining coefficients are equal to zero. There is an explicit formula for the expansion coefficients, (see, e.g.,[1]), but we shall not need it here. The usual definition of ultraspherical polynomials is only for $\lambda > -1/2$, a restriction prompted by the requirement for the weight function $(1-x^2)^{\lambda-1/2}$ to be integrable on $[-1,1]$. Actually, $P_n^{(\lambda)}$ is well-defined for $\lambda \leq -1/2$, too, but in this case some of the general properties of the orthogonal polynomials are not preserved. In particular, the ultraspherical polynomial $P_n^{(-1/2)}$ is related to the Legendre polynomial by the equation

$$n(n-1)P_n^{(-1/2)}(x) = (1-x^2)\frac{dP_{n-1}(x)}{dx}. \tag{3.1}$$

The following lemma reveals a property of ultraspherical polynomials, which is especially useful for derivation of DS- and DSS-type inequalities on the basis of Theorems 2.5, 2.5′ and 2.6. It was proved by Shadrin in [62] for the special case $\lambda = 0$.

Lemma 3.2. *Let $Q = P_n^{(\lambda)}$, where $\lambda \geq 0$. Then, for each $k \in \{1, 2, \dots, n\}$ and $s \geq k$,*

$$\left\| \frac{x^2-1}{s}\, Q^{(k+1)}(x) + xQ^{(k)}(x) \right\| = |Q^{(k)}(\pm 1)|. \tag{3.2}$$

Moreover, for $k < n$, the norm in the left-hand side of (3.2) is attained only at $x = \pm 1$. When $\lambda \in [-1/2, 0)$, (3.2) holds true for each $k \in \{2, 3, \dots, n\}$.

Proof. It suffices to prove the result for $s = k$, then for $s > k$ it follows from

$$
\begin{aligned}
\Big\| \frac{x^2-1}{s}\, Q^{(k+1)}(x) + xQ^{(k)}(x)\Big\|
&= \frac{1}{s}\, \|(x^2-1)Q^{(k+1)}(x) + kxQ^{(k)}(x) + (s-k)xQ^{(k)}(x)\| \\
&\le \frac{k}{s}\, \Big\|\frac{x^2-1}{k} Q^{(k+1)}(x) + xQ^{(k)}(x)\Big\| + \frac{s-k}{s}\, \|xQ^{(k)}(x)\| \\
&= \frac{k}{s}\, |Q^{(k)}(\pm 1)| + \frac{s-k}{s}\, |Q^{(k)}(\pm 1)| \\
&= |Q^{(k)}(\pm 1)|
\end{aligned}
$$

(here we used properties (ii) and (iii) in Lemma 3.1).

For the case $\lambda = 0$ we follow the proof of Shadrin [62]. We want to show that for every $n, k \in \mathbb{N}$,

$$
\Big\| \frac{x^2-1}{k}\, T_n^{(k+1)}(x) + xT_n^{(k)}(x)\Big\| = |T_n^{(k)}(\pm 1)|, \quad k = 1, \ldots, n, \tag{3.3}
$$

and, if $k < n$, then the norm is attained only at $x = \pm 1$. For $k = 1$ this follows from the differential equation for T_n (Lemma 3.1 (i)):

$$
|(x^2-1)T_n''(x) + xT_n'(x)| = |n^2 T_n(x)| \le n^2 = |T_n'(\pm 1)| \ \text{ for every } \ x \in [-1,1]
$$

(notice that $|T_n'(x)| < n^2$ when $x \in (-1,1)$). For $k > 1$, the proof of (3.3) proceeds by induction with respect to n. The cases $n = 2, 3$ are verified directly, so we may think that $n \ge 4$. Assume that (3.3) is true for $k-1$, and that for k, (3.3) is true for all $n < m$. After k-fold differentiation of the identity

$$
(m-2)T_m'(x) - mT_{m-2}'(x) = 2m(m-2)T_{m-1}(x),
$$

which follows easily from the explicit representation of the Chebyshev polynomials of first and of second kind and property (iii$'$), we obtain

$$
\begin{aligned}
T_m^{(k)}(x) &= \frac{m}{m-2}\, T_{m-2}^{(k)}(x) + 2mT_{m-1}^{(k-1)}(x), \\
T_m^{(k+1)}(x) &= \frac{m}{m-2}\, T_{m-2}^{(k+1)}(x) + 2mT_{m-1}^{(k)}(x).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{x^2-1}{k}\, T_m^{(k+1)}(x) + xT_m^{(k)}(x) &= \frac{m}{m-2}\Big[\frac{x^2-1}{k}\, T_{m-2}^{(k+1)}(x) + xT_{m-2}^{(k)}(x)\Big] \\
&\quad + \frac{2m(k-1)}{k}\Big[\frac{x^2-1}{k-1}\, T_{m-1}^{(k)}(x) + xT_{m-1}^{(k-1)}(x)\Big] \\
&\quad + \frac{2m}{k}\, xT_{m-1}^{(k-1)}(x).
\end{aligned}
$$

On using this representation, the triangle inequality and the induction hypothesis, we establish (3.3) for $n = m$ as follows:

$$\begin{aligned}
&\Big\| \frac{x^2-1}{k} T_m^{(k+1)}(x) + xT_m^{(k)}(x) \Big\| \\
&\quad \le \frac{m}{m-2} \Big\| \frac{x^2-1}{k} T_{m-2}^{(k+1)}(x) + xT_{m-2}^{(k)}(x) \Big\| \\
&\qquad + \frac{2m(k-1)}{k} \Big\| \frac{x^2-1}{k-1} T_{m-1}^{(k)}(x) + xT_{m-1}^{(k-1)}(x) \Big\| + \frac{2m}{k} \| xT_{m-1}^{(k-1)}(x) \| \\
&\quad = \frac{m}{m-2} |T_{m-2}^{(k)}(\pm 1)| + 2m |T_{m-1}^{(k-1)}(\pm 1)| = |T_m^{(k)}(\pm 1)|.
\end{aligned}$$

The induction step is completed, and so is the proof of (3.3). The proof of the case $\lambda > 0$ follows from Lemma 3.1(v) and the case we just considered:

$$\begin{aligned}
&\|(x^2-1)Q^{(k+1)}(x) + kxQ^{(k)}(x)\| \\
&\quad = \Big\| (x^2-1) \sum_{m=0}^{n} a(m,n,\lambda) T_m^{(k+1)}(x) + kx \sum_{m=0}^{n} a(m,n,\lambda) T_m^{(k)}(x) \Big\| \\
&\quad \le \sum_{m=0}^{n} a(m,n,\lambda) \|(x^2-1)T_m^{(k+1)}(x) + kxT_m^{(k)}(x)\| \\
&\quad = k \sum_{m=0}^{n} a(m,n,\lambda) T_m^{(k)}(1) = kQ^{(k)}(1).
\end{aligned}$$

Finally, for the proof of (3.2) when $-1/2 \le \lambda < 0$ we use Lemma 3.1(ii) as follows: we have $Q' = 2\lambda P_{n-1}^{(\lambda+1)}$, where $\lambda + 1 \ge 1/2 > 0$. For $k \ge 2$ and $P = Q'$, we have

$$\begin{aligned}
\Big\| \frac{x^2-1}{s} Q^{(k+1)}(x) + xQ^{(k)}(x) \Big\| &= \| \frac{x^2-1}{s} P^{(k)}(x) + xP^{(k-1)}(x) \| \\
&= |P^{(k-1)}(\pm 1)| = |Q^{(k)}(\pm 1)|.
\end{aligned}$$

Lemma 3.2 is proved. □

Corollary 2.1 and Lemma 3.2 yield immediately the following DS-type inequality.

Theorem 3.1. *Let $Q = P_n^{(\lambda)}$, where $\lambda > -1/2$. If $f \in \pi_n$ and $|f| \le |Q|$ at the zeros of $(x^2-1)Q'(x)$, then*

$$\|f^{(k)}\| \le \|Q^{(k)}\| \tag{3.4}$$

for $k = 1, \dots, n$, if $\lambda \ge 0$, and for $k = 2, \dots, n$, if $\lambda \in (-1/2, 0)$. The equality in (3.4) occurs if and only if $f = cQ$, $|c| = 1$.

Theorem 3.1 was established by Bojanov and the author in [11]. Notice that the case $\lambda = 0$ reproduces Theorem C. Passage to the limit $\lambda \to -1/2$ in Theorem 3.1, and usage of (3.1) implies the following DSS-type inequality:

Theorem 3.2. *Let $Q(x)=(1-x^2)P'_{n-1}(x)$. If $f\in\pi_n$ and $|f(x)|\le|Q(x)|$ whenever $(x^2-1)P_{n-1}(x)=0$, then*

$$\|f^{(k)}\|\le\|Q^{(k)}\|\quad\text{for}\quad k=2,\dots,n.$$

Notice the analogy between Theorem 3.2 and Theorem E of Rahman and Schmeisser: here the role of the Chebyshev polynomials of the first kind is played by Legendre's polynomials.

The next theorem also furnishes DS-type inequality for ultraspherical polynomials, but with different check-points.

Theorem 3.3. *Let $Q=P_n^{(\lambda)}$, $\lambda>-1/2$. If $f\in\pi_n$ and $|f(x)|\le|Q(x)|$ whenever $(x^2-1)P_{n-1}^{(\lambda)}(x)=0$, then*

$$\|f^{(k)}\|\le\|Q^{(k)}\|$$

for $k=1,\dots,n$, if $\lambda\ge1$, for $k=2,\dots,n$, if $\lambda\in[0,1)$, and for $k=3,\dots,n$, if $\lambda\in(-1/2,0)$. For the cases of k and λ, postulated above, the equality $\|f^{(k)}\|=\|Q^{(k)}\|$ takes place if and only if $f=c\,Q$, $|c|=1$.

Proof. To derive Theorem 3.3, we substitute in Theorem 2.5 $Q=P_n^{(\lambda)}$ and $\omega_*=P_{n-1}^{(\lambda)}$. From Lemma 3.1(iv) after k-fold differentiation we obtain

$$Q^{(k)}(x)=(n+2\lambda+k-2)\omega_*^{(k-1)}(x)+x\omega_*^{(k)}(x),\quad k=1,\dots,n.$$

The constants $c_1:=n+2\lambda+k-2$ and $c_2:=1$ satisfy the assumption (2.13). Application of Theorem 2.5 yields $|f^{(k)}(x)|\le\max\big\{|Q^{(k)}(x)|,|Z_{n,k}(x)|\big\}$ for $x\in[-1,1]$, where

$$Z_{n,k}(x):=(n+2\lambda-2)\Big[\frac{x^2-1}{k}\,\omega_*^{(k)}(x)+x\omega_*^{(k-1)}(x)\Big]-\omega_*^{(k)}(x).$$

For the values of k and λ, postulated in Theorem 3.3, we obtain with the help of Lemma 3.1(ii) that

$$\|Q^{(k)}\|=(n+2\lambda+k-2)|\omega_*^{(k-1)}(\pm1)|+|\omega_*^{(k)}(\pm1)|,$$

while for the norm of $Z_{n,k}$, Lemma 3.2 yields

$$\begin{aligned}\|Z_{n,k}\|&\le(n+2\lambda-2)\Big\|\frac{x^2-1}{k}\,\omega_*^{(k)}(x)+x\omega_*^{(k-1)}(x)\Big\|+\|\omega_*^{(k)}\|\\&=(n+2\lambda-2)|\omega_*^{(k-1)}(\pm1)|+|\omega_*^{(k)}(\pm1)|.\end{aligned}$$

This upper bound for $\|Z_{n,k}\|$ does not exceed $\|Q^{(k)}\|$, whence the DS-inequality is proved. The statement characterizing the cases of equality in Theorems 3.1 and 3.3, as well as in the subsequent theorems follows from the fact, that the norm of $Q^{(k)}$ is attained only at the end-points. □

It should be pointed out that in the cases of k and λ, which are not covered by Theorem 3.3, the real situation is not known. As we shall see later, when $\lambda = 0$, the result holds true also for $k = 1$.

Another application of Theorem 2.5 yields the following DSS-type inequality.

Theorem 3.4. *Let $\omega_* = P_{n-1}^{(\lambda)}$, and $Q(x) = (x^2-1)\omega_*'(x)$. If $f \in \pi_n$ and $|f(x)| \le |Q(x)|$ whenever $(x^2-1)\omega_*(x) = 0$, then*

$$\|f^{(k)}\| \le \|Q^{(k)}\| \tag{3.5}$$

for $k = 2, \dots, n$, if $\lambda \in [0, 1/2]$, and for $k = 3, \dots, n$, if $\lambda \in (-1/2, 0)$. For the above k and λ, the equality in (3.5) takes place if and only if $f = c\,Q$, $|c| = 1$.

Proof. $(k-1)$-fold differentiation of the differential equation for $P_{n-1}^{(\lambda)}$ yields the representation

$$Q^{(k)}(x) = \big[n(n+2\lambda-2) + k(1-2\lambda)\big]\omega_*^{(k-1)}(x) + (1-2\lambda)x\omega_*^{(k)}(x).$$

From Theorem 2.5, $|f^{(k)}(x)| \le \max\big\{|Q^{(k)}(x)|, |Z_{n,k}(x)|\big\}$ for every $x \in [-1,1]$, where

$$Z_{n,k}(x) = n(n+2\lambda-2)\Big[\frac{x^2-1}{k}\omega_*^{(k)}(x) + x\omega_*^{(k-1)}(x)\Big] + (2\lambda-1)\omega_*^{(k)}(x).$$

On using Lemma 3.2, we see that $\|Z_{n,k}\| \le \|Q^{(k)}\|$ for the values of λ and k, postulated in the statement of the theorem. This proves Theorem 3.4. Notice that the case $\lambda = 0$ reproduces Theorem E. □

Passage to the limit $\lambda \to -1/2$ in Theorems 3.3 and 3.4 yields two further DSS-type inequalities.

Theorem 3.5. *Let $Q(x) = (1-x^2)P_{n-1}'(x)$, and $f(x) = (1-x^2)q(x)$, where $q \in \pi_{n-2}$ satisfies the restriction*

$$|q(x)| \le |P_{n-1}'(x)| \quad \textit{whenever} \quad (1-x^2)P_{n-2}'(x) = 0.$$

Then

$$\|f^{(k)}\| \le \|Q^{(k)}\| \quad \textit{for} \quad k = 3, \dots, n.$$

Theorem 3.6. *Let $Q(x) = (1-x^2)P_{n-2}(x)$, and $f(x) = (1-x^2)q(x)$, where $q \in \pi_{n-2}$ satisfies the restriction*

$$|q(x)| \le |P_{n-2}(x)| \quad \textit{whenever} \quad (1-x^2)P_{n-2}'(x) = 0.$$

Then

$$\|f^{(k)}\| \le \|Q^{(k)}\| \quad \textit{for} \quad k = 3, \dots, n.$$

As was already mentioned, the transformed Chebyshev polynomial

$$\overline{T}_n(x) = T_n(\xi x), \quad \xi = \cos\frac{\pi}{2n},$$

is the extremizer in the Markov inequality for polynomials satisfying zero boundary conditions (Schur's Theorem B). This result was extended to higher order derivatives (for the cases $k = 2$ and $k = 3$, see also [31, 32]), and in the DS-sense in [35].

Theorem 3.7. *Let $\{y_\nu^*\}_{\nu=1}^{n-1}$ be the critical points of $\overline{T}_n$,*

$$y_\nu^* = \xi^{-1}\cos\frac{\nu\pi}{n}, \quad \nu = 1, \dots, n-1.$$

If $f \in \pi_n$ satisfies $f(-1) = f(1) = 0$, and $|f(y_\nu^*)| \le 1$ for $\nu = 1, \dots, n-1$, then

$$\|f^{(k)}\| \le \|\overline{T}_n^{(k)}\| \quad \text{for} \quad k = 2, \dots, n. \tag{3.6}$$

The equality $\|f^{(k)}\| = \|\overline{T}_n^{(k)}\|$ takes place if and only if $f = c\overline{T}_n$, $|c| = 1$.

For the proof, we make use of Corollary 2.1 with $Q = T_n$, and then show that

$$\Big\|\frac{(x^2-1)T_n^{(k+1)}(x)}{k+1} + xT_n^{(k)}\Big\|_{C[-\xi,\xi]} \le \|T_n^{(k)}\|_{C[-\xi,\xi]}.$$

The polynomial $\overline{T}_n$ is also extremal in the next DSS-inequality.

Theorem 3.8 **([39]).** *Let $f \in \pi_n$ satisfy $f(-1) = f(1) = 0$ and*

$$|f(x)| \le \sqrt{1-\xi^2x^2} \quad \text{for} \quad x = \xi^{-1}\cos\frac{(2\nu-1)\pi}{2n-2}, \ \nu = 1, \dots, n-1.$$

Then (3.6) holds true, and, in addition, the equality occurs in (3.6) if and only if $f = c\overline{T}_n$, $|c| = 1$.

There is also a DSS-type inequality for the transformed Chebyshev polynomial of the second kind $\overline{U}_n$,

$$\overline{U}_n(x) := U_n(\eta x), \quad \eta = \cos\frac{\pi}{n+1},$$

The check-points in this inequality are

$$y_\nu := \eta^{-1}\cos\frac{\nu\pi}{n}, \quad \nu = 1, \dots, n-1.$$

Theorem 3.9 **([38]).** *Let $f \in \pi_n$ satisfy the conditions $f(-1) = f(1) = 0$, and $|f(y_\nu)| \le 1$ for $\nu = 1, \dots, n-1$. Then*

$$\|f^{(k)}\| \le \|\overline{U}_n^{(k)}\| \quad \text{for} \quad k = 1, \dots, n.$$

Moreover, the equality takes place if and only if $f = c\,\overline{U}_n$, $|c| = 1$.

Theorems 3.8 and 3.9 are derived with the help of the pointwise estimate in Theorem 2.5′, applied to $Q = T_n$ and to $Q = U_n$ with $a = \xi$ and $a = \eta$, respectively. The application of Theorem 2.5′ is possible because of the representations

$$T_n^{(k)}(x) = \frac{n}{n-1}\Big[xT_{n-1}^{(k)}(x) + (n+k-2)T_{n-1}^{(k-1)}(x)\Big]$$

and

$$U_n^{(k)}(x) = (n+k)U_{n-1}^{(k-1)}(x) + xU_{n-1}^{(k)}(x).$$

In view of Theorem 2.5′, if $g \in \pi_n$ vanishes at $\pm\xi$ and $|g| \le |T_n|$ at the zeros of T_{n-1} (or, if g vanishes at $\pm\eta$ and $|g| \le |U_n|$ at the zeros of U_{n-1}, respectively), then for $k = 1, \dots, n$, $|g^{(k)}(x)| \le \max\big\{|T_n^{(k)}(x)|, |Z_{n,k}(x)|\big\}$ for every $x \in [-\xi, \xi]$ in the former case, and $|g^{(k)}(x)| \le \max\big\{|U_n^{(k)}(x)|, |\widetilde{Z}_{n,k}(x)|\big\}$ for every $x \in [-\eta, \eta]$ in the latter case, where

$$Z_{n,k}(x) := \frac{n(n-2)}{(n-1)k\xi}\Big[\Big(x^2 - \frac{n+k-2}{n-2}\xi^2\Big)T_{n-1}^{(k)}(x) + kxT_{n-1}^{(k-1)}(x)\Big]$$

and

$$\widetilde{Z}_{n,k} := \frac{1}{k\eta}\Big[\Big(x^2 - \frac{n+k}{n}\eta^2\Big)T_n^{(k+1)}(x) + kxT_n^{(k)}(x)\Big].$$

The rest of proof consists of verification that

$$\|Z_{n,k}\|_{C[-\xi,\xi]} \le \|T_n^{(k)}\|_{C[-\xi,\xi]}$$

and

$$\big\|\widetilde{Z}_{n,k}\big\|_{C[-\eta,\eta]} \le \|U_n^{(k)}\|_{C[-\eta,\eta]}.$$

The details are given in [39] and [38].

Theorems 3.1 - 3.9 represent DS- or DSS-type inequalities with some specific check-points, whose choice depends on the oscillating polynomial Q. Such a dependence is pre-assumed in the pointwise inequalities used for their derivation (Theorems 2.5, 2.5′, and 2.6). In contrast, in the pointwise inequalities provided by Theorems 2.1, 2.2, 2.3, and 2.4, the only assumption is that the check points interlace with the zeros of Q. This assumption appears to be enough to secure DS-type inequality for the ultraspherical polynomials (at least, we have shown this for $Q = P_n^{(\lambda)}$, $\lambda \in \mathbb{N}_0$). In particular, the following striking result holds true:

Theorem 3.10. *If $f \in \pi_n$ satisfies $|f| \le |T_n|$ at $n+1$ points in $[-1,1]$, which are separated by the zeros of T_n, then, for $k = 1, \dots, n$,*

$$\|f^{(k)}\| \le \|T_n^{(k)}\|. \tag{3.7}$$

Moreover, the equality in (3.7) takes place if and only if $f = cT_n$, $|c| = 1$.

Theorem 3.10 was proved for $k = 1$ in [40], and for $k \ge 2$ in [46]. While the proof of the case $k = 1$ relies on Theorem 2.1, the case $k \ge 2$ is proved with the help of Theorem 2.4. Without going into details, we only mark the main steps in the proof.

Steps to the proof of Theorem 3.10. Case k = 1

Step 1. Application of Theorem 2.1 to $Q = T_n$ with $k = 1$, yielding $\|f'\| \le \max\big\{\|T_n'\|, \|Q_\nu'\|,\ \nu = 1, \dots, n\big\}$, where

$$Q_\nu(x) = T(x)\,\frac{1 - \xi_\nu x}{x - \xi_\nu}, \quad \xi_\nu = \cos\frac{(2\nu - 1)\pi}{2n}.$$

Hence, it suffices to prove that $\|Q_\nu\| \le \|T_n'\| = n^2$, and because of symmetry, the verification may be restricted only to those Q_ν with $1 \le \nu \le \lfloor (n+1)/2 \rfloor$.

Step 2. Proof of the inequality $|Q_\nu(x)| \le R_\nu(x)$, where

$$R_\nu(x) = \left[\frac{(1 - \xi_\nu^2)^2}{(x - \xi_\nu)^4} + \frac{n^2(1 - \xi_\nu x)^2}{(1 - x^2)(x - \xi_\nu)^2}\right]^{1/2} =: \big[g_{1,\nu}(x)^2 + g_{2,\nu}(x)^2\big]^{1/2}.$$

It follows from $T_n^2(x) + (1 - x^2)[T_n'(x)]^2/n^2 = 1$, and Cauchy's inequality.

Step 3. Proof that $R(x)$ is convex in $(-1, \xi)$ and in $(\xi, 1)$ (the subscript ν is omitted). Since $R^3R'' = (g_1g_2' - g_1'g_2)^2 + R^2(g_1g_1'' + g_2g_2'')$, the result follows from $g_ig_i'' \ge 0$ in $(-1, \xi)$ and $(\xi, 1)$, $i = 1, 2$. Hence, Q_ν' possesses a majorant R_ν, which is convex in $(-1, \xi_\nu)$ and in $(\xi_\nu, 1)$. As $R_\nu(x)$ is unbounded near the endpoints of those intervals, we need

Step 4. Determination of a right neighbourhood of $x = -1$, a left neighbourhood of $x = 1$, and an interval surrounding ξ_ν, in which Q_ν' is monotone.

Step 5. Proof that at the endpoints of the intervals mentioned in Step 4, $|Q_\nu'| \le n^2$ and $R_\nu < n^2$ (the latter does not apply to $x = \pm 1$). This step accomplishes the proof of the case $k = 1$. For more details see [40].

Steps to the proof of Theorem 3.10. Case k ≥ 2

Step 1. Application of Theorem 2.4 to $Q = T_n$, yielding for $r = 0, \dots, n-k$,

$$\max_{t \in [\tau_{r+1}, \tau_r]} |f^{(k)}(t)| \le \max\Big\{\big\|\hat{Q}_r\big\|_{C[\tau_{r+1}, \tau_{r-1}]}, \big\|\hat{Q}_{r+1}\big\|_{C[\tau_{r+2}, \tau_r]}\Big\},$$

where $\tau_{-1} = \tau_0 = 1$, $\tau_{n-k+1} = \tau_{n-k+2} = -1$, $\tau_1 > \dots > \tau_{n-k}$ are the zeros of $T_n^{(k)}$, and

$$\hat{Q}_\nu(x) = T_n^{(k)}(x)\,\frac{1 - \tau_\nu x}{x - \tau_\nu}.$$

All we need is to prove that $\|\hat{Q}_r\|_{C[\tau_{r+1},\tau_{r-1}]} \leq \|T_n^{(k)}\|$ for $k = 0, \dots, n-k+1$.

Step 2. Proof of the inequality

$$|T_n^{(k)}(\tau)| \leq \frac{1}{2k+1}\|T_n^{(k)}\|, \quad k = 0, \dots, n-2, \tag{3.8}$$

where τ is the rightmost critical point (and hence, $|T_n^{(k)}(\tau)|$ is the largest local maximum) of $|T_n^{(k)}|$. This result was also established in [19].

Step 3. Proof that the function $\psi_k(x) := (x^2-1)T_n^{(k+1)}(x)$ satisfies the inequality

$$\|\psi_k\| \leq \Big(1 - \frac{1}{(2k+1)^2}\Big)\|T_n^{(k)}\|, \quad k = 2, \dots, n-2. \tag{3.9}$$

The proof of (3.9) is rather technical. By a result of Sonin-Pólya ([65], [66, §7.31]) we conclude that the relative maxima of $|\psi_k(x)|$ increase as $|x|$ increases, and then a careful estimation of the largest local maximum yields (3.9) (actually, for $k \geq 3$ a sharper inequality than (3.9) was proved).

Step 4. Proof that the polynomials $\{\hat{Q}_\nu\}_{\nu=2}^{n-k-1}$ satisfy

$$\|\hat{Q}_\nu\|_{C[\tau_{\nu+1},\tau_{\nu-1}]} < \|T_n^{(k)}\|.$$

By symmetry, it suffices to consider only half of polynomials $\hat{Q}_\nu$, e.g., those with $2 \leq \nu \leq \lfloor (n-k+1)/2 \rfloor$. Clearly, $\|\hat{Q}_\nu\|_{C[\tau_{\nu+1},\tau_{\nu-1}]}$ is attained at the unique zero $\theta = \theta_\nu$ of ψ'_ν, located in $(\tau_{\nu+1}, \tau_{\nu-1})$. The differential equation for $T_n^{(k)}$ allows us to confine that $\theta \in [\tau_\nu, \tau_{\nu-1})$, $\theta \geq 0$, and to conclude that $T_n^{(k)}$ has a local extremum in $[\theta, \tau_{\nu-1})$. At the critical point θ we have the representations

$$|\hat{Q}_\nu(\theta)| = \frac{1-\tau_\nu\theta}{\theta-\tau_\nu}\,|T_n^{(k)}(\theta)| = \Big(1 - \Big(\frac{\theta-\tau_\nu}{1-\tau_\nu\theta}\Big)^2\Big)^{-1}\psi_k(\theta).$$

If $(1-\tau_\nu\theta)/(\theta-\tau_\nu) < 2k+1$, then the first representation and (3.8) imply

$$|\hat{Q}_\nu(\theta)| = \frac{1-\tau_\nu\theta}{\theta-\tau_\nu}\,|T_n^{(k)}(\theta)| < (2k+1)\frac{1}{2k+1}\,\|T_n^{(k)}\| = \|T_n^{(k)}\|,$$

while in the case $(1-\tau_\nu\theta)/(\theta-\tau_\nu) \geq 2k+1$, the second representation and (3.9) yield the same conclusion,

$$|\hat{Q}_\nu(\theta)| = \Big(1 - \Big(\frac{\theta-\tau_\nu}{1-\tau_\nu\theta}\Big)^2\Big)^{-1}\psi_k(\theta) \leq \Big(1 - \frac{1}{(2k+1)^2}\Big)^{-1}\|\psi_k\| < \|T_n^{(k)}\|.$$

Step 5. Proof that $\hat{Q}_1$ is monotone in $[\tau_2, \tau_0] = [\tau_2, 1]$, and, by symmetry, $\hat{Q}_{n-k}$ is monotone in $[\tau_{n-k+1}, \tau_{n-k-1}] = [-1, \tau_{n-k-1}]$. As a consequence, we get

$$\|\hat{Q}_1\|_{C[\tau_2,1]} = T_n^{(k)}(1) = \|T_n^{(k)}\|,$$

$$\|\hat{Q}_{n-k}\|_{C[-1,\tau_{n-k-1}]} = |T_n^{(k)}(-1)| = \|T_n^{(k)}\|.$$

The monotonicity is equivalent to showing that τ_1, the largest zero of $T_n^{(k)}$, satisfies the inequality

$$\frac{1+\tau_1}{1-\tau_1} < \frac{n^2-k^2}{2k+1},$$

and the latter is established by some delicate estimates for τ_1.

Step 6. Since $-\hat{Q}_0 = \hat{Q}_{n-k+1} = T_n^{(k)}$, we have $\|\hat{Q}_0\|_{C[\tau_1,1]} = \|T_n^{(k)}\|$ and $\|\hat{Q}_{n-k+1}\|_{C[-1,\tau_{n-k}]} = \|T_n^{(k)}\|$. With this step the inequality $\|f^{(k)}\| \le \|T_n^{(k)}\|$ is established for the case $2 \le k \le n-2$. The characterization of the equality cases follows from the observation, that $|f^{(k)}(x)| = \|T_n^{(k)}\|$ is only possible when $x = \pm 1$. The reader is referred to [46] for a detailed proof.

Our method of proof of the case $k \ge 2$ in Theorem 3.10 actually allows us to obtain a more general statement, concerning the ultraspherical polynomials $P_n^{(\lambda)}$ with $\lambda \in \mathbb{N}_0$:

Theorem 3.11. *Let $Q = P_n^{(\lambda)}$, where $\lambda \in \mathbb{N}_0$, and let $f \in \pi_n$ satisfy $|f| \le |Q|$ at $n+1$ distinct points in $[-1,1]$, which are separated by the zeros of Q. Then*

$$\|f^{(k)}\| \le \|Q^{(k)}\| \quad \textit{for} \quad k = 1, \dots, n,$$

and the equality takes place if and only if $f = c\,Q$, $|c| = 1$.

The choice $\lambda = 0$ in Theorem 3.11 corresponds to Theorem 3.10, while for the case $\lambda = m$, $m \in \mathbb{N}$, we may resort to the representation

$$P_n^{(\lambda)}(x) = c(m)\frac{d^m}{dx^m}\{T_{n+m}(x)\},$$

to reduce the proof to the case $k \ge 2$ in Theorem 3.10.

The extension of Schur's Theorem B in the DS-sense, which was given in Theorem 3.7, does not cover the case $k = 1$. However, in this particular case we were able to prove much more [45].

Theorem 3.12. *Let $\{t_\nu\}_{\nu=1}^{n-1}$ be arbitrary $n-1$ points on the real line, which separate the zeros of $\overline{T}_n$. If $f \in \pi_n$ satisfies the assumptions*

$$(i) \quad f(-1) = f(1) = 0;$$
$$(ii) \quad |f(t_\nu)| \le |\overline{T}_n(t_\nu)|, \quad \nu = 1, \dots, n-1,$$

then

$$\|f'\| \le \|\overline{T}_n'\| \ \left(= n\cot\frac{\pi}{2n}\right), \tag{3.10}$$

and the equality occurs if and only if $f = c\,\overline{T}_n$, $|c| = 1$.

The proof of Theorem 3.12 is based on the pointwise estimates, given by Theorem 2.2 with $Q = T_n$. It follows the same scheme as that of the case $k = 1$ in Theorem 3.10, but the technical details are more involved. Notice that Theorem 3.12 fills up also the gap in Theorem 3.8, concerning the case $k = 1$.

4. An Extension of Theorem D

To prove their refinement of Markov's inequality (Theorems C and D), Duffin and Schaeffer put together in a very clever way classical tool from complex analysis as Gauss-Lucas theorem and Rousche's theorem, and geometry of zeros of Chebyshev polynomials. A crucial observation in their proof is the following property of T_n:

$$|T_n(x+iy)| \le |T_n(1+iy)| \quad \text{for every} \quad x \in [-1,1] \quad \text{and} \quad y \in \mathbb{R}. \tag{4.1}$$

We sketch below the argument of Duffin and Schaeffer leading to the proof of (4.1). If we set $\theta_\nu = (\nu - 1/2)\pi/n$ for $\nu = 1, \ldots, n$, $x = \cos\theta$, $0 \le \theta \le \pi$, and $c = 2^{n-1}$, then

$$|T_n(x+iy)|^2 = c^2 \prod_{\nu=1}^{n} \Big(\frac{1}{4}|e^{i\theta} - e^{-i\theta_\nu}|^2 |e^{i\theta} - e^{i\theta_\nu}|^2 + y^2\Big). \tag{4.2}$$

The quantities $|e^{i\theta} - e^{\pm i\theta_\nu}|$ define the lengths of the chords, connecting the vertices of a equilateral $2n$-polygon, inscribed in the unit circle in the complex plane, with the point $e^{i\theta}$. If we alter θ by $k\pi/n$, $k \in \mathbb{Z}$, then the chords change, but the aggregate of their lengths remains the same. Duffin and Schaeffer refer to the following simple fact: if $\{\alpha_j\}_{j=1}^{2n}$ are non-negative real numbers, and $\{\alpha_j'\}_{j=1}^{2n}$ is their permutation such that $\alpha_1' \ge \alpha_2' \ge \cdots \ge \alpha_{2n}'$, then for every $t \ge 0$,

$$\prod_{j=1}^{n-1} (\alpha_{2j-1}'\alpha_{2j}' + t) \ge \prod_{j=1}^{n-1} (\alpha_{2j-1}\alpha_{2j} + t).$$

This observation applied to (4.2) leads to the following conclusion: if $x = \cos\theta$ is fixed, and $x^* = \cos\phi$, where

$$\phi \equiv \theta \pmod{\pi/n}, \quad -\frac{\pi}{2n} < \phi \le \frac{\pi}{2n},$$

then $|T_n(x^* + iy)| \ge |T_n(x+iy)|$. Hence, for a fixed $y \in \mathbb{R}$,

$$\max_{x\in[-1,1]} |T_n(x+iy)| = \max_{x\in[\xi_1,1]} |T_n(x+iy)|,$$

where $\xi_1 = \cos\theta_1$ is the largest zero of T_n. If $x^* \in [\xi_1, 1]$, then the distance between either zero of T_n and the point $x^* + iy$ increases as x^* increases. This yields $|T_n(x^* + iy)| \le |T_n(1 + iy)|$, and proves (4.2).

As was already mentioned, the Chebyshev polynomial T_n is a representative of the ultraspherical polynomials $P_n^{(\lambda)}$, so it seems a natural task to extend Theorem D to the class of ultraspherical polynomials, just in the same way as Theorem 3.1 extends Theorem C. And as Bojanov wrote in [7], such an extension would require knowledge of an analogue of (4.1) for ultraspherical polynomials. Unfortunately, the simple geometrical argument used for derivation of (4.1) does no longer work. In [44] we proposed another approach, which enabled us to prove

Theorem 4.1. *The ultraspherical polynomials $P_n^{(\lambda)}$, ($n \in \mathbb{N}$, $\lambda \ge 0$), satisfy*

$$|P_n^{(\lambda)}(x + iy)| \le |P_n^{(\lambda)}(1 + iy)|, \quad x \in [-1, 1], \quad y \in \mathbb{R}. \tag{4.3}$$

Moreover, if either $\lambda > 0$ or $\lambda = 0$ and $y \ne 0$, then the equality in (4.3) *is attained only at $x = \pm 1$.*

Denote by $\mathcal{P}^r$ the class of polynomials, having only real zeros and coefficients, and by $\mathcal{P}_n^r$ the subset of $\mathcal{P}^r$ of polynomials of degree at most n. Our approach to the proof of Theorem 4.1 is based on the following expansion formula:

$$|f(x + iy)|^2 = \sum_{k=0}^{n} L_k(f; x) y^{2k}, \quad f \in \mathcal{P}_n^r, \tag{4.4}$$

where L_k is a non-linear differential operator, defined by

$$L_k(f; x) := \sum_{j=0}^{2k} (-1)^{k+j} \, \frac{f^{(j)}(x)}{j!} \frac{f^{(2k-j)}(x)}{(2k-j)!} \, . \tag{4.5}$$

Apparently, Jensen [26] was the first to introduce operators L_k. It is well-known, that

$$L_k(f; x) \ge 0 \ \text{ for every } \ x \in \mathbb{R}, \quad \text{if } \ f \in \mathcal{P}^r \tag{4.6}$$

(actually, (4.6) holds true for every function from the Laguerre-Pólya class). If $f \in \mathcal{P}_n^r$, $f(x) = c\,(x - x_1)(x - x_2) \dots (x - x_n)$, it is not difficult to see that (4.5) is equivalent to

$$L_k(f; x) = f^2(x) \sum \frac{1}{(x - x_{i_1})^2 \cdots (x - x_{i_k})^2},$$

the sum being extended over all k-combinations $(i_1, \dots, i_k)$ of $\{1, \dots, n\}$. This representation proves (4.6), and allows us to specify that if $f \in \mathcal{P}_n^r$ has only simple zeros and $k \le n$, then $L_k(f; x) > 0$ for every $x \in \mathbb{R}$.

Let $P = P_n^{(\lambda)}$. For $k = 1, 2, \dots, n$, we introduce an auxiliary function $F_k(P; x)$,

$$F_k(P;x) := L_k(P;x) + \frac{(2\lambda+1)(1-x^2)}{(2\lambda+2k+1)n(n+2\lambda)} L_k(P';x).$$

Theorem 4.1 is proved with the help of the following lemma.

Lemma 4.1. *The derivative of $F_k(P;x)$ can be represented as*

$$F_k'(P;x) = \frac{4x}{(2\lambda+2k+1)n(n+2\lambda)} L_{k-1}(P'';x) + \frac{2(2\lambda+1)(2\lambda+3k)x}{(2\lambda+2k+1)n(n+2\lambda)} L_k(P';x).$$

Let us mention that, although the proof of Lemma 4.1 is rather technical, it uses solely the representation (4.5) and the differential equation

$$(n-j)(n+j+2\lambda)P^{(j)}(x) - (2\lambda+2j+1)xP^{(j+1)}(x) + (1-x^2)P^{(j+2)}(x) = 0.$$

It follows from Lemma 4.1 that $F_k(P;x)$ is monotone decreasing in $(-\infty, 0]$, and monotone increasing in $[0, \infty)$, and, in addition, the monotonicity is strict if $k < n$. As $F_k(P;\cdot)$ is even function,

$$L_k(P;x) \le F_k(P;x) \le F_k(P;1) = L_k(P;1) \quad \text{for every} \quad x \in [-1,1].$$

Then the claim of Theorem 4.1 follows from

$$|P(1+iy)|^2 - |P(x+iy)|^2 = \sum_{k=0}^{n} [L_k(P;1) - L_k(P;x)] y^{2k}$$

(here, $L_0(P;x) = P^2(x)$, and for $P = P_n^{(\lambda)}$ with $\lambda \ge 0$, $P^2(x) \le P^2(1)$ whenever $x \in [-1,1]$).

Theorem 4.1 allows us to obtain the following extension of Theorem D:

Theorem 4.2. *Let $Q = P_n^{(\lambda)}$, where $n \in \mathbb{N}$ and $\lambda \in [0, 1/2]$. If $f \in \pi_n^r$ and $|f(x)| \le |Q(x)|$ whenever $(1-x^2)Q'(x) = 0$, then, for $k = 1, \dots, n$,*

$$|f^{(k)}(x+iy)| \le |Q^{(k)}(1+iy)| \quad \textit{for every} \quad x \in [-1,1],\ y \in \mathbb{R}.$$

The equality occurs of and only if $f = \pm Q$.

The restriction $\lambda \le 1/2$ drops, if we replace the requirement $|f| \le |Q)|$ at the zeros of $(1-x^2)Q'(x)$ by $|f'| \le |Q'|$ at the zeros of Q.

Theorem 4.3. *Let $Q = P_n^{(\lambda)}$, where $n \in \mathbb{N}$ and $\lambda \ge 0$. If $f \in \pi_n^r$ and $\|f'(x)\| \le \|Q'(x)\|$ whenever $Q(x) = 0$, then, for $k = 1, \dots, n$,*

$$|f^{(k)}(x+iy)| \le |Q^{(k)}(1+iy)| \quad \textit{for every} \quad x \in [-1,1],\ y \in \mathbb{R}.$$

It can be said that Theorem 4.2 makes sense only when $x^2+y^2 \le 1$, because of the following result of Bernstein [2].

Theorem 4.4. *Let Q be a polynomial of degree n, whose zeros are all simple and located in $(-1,1)$. If $f \in \pi_n$ satisfies $|f| \le |Q|$ at $n+1$ points in $[-1,1]$, which are separated by the zeros of Q, then for $k=1,\dots,n$,*

$$|f^{(k)}(x+iy)| \le |Q^{(k)}(x+iy)| \quad \textit{whenever} \quad x^2+y^2 \ge 1.$$

5. DS-type Inequalities in L_2-norms

Let us recall the claim of Lemma 2.2. Assume that $\Delta = \{t_\nu\}_{\nu=0}^n$ consists of $n+1$ distinct points on the real line,

$$t_0 > t_1 > \cdots > t_n,$$

and Q is a polynomial of degree n, whose zeros $\{x_\nu\}_{\nu=1}^n$ separate the points from Δ, i.e.,

$$t_0 > x_1 > t_1 > \cdots > t_{n-1} > x_n > t_n.$$

For $k=1,\dots,n$, we consider the class of polynomials

$$\Omega(Q,\Delta) := \{f \in \pi_n \,:\, |f| \le |Q| \ \text{ on } \ \Delta\}.$$

According to Lemma 2.2, there exists a set $I_{n,k}(\Delta) \subset \mathbb{R}$, with the property

$$f \in \Omega(Q,\Delta) \Rightarrow |f^{(k)}(x)| \le |Q^{(k)}(x)| \ \text{ for every } \ x \in I_{n,k}(\Delta). \tag{5.1}$$

The set $I_{n,k}(\Delta)$ coincides with $\mathbb{R}$ when $k=n$, and consists of $n-k+1$ non-overlapping intervals, when $1 \le k \le n-1$. In the latter case, we have the following necessary and sufficient condition for $t \in \mathbb{R}$ to belong to $I_{n,k}(\Delta)$:

$$t \in I_{n,k}(\Delta) \iff \omega_0^{(k)}(t)\,\omega_n^{(k)}(t) \ge 0, \tag{5.2}$$

where $\omega_\nu(x) = \omega(x)/(x-t_\nu)$, $\nu=0,\dots,n$. Moreover, $t \in \mathbb{R}$ is interior point for $I_{n,k}(\Delta)$ exactly when the inequality in (5.2) is strict. And, finally, we have the equivalence

$$\begin{aligned} f \in \Omega(Q,\Delta) \ \text{ and } \ |f^{(k)}(t)| = |Q^{(k)}(t)| \ \text{ at an interior point } \ t \in I_{n,k}(\Delta) \\ \iff f = c\,Q \ \text{ with some constant } \ c, \ |c|=1. \end{aligned} \tag{5.3}$$

The following is idea due to Bojanov since 1986 (see [4]). If $f, Q \in \pi_n$, then $|f^{(k)}|^2$ and $|Q^{(k)}|^2$ belong to π_{2n-2k}, and therefore any weighted L_2-norm of $f^{(k)}$ and $Q^{(k)}$ can be calculated exactly with the help of appropriate $(n-k+1)$-point Gauss-type quadrature formula. It is well-known that the coefficients of such quadrature formulae are all positive. Therefore, should all the nodes of this quadrature formula belong to $I_{n,k}(\Delta)$, one immediately obtains a DS-type inequality in L_2-norm. Bojanov's idea has been exploited in [4], [67], [15], [24], [25], and [42] for derivation of sharp polynomial inequalities in some weighted L_2-norms, generated by the classical weight functions of Gegenbauer $w_\mu(x) = (1-x^2)^{\mu-1/2}$ ($\mu > -1/2$, $x \in [-1,1]$), Laguerre $w_{L,\alpha}(x) = x^\alpha e^{-x}$ ($\alpha > -1$, $x \in \mathbb{R}_+$), and Hermite $w_H(x) = e^{-x^2}$ ($x \in \mathbb{R}$). What makes Bojanov's idea applicable in these cases is the fact, that derivatives of the corresponding orthogonal polynomials are again orthogonal polynomials associated with the same (Hermite) or alien (Gegenbauer, Laguerre) weight function.

The theorem we quote below summarizes some inequalities, essentially obtained by Bojanov [4], and by Guessab and Rahman [24].

Theorem 5.1. *Let Q be a polynomial of degree n, whose zeros separate the zeros of $(1-t^2)\dfrac{d}{dt}P_n^{(\lambda)}(t)$, $-1/2 < \lambda \le 1/2$. If $p \in \pi_n$ and $|f(x)| \le |Q(x)|$ whenever $(1-x^2)\dfrac{d}{dx}P_n^{(\lambda)}(x) = 0$, then*

$$\int_{-1}^{1} w_{\lambda+k-1}(t)|p^{(k)}(t)|^2\,dt \le \int_{-1}^{1} w_{\lambda+k-1}(t)|Q^{(k)}(t)|^2\,dt \tag{5.4}$$

for $k = 1,\dots,n$, and

$$\int_{-1}^{1} w_{\lambda+k-2}(t)|p^{(k)}(t)|^2\,dt \le \int_{-1}^{1} w_{\lambda+k-2}(t)|Q^{(k)}(t)|^2\,dt \tag{5.5}$$

for $k = 2,\dots,n$. The equality in either (5.4) *or* (5.5) *is attained if and only if $p(x) = c\,Q(x)$, $|c| = 1$.*

Proof. Let Δ consists of the zeros of $(1-t^2)\dfrac{d}{dt}P_n^{(\lambda)}(t)$. We shall verify that the zeros of $\dfrac{d^{k-1}}{dx^{k-1}}\{P_n^{(\lambda)}(x)\} = c\,P_{n-k+1}^{(\lambda+k-1)}(x)$, where $c = const.$, are interior points for $I_{n,k}(\Delta)$. We start with the case $k=1$. According to (5.2), we have to show that if $P_n^{(\lambda)}(x_*) = 0$, then $\omega_0'(x_*)\,\omega_n'(x_*) > 0$. Since $y = P_n^{(\lambda)}$ satisfies

$$(1-x^2)y'' - (2\lambda+1)xy' + n(n+2\lambda)y = 0,$$

and $y(x_*) = 0$, we find $(1-x_*^2)y''(x_*) = (2\lambda+1)x_*y'(x_*)$, whence

$$\omega_0'(x_*) = \Big[1 + (2\lambda+1)\frac{x_*}{1-x_*}\Big]y'(x_*), \quad \omega_n'(x_*) = \Big[1 - (2\lambda+1)\frac{x_*}{1+x_*}\Big]y'(x_*).$$

The terms in square brackets are easily seen to be positive, as $-1/2 < \lambda \le 1/2$ and $x^* \in (-1,1)$. Since $y'(x_*) \ne 0$ (the zeros of $P_n^{(\lambda)}$ are all simple), we conclude that $\omega_0'(x_*)\,\omega_n'(x_*) > 0$, i.e., x_* is interior point for $I_{n,1}(\omega)$. Even more, if x_* and x_{**} are two consecutive zeros of $y = P_n^{(\lambda)}$, then it follows from the above representation of $\omega_0'(x_*)$ and $\omega_n'(x_*)$ that

$$\omega_0'(x_*)\,\omega_0'(x_{**}) < 0, \qquad \omega_n'(x_*)\,\omega_n'(x_{**}) < 0.$$

This means that the zeros of $P_n^{(\lambda)}$ interlace strictly with the zeros of both ω_0' and ω_n'. According to Lemma 2.1, the interlacing property is inherited by the zeros of derivatives. Specifically, it follows that, for $k = 1, \dots, n-1$, the zeros of $\dfrac{d^{k-1}}{dx^{k-1}}\{P_n^{(\lambda)}\}$ interlace strictly with the zeros of both $\omega_0^{(k)}$ and $\omega_n^{(k)}$. In view of the definition of $I_{n,k}(\Delta)$, this means that the zeros of $\dfrac{d^{k-1}}{dx^{k-1}}\{P_n^{(\lambda)}\}$ are interior points for $I_{n,k}(\Delta)$ (more precisely, each of the $n-k+1$ Chebyshev intervals forming $I_{n,k}(\Delta)$ contains exactly one zero of $\dfrac{d^{k-1}}{dx^{k-1}}\{P_n^{(\lambda)}\}$).

The zeros $\{z_j\}_{j=1}^{n-k+1}$ of $\dfrac{d^{k-1}}{dx^{k-1}}\{P_n^{(\lambda)}\}$ are the abscissae in the $(n-k+1)$-point Gaussian quadrature formula

$$\int_{-1}^{1} w_{\lambda+k-1}(t)\, f(t)\, dt = \sum_{j=1}^{n-k+1} w_j^G\, f(z_j) + R_{n-k+1}^G(f),$$

which is uniquely determined by the property that $R_{n-k+1}^G(f) = 0$ whenever $f \in \pi_{2n-2k+1}$. If $p \in \pi_n$ satisfies the assumption of Theorem 5.1, then

$$|p^{(k)}(z_j)| \le |Q^{(k)}(z_j)|, \quad j = 1, \dots, n-k+1, \tag{5.6}$$

and all the inequalities are strict unless $p = c\,Q$, $|c| = 1$. Since $|p^{(k)}|^2$ and $|Q^{(k)}|^2$ belong to π_{2n-2k}, we have

$$\begin{aligned}\int_{-1}^{1} w_{\lambda+k-1}(t)\, |p^{(k)}(t)|^2 dt &= \sum_{j=1}^{n-k+1} w_j^G\, |p^{(k)}(z_j)|^2 \\ &\le \sum_{j=1}^{n-k+1} w_j^G\, |Q^{(k)}(z_j)|^2 = \int_{-1}^{1} w_{\lambda+k-1}(t)\, |Q^{(k)}(t)|^2 dt\,.\end{aligned}$$

Hence, (5.4) holds true, and the equality occurs if and only if $p = c\,Q$, $|c| = 1$.

For the proof of (5.5)) we apply the generalized Gauss-Lobatto quadrature formula

$$\int_{-1}^{1} w_\lambda(t) f(t)\, dt = \sum_{\nu=0}^{k-2} a_\nu \left[f^{(\nu)}(-1) + (-1)^\nu f^{(\nu)}(1)\right] + \sum_{j=1}^{n-k+1} w_j^{Lo} f(z_j) + R^{Lo}(f),$$

which has the same interior nodes as the Gaussian quadrature formula we used above, and whose coefficients $\{a_\nu\}_{\nu=0}^{k-2}$ and $\{w_j^{Lo}\}_{j=1}^{n-k+1}$ are all positive.

The remainder R^{Lo} of this quadrature formula satisfies $R^{Lo}(f) = 0$ whenever $f \in \pi_{2n-1}$, therefore it calculates exactly the (weighted with $w_\lambda(t)$) integrals of $(1-x^2)^{k-2}|p^{(k)}(x)|^2$ and $(1-x^2)^{k-2}|Q^{(k)}(x)|^2$. Hence,

$$\begin{aligned}
\int_{-1}^{1} w_{\lambda+k-2}(t)\,|p^{(k)}(t)|^2 dt &= \int_{-1}^{1} w_\lambda(t)(1-t^2)^{k-2}\,|p^{(k)}(t)|^2 dt \\
&= (2k-4)!!\,a_{k-2}\big[|p^{(k)}(-1)|^2 + |p^{(k)}(1)|^2\big] + \sum_{j=1}^{n-k+1} w_j^{Lo}(1-z_j^2)^{k-2}\,|p^{(k)}(z_j)|^2 \\
&\le (2k-4)!!\,a_{k-2}\big[|Q^{(k)}(-1)|^2 + |Q^{(k)}(1)|^2\big] + \sum_{j=1}^{n-k+1} w_j^{Lo}(1-z_j^2)^{k-2}\,|Q^{(k)}(z_j)|^2 \\
&= \int_{-1}^{1} w_{\lambda+k-2}(t)\,|Q^{(k)}(t)|^2 dt.
\end{aligned}$$

Again, the equality occurs in (5.6) if and only if $p = c\,Q$, $|c| = 1$. □

Since the nodes of the Gauss-type quadrature formulae, used in the proof of Theorem 5.1, are interior points for the corresponding Chebyshev sets $I_{n,k}(\Delta)$, one might be tempted to apply the same approach to obtain DS-type inequalities for the same check points, but for different L_2-norms. This turns out to be possible, though only for small (depending on n) variation in the weight function. The theorem below demonstrates this effect in the special case $\lambda = 1/2$.

Theorem 5.2. *Let Q be a polynomial of degree n, whose zeros separate the zeros of $(1-t^2)P_n'(t)$. If $p \in \pi_n$ and $|p(x)| \le |Q(x)|$ whenever $(1-x^2)P_n'(x) = 0$, then, for every $\lambda \in [\frac{1}{2} - \frac{2}{2n+5}, \frac{1}{2}]$,*

$$\int_{-1}^{1} w_{\lambda+k-1}(t)\,|p^{(k)}(t)|^2 dt \le \int_{-1}^{1} w_{\lambda+k-1}(t)\,|Q^{(k)}(t)|^2 dt \quad \text{for} \quad k = 1,\dots,n,$$

and

$$\int_{-1}^{1} w_{\lambda+k-2}(t)\,|p^{(k)}(t)|^2 dt \le \int_{-1}^{1} w_{\lambda+k-2}(t)\,|Q^{(k)}(t)|^2 dt \quad \text{for} \quad k = 2,\dots,n.$$

The equality occurs in either case if and only if $p = c\,Q_n$, $|c| = 1$.

Theorem 5.2 was proved in [25] by means of Sturm's comparison theorem. Also, it was shown in [25] that, for higher order derivatives, the variation in the weight can be significant.

We proceed with a DS-type inequality in the L_2-norm, generated by the Hermite weight function $w_H(x) = e^{-x^2}$, i.e.,

$$\|f\| := \Big(\int_{\mathbb{R}} e^{-x^2}|f(x)|^2 dx\Big)^{1/2}.$$

As is well-known, the associated with $w_H(x)$ orthogonal polynomials are the Hermite polynomials

$$H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m}\{e^{-x^2}\}, \quad m = 0, 1, \dots$$

The following theorem was proved in [42].

Theorem 5.3. *If $f \in \pi_n$ and $|f(x)| \le |H_n(x)|$ whenever $H_{n+1}(x) = 0$, then, for $k = 1, \dots, n$,*

$$\|f^{(k)}\| \le \|H_n^{(k)}\|. \tag{5.7}$$

Moreover, the equality in (5.7) *occurs if and only if $f = cH_n$, $|c| = 1$.*

Proof. Let $\Delta = \{t_\nu\}_{\nu=0}^n$ be formed by the zeros of H_{n+1}, they obviously are separated by the zeros of H_n. We shall show that if $H_n(\tau) = 0$, then τ is an interior point for $I_{n,1}(\Delta)$. In this case, apart from a constant multiplier, we have

$$\omega_0(x) = \frac{H_{n+1}(x)}{x - t_0}, \quad \omega_n(x) = \frac{H_{n+1}(x)}{x - t_n}.$$

We make use of the property $H_m'(x) = 2mH_{m-1}(x)$ (see [66, Chapt. 5.5]) to obtain

$$\omega_0'(\tau) = 2(n+1)\frac{H_n(\tau)}{\tau - t_0} - \frac{H_{n+1}(\tau)}{(\tau - t_0)^2} = -\frac{H_{n+1}(\tau)}{(\tau - t_0)^2}$$

and

$$\omega_n'(\tau) = 2(n+1)\frac{H_n(\tau)}{\tau - t_n} - \frac{H_{n+1}(\tau)}{(\tau - t_n)^2} = -\frac{H_{n+1}(\tau)}{(\tau - t_n)^2},$$

whence $\omega_0'(\tau)\,\omega_n'(\tau) > 0$. This means that $x = \tau$ is interior point for $I_{n,1}(\Delta)$, and we conclude in the same fashion as in the proof of Theorem 5.1 that, for $k = 1, \dots, n$, the zeros of $H_n^{(k-1)} = c(k,n)H_{n-k+1}$ are interior points for $I_{n,k}(\Delta)$. Hence,

$$|f^{(k)}| \le |H_n^{(k)}| \ \text{ at the zeros of } \ H_{n-k+1},$$

and the inequality is strict unless $f = cH_n$ with some constant c, $|c| = 1$. The quantities $\|f^{(k)}\|^2$ and $\|H_n^{(k)}\|^2$ can be calculated exactly by the $(n-k+1)$-point Gaussian quadrature formula Q_{n-k+1}^G, associated with $w_H(x)$. Since Q_{n-k+1}^G is positive quadrature formula, whose nodes are the zeros of H_{n-k+1}, we obtain from the above inequalities

$$\begin{aligned}\|f^{(k)}\|^2 &= \int_{\mathbb{R}} e^{-x^2}|f^{(k)}(x)|^2 dx = Q_{n-k+1}^G|f^{(k)}|^2 \\ &\le Q_{n-k+1}^G|H_n^{(k)}|^2 = \int_{\mathbb{R}} e^{-x^2}|H_n^{(k)}(x)|^2 dx = \|H_n^{(k)}\|^2.\end{aligned}$$

The inequality $\|f^{(k)}\|^2 \le \|H_n^{(k)}\|^2$ is strict unless $f = c\,H_n$ for some constant c, $|c| = 1$. □

From Theorem 5.2 we obtain the following interesting consequence:

Corollary 5.1. *If* $f(x) = \sum_{m=0}^{n} c_m H_m(x)$ *satisfies* $|f| \le |H_n|$ *at the zeros of* H_{n+1}*, then*

$$|c_m|^2 \le 2^{n-m} \frac{n}{m} \frac{n!}{m!} \quad \textit{for} \quad m = 1, \ldots, n.$$

Proof. Theorem 5.2, applied with $k = 1$, together with the properties (see [66]) $H_m'(x) = 2mH_{m-1}(x)$ and $\|H_m\|^2 = \pi^{1/2} 2^m m! =: h_m^2$ yields

$$\|f'\|^2 = \sum_{m=1}^{n} 4m^2 |c_m|^2 h_{m-1}^2 \le \|H_n'\|^2 = 4n^2 h_{n-1}^2\,,$$

whence

$$|c_m|^2 \le \frac{n^2}{m^2} \frac{h_{n-1}^2}{h_{m-1}^2} = 2^{n-m} \frac{n}{m} \frac{n!}{m!}\,.$$

In particular, $|c_n| \le 1$, i.e., $f \in \pi_n$ cannot have bigger leading coefficient than H_n, if $|f| \le |H_n|$ at the zeros of H_{n+1}. Let us mention that the estimates for $|c_m|$, which are obtained from Theorem 5.2 with $k \ge 2$, are less precise. □

We conclude this section with some DS-type inequalities in the L_2-norms, generated by the Laguerre weight functions $w_{L,\alpha}(x) = x^\alpha e^{-x}$, $\alpha > -1$, $x \in \mathbb{R}_+$. It is not surprising that the extremizer in these inequalities is the associated with $w_{L,\alpha}(x)$ orthogonal polynomial, namely, the Laguerre polynomial

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\, e^x x^{-\alpha} \Big(\frac{d}{dx}\Big)^n \{e^{-x} x^{n+\alpha}\}.$$

Theorem 5.4. *(i) Let* $Q := L_n^{(\alpha)}$*, and let* $f \in \pi_n$ *satisfy* $|f| \le |Q|$ *at the zeros of* $L_{n+1}^{(\beta)}(x)$*, where either* $\beta = \alpha$*, or* $\beta = \alpha - 1$ *and* $\beta > -1$*. Then, for* $k = 1, \ldots, n$*,*

$$\int_0^\infty x^{k+\beta} e^{-x} |f^{(k)}(x)|^2\, dx \le \int_0^\infty x^{k+\beta} e^{-x} |Q^{(k)}(x)|^2\, dx\,, \tag{5.8}$$

and the equality occurs in (5.8) *if and only if* $f = c\,Q$ *with some constant* c*,* $|c| = 1$*.*

(ii) Let $Q := L_n^{(\alpha)}$*, and let* $f \in \pi_n$ *satisfy* $|f| \le |Q|$ *at the zeros of* $xL_n^{(\beta+1)}(x)$*, where either* $\beta = \alpha$ *or* $\beta = \alpha + 1$*. Then* (5.8) *holds true, and the equality occurs if and only if* $f = c\,Q$ *with some constant* c*,* $|c| = 1$*.*

The proof of Theorem 5.4 follows the same scheme as that of Theorems 5.1 and 5.3 (for part (ii), an appropriate Radau quadrature formula is used instead of the Gaussian quadrature formula). We omit the details.

6. Remarks

1. The requirement in Section 2 for the zeros of oscillating polynomial Q to separate the points in Δ (Assumption B) was imposed only for reasons of clarity. Actually, instead of strict interlacing, we may assume only that the zeros of Q and the points from Δ interlace.

2. As was mentioned in the introduction, the genesis of the Markov inequalities was inspired by a question about the magnitude of the coefficients of a polynomial $P \in \pi_n$, given its uniform deviation from zero in some interval $[a, b]$. Of course, the same question can be asked for the class of polynomials $\Omega_n(Q, \Delta) = \{f \in \pi_n : |f| \le |Q| \text{ on } \Delta\}$, where Q and Δ satisfy Assumptions A and B, namely, how large can be

$$\sup\left\{|P^{(k)}(0)|/k! : P \in \Omega_n(Q, \Delta)\right\} ? \tag{6.1}$$

The pointwise estimates in Section 2 can be used for answering this question, or at least for providing some upper bounds for this quantity. For instance, when Δ is symmetric with respect to the origin and $n - k$ is even, then the suppremum in (6.1) is attained for $P = Q$, for, in this case $I_{n,k}(\Delta)$ is also symmetric and consists of $n - k + 1$ intervals, hence $x = 0$ belongs to $I_{n,k}(\Delta)$. For some similar results, as well as for pointwise estimates in $\mathbb{C}$ for polynomials which are bounded on a fixed discrete set of points in $\mathbb{R}$, see also [17].

3. The DS-type inequalities in Section 3 can find application to the estimation of the round-off error on Lagrange's differentiation formula. Namely, let the pair (Q, Δ) admit DS-type inequality, where $\Delta = \{t_\nu\}_{\nu=0}^n$. Consider the approximation

$$f^{(k)}(x) \approx L_n^{(k)}(f; x),$$

where $L_n(f; \cdot)$ is the Lagrange interpolating polynomial for f with interpolation nodes defined by Δ. Assume that, instead of the exact values $\{f(t_\nu)\}_{\nu=0}^n$, inaccurate data $\{\widetilde{f}(t_\nu)\}_{\nu=0}^n$ is available, and

$$|\widetilde{f}(t_\nu) - f(t_\nu)| \le \varepsilon_\nu, \quad \nu = 0, \dots, n.$$

Then the error in the Lagrange differentiation formula due to the inaccuracy of the data is $R_k(f; x) = L_n^{(k)}(\widetilde{f} - f; x)$, and the following estimate holds true:

$$\|R_k(f; \cdot)\| \le M\, \|Q^{(k)}\|, \quad \text{where} \quad M = \max_{0 \le \nu \le n} \frac{\varepsilon_\nu}{|Q(t_\nu)|}\,.$$

In the DSS-type inequalities, the zero boundary conditions could be interpreted as the exact values of f at the end-points are given.

Similarly, the results from Section 2 can be applied for derivation of pointwise bounds for $R_k(f; x)$.

4. Our proof of the inequality $L_k(P;x) \le L_k(P;\pm 1)$, $P = P_n^{(\lambda)}$, $x \in [0,1]$, was influenced by the work of Patrick [47], who proved this result for $1 \le k \le 3$. We believe that the following stronger statement holds true:

   Conjecture. *For $P = P_n^{(\lambda)}$ and $k = 1, \dots, n-1$, $L_k(P;\cdot)$ is a strictly monotone increasing function in $[0,\infty)$.*

   Recently we found a refinement of Jensen's inequalities $\mathcal{L}_m(f;x) \ge 0$, $x \in \mathbb{R}$, where $\mathcal{L}_m(f;x) = (2m)!\, L_m(f;x)$, i.e.,

$$\mathcal{L}_m(f;x) := \sum_{j=0}^{2m} (-1)^{m+j} \binom{2m}{j} f^{(j)}(x) f^{(2m-j)}(x).$$

   Jensen's inequalities hold true for every real-valued entire function f from the Laguerre-Pólya class. For $m, n \in \mathbb{R}$, $0 \le 2m \le n$, let

$$U_{2m}^n(f;x) := \sum_{j=0}^{2m} (-1)^{m+j} \binom{2m}{j} \frac{(n-j)!\,(n-2m+j)!}{(n-m)!(n-2m)!} f^{(j)}(x)\, f^{(2m-j)}(x).$$

   Foster and Krasikov formulated in [20] a conjecture, stating that

$$U_{2m}^n(f;x) \ge 0 \quad \text{for every} \quad f \in \mathcal{P}_n^r \quad \text{and} \quad x \in \mathbb{R}. \tag{6.2}$$

   This conjecture was validated by Uluchev and the author in [43]. A step towards the proof of (6.2) is the following identity [43, Lemma 4],

$$(n+1-m)\, U_{2m}^{n+1}(f;x) = (n+1)\, U_{2m}^n(f;x) + 2m(2m-1)\, U_{2m-2}^{n-1}(f';x).$$

   Clearly, for a fixed $m \in \mathbb{N}$, $\lim_{n\to\infty} n^{-m}\, U_{2m}^n(f;x) = \mathcal{L}_m(f;x)$. The above identity allows us to conclude that, for a fixed $f \in \mathcal{P}_k^r$, $k \ge 2m$, the sequence $\{n^{-m}\, U_{2m}^n(f;x)\}_{n=k}^{\infty}$ tends to $\mathcal{L}_m(f;x)$ *monotonically,*

$$n^{-m}\, U_{2m}^n(f;x) \nearrow \mathcal{L}_m(f;x) \quad \text{as} \quad n \to \infty.$$

   Indeed, it follows from the above identity and (6.2) that

$$(n+1-m)\, U_{2m}^{n+1}(f;x) \ge (n+1)\, U_{2m}^n(f;x),$$

   whence

$$\frac{(n+1)^{-m}\, U_{2m}^{n+1}(f;x)}{n^{-m}\, U_{2m}^n(f;x)} \ge \Big(1 - \frac{1}{n+1}\Big)^m \Big(1 - \frac{m}{n+1}\Big)^{-1} \ge 1,$$

   where in the last step we have used the Bernoulli inequality. Thus, for polynomials with only real zeros, *Jensen's inequalities are a consequence from* (6.2).

5. The statement of Theorem 3.11 could be equivalently formulated as follows: *Whenever $P_n^{(\lambda)}$ ($\lambda \in \mathbb{N}_0$) is the snake polynomial associated with some (unknown) curve $\varphi(x)$, $P_n^{(\lambda)}$ is also the extremizer in the Markov-type inequality for the class of polynomials majorized by $\varphi(x)$.* The same applies to the stretched Chebyshev polynomial $\overline{T}_n(x)$, but only for $k = 1$ (see Theorem 3.12). Actually, the method of proof in [46] allows the result of Theorem 3.11 to be extended without difficulties to any $\lambda \geq 2$.

   As was already mentioned in the introduction, it is not clear whether for any majorant φ, the snake-polynomial Q is always the extremizer in the corresponding Markov-type inequality. Most probably, it isn't. The snake-polynomial Q is obviously extremal for the highest derivatives $k = n - 1$ and $k = n$ (when $\deg Q = n$). In a recent paper [41] this was shown to be true for $k = n - 2$, too. A conjecture of Shadrin states that, for any φ, the snake-polynomial of degree n is the extremizer in the Markov-type inequality for derivatives of order $k > n/2$.

6. Recently Dryanov [16] established an inequality in the $C[-1, 1]$-norm, which reads as

$$\|p'\| \leq n \max_{1 \leq j \leq n} |\Lambda_j(p)|, \quad p \in \pi_n. \tag{6.3}$$

   Here $\{\Lambda_j\}_{j=1}^n$ are some linear functionals satisfying

$$\max_{1 \leq j \leq n} |\Lambda_j(p)| \leq n \max_{1 \leq j \leq n} \big|p\big(\cos(j\pi/n)\big)\big|.$$

   Therefore, (6.3) furnishes another DS-type extension of A. Markov's inequality $\|p'\| \leq n^2 \|p\|$.

7. Bernstein's inequality on the unit disc of $\mathbb{C}$ was extended in the DS-sense by Frappier, Rahman and Ruscheweich [22], who proved that

$$\max_{|z|=1} |p'(z)| \leq n \max_{1 \leq m \leq 2n} \big|p\big(e^{im\pi/n}\big)\big| \quad \text{for every} \quad p \in \pi_n^r.$$

8. There is a huge number of publications on the inequality of brothers Markov and its various generalization. Every monograph on approximation theory or polynomial inequalities contains at least a chapter devoted to this topic (see, e.g., [13], [14], [36], [58], [56], [60]). Twelve proofs of the classical Markov inequality ("...to satisfy any taste: long, short, elementary, complex, erroneous, incomplete...") along with a lot of historical remarks can be found in the recent survey of Shadrin [64]. Another survey on this subject was written by Bojanov [7].

   We only outline some developments on this subject, which do not fall into the DS-concept. An important property of oscillating polynomials was established by Bojanov and Rahman [12]. It says that, for a polynomial p with only real zeros with prescribed multiplicities, the magnitude of every local extremum of p' is a monotone increasing function on the magnitude of each local extremum of p. This remarkable property was exploited in a series of papers [3], [5], [33], [34] for derivation of various Markov- or Turán-type inequalities for oscillating algebraic or trigonometric polynomials. Bojanov and Naidenov proved in [9] Markov-type inequality for oscillating perfect splines. Sufficient conditions on a general Chebyshev system of functions to secure the role of extremizer of the Chebyshev polynomial in this system (i.e., the maximally equioscillating polynomial) in the Markov inequality were found by Bojanov and Naidenov in [10]. In a recent paper [8] Bojanov studied pointwise estimates in $\mathbb{C}$ for polynomials with a polynomial majorant in $[-1,1]$. In particular, his result yields yet another proof of the inequality of brothers Markov.

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