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On the Discriminants of the Hypergeometric Polynomials from the Askey Scheme

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It has already been proven that all the hypergeometric polynomials from the Askey scheme have a lowering operator of the form

$$p_n'(x) = A_n(x)\,p_{n-1}(\lambda(x)) - B_n(x)\,p_n(\lambda(x)),$$

which leads to an alternative way of computing the discriminants of these polynomial sets. In this note we shall discuss a constructive way of obtaining the rational functions $A_n(x)$ in the above representation.

One can prove a theorem about the representation of the form

$$\Delta p_n(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x) \tag{1}$$

where

$$A_n(x) = \frac{p_n(t+1)p_n(t)w(t)}{a_n\kappa_{n-1}(t-x)} + \frac{1}{a_n\kappa_{n-1}}\sum_{l=s}^{t} p_n(l)p_n(l-1)\frac{u(x+1)-u(l)}{x+1-l}w(l)$$

$$B_n(x) = \frac{p_{n-1}(t)p_n(t+1)w(t)}{a_n\kappa_{n-1}(t-x)} + \frac{1}{a_n\kappa_{n-1}}\sum_{l=s}^{t} p_n(l)p_{n-1}(l-1)\frac{u(x+1)-u(l)}{x+1-l}w(l)$$

and explicitly show the expression for $A_n(x)$ and $B_n(x)$ for all polynomials orthogonal w.r.t. a measure supported on equidistant subset of integers bounded from below. This theorem is constructive in the sense that if one applies the formula in it and works using series manipulations, summations theorems and so on, as an application of it one can obtain the coefficient functions $A_n(x)$ and $B_n(x)$. However it is much more convenient to find the coefficient functions $A_n(x)$ and $B_n(x)$ for all the cases from the Askey scheme directly using only the definition of the polynomials. The answers are the following:

Charlier polynomials

$$\Delta C_n(x) = -\frac{n}{a}\,C_{n-1}(x).$$

Kravchuck polynomials

$$\Delta K_n(x) = \frac{n(1-p)}{p(x-N)} K_{n-1}(x) + \frac{n}{x-N} K_n(x).$$

Hahn polynomials

$$\begin{aligned} \Delta Q_n(x) = {} & \frac{n(\beta+n)(\alpha+\beta+n+N+1)}{(\alpha+\beta+2n)(\alpha+x+1)(x-N)} Q_{n-1}(x) \\ & + \frac{n(\alpha+n)(\alpha+x+1)+n(\beta+n)(\alpha+n+x-N)}{(x-N)(\alpha+x+1)(\alpha+\beta+2n)} Q_n(x). \end{aligned}$$

Dual Hahn polynomials

$$\begin{aligned} \frac{\Delta R_n(\lambda(x))}{\Delta\lambda(x)} = {} & \frac{n(N+1+\delta-n)}{(x+\gamma+\delta+1)(x-N)(x+\gamma+1)} R_{n-1}(\lambda(x)) \\ & - \frac{n(N-n-x-\gamma)}{(x+\gamma+\delta+1)(x-N)(x+\gamma+1)} R_n(\lambda(x)). \end{aligned}$$

Racah polynomials

$$\begin{aligned} A_n(x) &= -\frac{n(\beta+n)(\alpha+n-\delta)(\alpha+\beta+n-\gamma)}{(\alpha+\beta+2n)(x+\gamma+\delta+1)(x+\beta+\delta+1)(x+\gamma+1)(\alpha+x+1)}, \\ B_n(x) &= -\frac{n[(x+\gamma+\delta+1)(\alpha+\beta+n+x+1)(\alpha+\beta+2n)+(\beta+n)(\alpha+n-\delta)(\alpha+\beta+n-\gamma)]}{(\alpha+\beta+2n)(x+\gamma+\delta+1)(x+\beta+\delta+1)(x+\gamma+1)(\alpha+x+1)}. \end{aligned}$$

Similar expression can be obtained for the differences backwards using the same technique in the proof as the one in [4]. We define $\xi(x)$ by

$$w(x) - w(x+1) = \xi(x)w(x). \tag{2}$$

One can receive that

$$\nabla p_n(x) = C_n(x)p_{n-1}(x) - D_n(x)p_n(x) \tag{3}$$

where

$$\begin{aligned} C_n(x) = {} & \sum_{l=s}^{t} \frac{\gamma_{n-1}[\xi(x-1)-\xi(l)]}{\gamma_n \kappa_{n-1}(x-1-l)} p_n(l)p_n(l+1)w(l) \\ & + \frac{\gamma_{n-1}p_n(s-1)p_n(s)w(s)}{\gamma_n\kappa_{n-1}(x-s)}, \\ D_n(x) = {} & \sum_{l=s}^{t} \frac{\gamma_{n-1}[\xi(x-1)-\xi(l)]}{\gamma_n \kappa_{n-1}(x-l-1)} p_n(l)p_{n-1}(l+1)w(l) \\ & + \frac{\gamma_{n-1}p_n(s-1)p_{n-1}(s)w(s)}{\gamma_n\kappa_{n-1}(x-s)}, \end{aligned}$$

and $\xi(x)$ is defined by (2). The proof of this theorem is similar to the proof of the theorem in [2], so we omit it here. However, the direct application of this theorem leads to series manipulations that are not computable. The theorem does not apply for dual Hahn and Racah polynomials. Instead, similar expressions can be obtained directly.

Charlier polynomials

$$\nabla C_n(x) = -\frac{n}{x}\, C_{n-1}(x) + \frac{n}{x}\, C_n(x).$$

Kravchuck polynomials

$$\nabla K_n(x) = -\frac{n}{x}\, K_{n-1}(x) + \frac{n}{x}\, K_n(x).$$

Hahn polynomials

$$\begin{aligned}\nabla Q_n(x) = {} & \frac{n(\beta+n)(\alpha+\beta+n+N+1)}{x(x-\beta-N-1)(\alpha+\beta+2n)}\, Q_{n-1}(x) \\ & + \frac{n[x(\alpha+\beta+2n)-(\beta+n)(\alpha+\beta+N+n+1)]}{x(\alpha+\beta+2n)(x-\beta-N-1)}\, Q_n(x).\end{aligned}$$

Dual Hahn polynomials

$$\begin{aligned}\frac{\nabla R_n(\lambda(x))}{\nabla \lambda(x)} = {} & -\frac{n(N+1+\delta-n)}{x(x+\gamma+\delta+N+1)(x+\delta)}\, R_{n-1}(\lambda(x)) \\ & + \frac{n(N+1+\delta-n+x)}{x(x+\gamma+\delta+N+1)(x+\delta)}\, R_n(\lambda(x)).\end{aligned}$$

Racah polynomials

$$\begin{aligned} C_n(x) &= -\frac{n(\beta+n)(\alpha+n-\delta)(\alpha+\beta+n-\gamma)}{(\alpha+\beta+2n)x(x+\gamma-\beta)(x+\delta)(x+\gamma+\delta-\alpha)}, \\ D_n(x) &= -\frac{[nx(\alpha+\beta+2n)(x+\gamma+\delta-\alpha-\beta-n)+n(\beta+n)(\alpha+n-\delta)(\alpha+\beta+n-\gamma)}{(\alpha+\beta+2n)x(x+\gamma-\beta)(x+\delta)(x+\gamma+\delta-\alpha)}.\end{aligned}$$

Subtracting equation (3) from (1), after some elementary simplification, one gets the following expression for the second order difference equations:

$$\frac{\Delta p_n(x)}{A_n(x)} - \frac{\nabla p_n(x)}{C_n(x)} = \frac{B_n(x)p_n(x)}{A_n(x)} + \frac{D_n(x)p_n(x)}{C_n(x)}\,.$$

Taking differences from both sides and applying induction one can get an expression involving k-th differences. The computations for Charlier, Kravchuck, and Hahn polynomials lead to the following expressions:

Charlier polynomials

$$a\Delta^{p+1}C_n(x) - (x+p-a-n)\Delta^p C_n(x) + (n-p+1)\Delta^{p-1}C_n(x) = 0.$$

Kravchuck polynomials

$$\begin{aligned}(1-p)(x-s)\nabla^{s+1}K_n(x) + (1-p)\nabla^{s+1}K_n(x) \\ - [x-2s-p(N-s+1)+n]\nabla^s + (n-s+1)\Delta^s K_n(x) = 0.\end{aligned}$$

Hahn polynomials

$$(x-s)(x-\beta-N-s-1)\nabla^{s+1}Q_n(x)$$
$$\times\ [(s-1)(2x-\beta-N-2s)+(x-s-1)(\alpha+\beta+2)-N(\alpha+1)+n(\alpha+\beta+n+1)]$$
$$\times\ \nabla^s Q_n(x)-[n(\alpha+\beta+n+1)+(s-1)(\alpha+\beta+s)]\nabla^{s-1}Q_n(x)=0.$$

The expressions for dual Hahn and Racah polynomials are more complicated and involve more terms since the coefficients in front of $p_n(\lambda(x))$ in the difference equation of the second order is a polynomial of x (not a constant as the considered above cases). Using these expressions repeatedly and applying induction leads us to expressions of the form

$$\Delta^p p_n(x)=\frac{q_{n,k}(x)}{s_{n,k}(x)}\Delta p_n(x)+\frac{m_{n,k}(x)}{s_{n,k}(x)}\,p_n(x)$$

where the denominators are explicitly known and one can obtain a recurrence relation for the numerators $q_{n,p}(x)$ and $m_{n,p}(x)$. The recurrence for these polynomials has the form

$$q_{n,k+1}(x)=r_{n,k}(x)q_{n,k}(x)+l_{n,k}(x)q_{n,k-1}(x).$$

The degree of $r_{n,k}(x)$ and $l_{n,k}(x)$ depends on the particular example. Using induction one can prove that

$$p_n(x-p)=\sum_{k=0}^{p}\binom{p}{k}(-1)^k\nabla^k p_n(x)$$

and

$$p_n(x+p)=\sum_{k=0}^{p}\binom{p}{k}\Delta^k p_n(x).$$

Let $p_m(x)$ be a polynomial of degree m w.r.t. x. Let

$$n=\lfloor\frac{m+1}{2}\rfloor.$$

Using the Taylor expansion for $p_m(x)$ we obtain

$$p_m(x+k)-p_m(x-k)=\frac{2k}{1!}\,p_m'(x)+\frac{2k^3}{3!}\,p_m^{(3)}(x)+\cdots+\frac{2k^{(2n-1)}}{(2n-1)!}\,p_m^{(2n-1)}(x)$$

for $k=\pm1,\pm2,\ldots,\pm n$, and thus

$$\frac{\Delta+\nabla}{2}\,p_m(x)=\tfrac{1}{1!}\,p_m'(x)+\tfrac{1}{3!}\,p_m^{(3)}(x)+\cdots+\tfrac{1}{(2n-1)!}\,p_m^{(2n-1)}(x),$$

$$\tfrac{1}{2}\sum_{i=1}^{2}\tbinom{2}{i}[\Delta^i+(-1)^{i+1}\nabla^i]\,p_m(x)=\tfrac{2}{1!}\,p_m'(x)+\tfrac{2^3}{3!}\,p_m^{(3)}(x)+\cdots+\tfrac{2^{2n-1}}{(2n-1)!}\,p_m^{(2n-1)}(x),$$

$$\tfrac{1}{2}\sum_{i=1}^{k}\tbinom{k}{i}[\Delta^i+(-1)^{i+1}\nabla^i]\,p_m(x)=\tfrac{k}{1!}\,p_m'(x)+\tfrac{k^3}{3!}\,p_m^{(3)}(x)+\cdots+\tfrac{k^{(2n-1)}}{(2n-1)!}\,p_m^{(2n-1)}(x),$$

$$\tfrac{1}{2}\sum_{i=1}^{n}\tbinom{n}{i}[\Delta^i+(-1)^{i+1}\nabla^i]\,p_m(x)=\tfrac{n}{1!}\,p_m'(x)+\tfrac{n^3}{3!}\,p_m^{(3)}(x)+\cdots+\tfrac{n^{2n-1}}{(2n-1)!}\,p_m^{(2n-1)}(x).$$

Solving this system with respect to $p'_m(x), p_m^{(3)}, \dots, p_m^{(2n-1)}$ using Cramer's rule leads us to the expression

$$p'_m(x) = \frac{1}{2\Delta_n} \begin{vmatrix} (\Delta+\nabla)p_m(x) & \frac{1}{3!} & \frac{1}{5!} & \dots & \frac{1}{(2n-1)!} \\ \dots & \dots & \dots & \dots & \dots \\ \sum_{i=1}^{k} \binom{k}{i} [\Delta^i + (-1)^{i+1}\nabla^i]\, p_m(x) & \frac{k^3}{3!} & \frac{k^5}{5!} & \dots & \frac{k^{2n-1}}{(2n-1)!} \\ \dots & \dots & \dots & \dots & \dots \\ \sum_{i=1}^{n} \binom{n}{i} [\Delta^i + (-1)^{i+1}\nabla^i]\, p_m(x) & \frac{n^3}{3!} & \frac{n^5}{5!} & \dots & \frac{n^{2n-1}}{(2n-1)!} \end{vmatrix}.$$

where

$$\Delta_n := \begin{vmatrix} 1 & \frac{1}{3!} & \frac{1}{5!} & \dots & \frac{1}{(2n-1)!} \\ \dots & \dots & \dots & \dots & \dots \\ k & \frac{k^3}{3!} & \frac{k^5}{5!} & \dots & \frac{k^{2n-1}}{(2n-1)!} \\ \dots & \dots & \dots & \dots & \dots \\ n & \frac{n^3}{3!} & \frac{n^5}{5!} & \dots & \frac{n^{2n-1}}{(2n-1)!} \end{vmatrix} = \frac{n!}{3!\,5!\dots(2n-1)!}\, W_n(1, 2^2, \dots, n^2).$$

Set

$$\Delta_{n,j} = \begin{vmatrix} \frac{1}{3!} & \frac{1}{5!} & \dots & \frac{1}{(2n-1)!} \\ \dots & \dots & \dots & \dots \\ \frac{(j-1)^3}{3!} & \frac{(j-1)^5}{5!} & \dots & \frac{(j-1)^{2n-1}}{(2n-1)!} \\ \frac{(j+1)^3}{3!} & \frac{(j+1)^5}{5!} & \dots & \frac{(j+1)^{2n-1}}{(2n-1)!} \\ \dots & \dots & \dots & \dots \\ \frac{n^3}{3!} & \frac{n^5}{5!} & \dots & \frac{n^{2n-1}}{(2n-1)!} \end{vmatrix}$$
$$= \frac{(n!)^3}{j^3\, 3!\,5!\dots(2n-1)!}\, W_{n-1}(1, 2^2, \dots, (j-1)^2, (j+1)^2, \dots, n^2),$$

then expansion of the above determinant representation along the first column yields

$$\begin{aligned} p'_m(x) =& \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{k} \binom{k}{j} (-1)^{k+1} [\Delta^j + (-1)^{j+1}\nabla^j]\, p_m(x) \frac{\Delta_{k,n}}{\Delta_n} \\ =& \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{k} \binom{k}{j} (-1)^{k+1} [\Delta^j + (-1)^{j+1}\nabla^j]\, p_m(x) \\ & \times \frac{(n!)^3\, W_{n-1}(1, 2^2, \dots, (j-1)^2, (j+1)^2, \dots, n^2)}{k^3\, n!\, W_n(1, 2^2, \dots, n^2)}. \end{aligned}$$

But

$$\begin{aligned}
&\frac{W_n(1,2^2,\dots,n^2)}{W_{n-1}(1,2^2,\dots,(k-1)^2,(k+1)^2,\dots,n^2)}\\
&= \frac{(2^2-1)\dots(k^2-1^2)\dots(n^2-1)}{(2^2-1)\dots[(k-1)^2-1][(k+1)^2-1]\dots(n^2-1)}\\
&\quad\times \frac{W_{n-1}(2^2,\dots,k^2,\dots,n^2)}{W_{n-2}(2^2,\dots,(k-1)^2,(k+1)^2,\dots,n^2)}\\
&= (k^2-1)\frac{W_{n-1}(2^2,\dots,(k-1)^2,k^2,(k+1)^2,\dots,n^2)}{W_{n-2}(2^2,\dots,(k-1)^2,(k+1)^2,\dots,n^2)}\\
&= (k^2-1)(k^2-2^2)\frac{W_{n-2}(3^2,\dots,k^2,\dots,n^2)}{W_{n-3}(3^2,\dots,(k-1)^2,(k+1)^2,\dots,n^2)}\\
&= (k^2-1)(k^2-2^2)\dots[k^2-(k-1)^2]\frac{W_{n-k+1}(k^2,\dots,n^2)}{W_{n-k}((k+1)^2,\dots,n^2)}\\
&= (k^2-1)\dots[k^2-(k-1)^2][(k+1)^2-k^2]\dots(n^2-k^2)\frac{W_{n-k}((k+1)^2,\dots,n^2)}{W_{n-k}((k+1)^2,\dots,n^2)}\\
&= (k-1)(k+1)(k-2)(k+2)\dots(k-k+1)(k+k-1)\\
&\quad\times (k+1-k)(k+k+1)(k+2-k)(k+2+k)\dots(n-k)(n+k)\\
&= (k-1)!\,(k+1)_{k-1}(n-k)!\,(2k+1)_{n-k}\\
&= \frac{(1)_{2k-1}\,2k(2k+1)_{n-k}\,(n-k)!}{k2k}\\
&= \frac{(1)_{2k-1+1+n-k}\,(1)_{n-k}\,(1)_{n-k}}{2k^2}\\
&= \frac{(1)_{n+k}\,(1)_{n-k}}{2k^2}.
\end{aligned}$$

In the above equations we used several times the property of the Vandermonde determinant that

$$W_n(x_1,x_2\dots,x_n) = (x_2-x_1)(x_3-x_1)\dots(x_n-x_1)W_{n-1}(x_2,\dots,x_n).$$

Substituting the above expression into the equation for the first derivative we obtain

$$p'_m(x) = \frac{(n!)^2}{2}\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{(1)_k\,2k^2(-1)^{k+1}[\Delta^j+(-1)^{j+1}\nabla^j]\,p_m(x)}{(1)_j\,(1)_{k-j}\,k^3\,(1)_{n-k}\,(1)_{n+k}}\,.$$

But

$$(1)_{n-k} = \frac{(-1)^k(1)_n}{(-n)_k}\,,$$

So

$$\begin{aligned}
p'_m(x) &= -(n!)^2 \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{(-n)_k (1)_{k-1} k [\Delta^j + (-1)^{j+1} \nabla^j]\, p_m(x)}{(1)_j\, (1)_{k-j}\, k (1)_n\, (1)_{n+k}} \\
&= -n! \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{(-n)_k (1)_{k-1} [\Delta^j + (-1)^{j+1} \nabla^j]\, p_m(x)}{(1)_j\, (1)_{k-j}\, (1)_{n+k}} \\
&= -n! \sum_{j=1}^{n} \sum_{k=j}^{n} \frac{(-n)_k\, (1)_{k-1} [\Delta^j + (-1)^{j+1} \nabla^j]}{(1)_j\, (1)_{k-j}\, (1)_{n+k}}\, p_m(x) \\
&= -n! \sum_{j=1}^{n} \frac{[\Delta^j + (-1)^{j+1} \nabla^j] p_m(x)}{(1)_j} \sum_{k=0}^{n-j} \frac{(-n)_{k+j}\, (1)_{k+j-1}}{(1)_k\, (1)_{n+k+j}} \\
&= -n! \sum_{j=1}^{n} \frac{[\Delta^j + (-1)^{j+1} \nabla^j] p_m(x)}{(1)_j} \sum_{k=0}^{n-j} \frac{(-n)_j\, (-n+j)_k\, (1)_{j-1}\, (1+j-1)_k}{k!\, (1)_{n+j}\, (1+n+j)_k} \\
&= -n! \sum_{j=1}^{n} \frac{[\Delta^j + (-1)^{j+1} \nabla^j]\, p_m(x) (-n)_j\, (1)_{j-1}}{j!\, (1)_{n+j}} \sum_{k=0}^{n-j} \frac{(-n+j)_k\, (j)_k}{k! (1+n+j)_k} \\
&= -n! \sum_{j=1}^{n} \frac{[\Delta^j + (-1)^{j+1} \nabla^j]\, p_m(x) (-n)_j\, (1)_{j-1} (1+n+j-j)_{n-j}}{j! (1)_{n+j}\, (1+n+j)_{n-j}} \\
&= -n! \sum_{j=1}^{n} \frac{[\Delta^j + (-1)^{j+1} \nabla^j]\, p_m(x) (-n)_j\, (n+1)_{n-j}}{j (1)_{2n}} \\
&= -(1)_n \sum_{j=1}^{n} \frac{(-n)_j\, (n+1)_{n-j} [\Delta^j + (-1)^{j+1} \nabla^j]}{j (1)_n\, (n+1)_{n-j}\, (2n+1-j)_j} \\
&= -\sum_{j=1}^{n} \frac{(-n)_j\, (-1)^j [\Delta^j + (-1)^{j+1}]}{j (2n+1-j)_j} \\
&= -\sum_{j=1}^{n} \frac{(-n)_j\, (-1)^j [\Delta^j + (-1)^{j+1} \nabla^j] p_m(x)}{j (-2n)_j}.
\end{aligned}$$

In the above computations the Chu-Vandermonde theorem was used. Therefore

$$p'_m(x) = -\sum_{j=1}^{\lfloor (m+1)/2 \rfloor} \frac{\left(-\lfloor \frac{m+1}{2} \rfloor\right)_j (-1)^j}{j \left(-2\lfloor \frac{m+1}{2} \rfloor\right)_j} [\Delta^j + (-1)^{j+1} \nabla^j]\, p_m(x)\,.$$

Now substituting the expressions for the difference operators inside the formula for the first derivative one can obtain the functions $A_n(x)$ and $B_n(x)$ of the representation

$$p'_n(x) = A_n(x) p_{n-1}(x) - B_n(x) p_n(x).$$

We would like to point out that the described algorithm is not unique. For example, if one writes a different system of linear equations one will arrive at another expression for the functions $A_n(x)$ and $B_n(x)$. If one writes a system of linear equations obtained from the Taylor expansion of the polynomial with step one in forward direction one will obtain $A_n(x)$ and $B_n(x)$ as functions of $\Delta^k p_n(x)$, $k = 1, 2, \ldots, n$. Similarly, one can write a system of linear equations obtained from the Taylor expansions with step one backwards, then one can obtain the functions $A_n(x)$ and $B_n(x)$ as a functions of the backwards differences. So the functions $A_n(x)$ and $B_n(x)$ are not unique. Each system leads to numerators and denominators of degree proportional to the degree of $p_n(x)$. The discriminant of a sequence of orthogonal polynomials can be expressed in terms of the coefficients of the recurrence relation and the functions $A_n(x)$ of the considered representation of the first derivative [1]. Then

$$D(p_n(x)) = \prod_{j=1}^{n} A_n(x_{j,n}) \prod_{i=1}^{n} a_i^{2n-i-2} \prod_{i=2}^{n} \left(\frac{c_i}{a_i}\right)^{i-1}.$$

Here we denote by a_n, b_n, and c_n the coefficients from the recurrence relation:

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= a_1 x + b_1, \\ p_n(x) &= (a_n x + b_n) p_{n-1}(x) - c_n p_{n-2}(x). \end{aligned}$$

When the sequence of orthogonal polynomials is a polynomial of degree n with respect to the variable $\lambda(x) = x(x + \gamma + \delta + 1)$ and the coefficient functions in the representation [3] are functions of x as in the cases of dual Hahn and Racah polynomials, the square of the discriminant can be represented in the following way:

$$D^2(R_n) = \prod_{i=1}^{n} A_n(x_{n,i,1}) A_n(-x_{n,i,1} - \gamma - \delta - 1) a_i^{4n-2i-4} \prod_{i=2}^{n} \left(\frac{c_i}{a_i}\right)^{2i-2}.$$

This expression is obtained in [3]. In this case they satisfy a recurrence relation of the form

$$R_0(x) = 1, \qquad R_1(x) = a_1 \lambda(x) + b_1$$

and the three term recurrence relation

$$R_n(\lambda(x)) = (a_n \lambda(x) + b_n) R_{n-1}(\lambda(x)) - c_n R_{n-1}(\lambda(x)).$$

One can create an algorithm for computation of the functions $A_n(x)$ of the representation for all the cases, however the degree of the numerator of $A_n(x)$ for the cases of Hahn, dual Hahn, and Racah polynomials exceeds the degree of the first derivative. For these cases one can use division of polynomials to obtain a representation of the polynomials in the numerator of degree equal to the first derivative. Then, using again division of polynomials and partial fraction decomposition of the remainder leads to the following representation of $A_n(x)$, for $n = 3$,

$$A_3(x) = A + \sum_{k=0}^{1} \frac{B_k}{x-k}\,.$$

For $n \geq 3$ using division of polynomials and partial fraction decomposition one can obtain similar representations. One can approach the problem of computing the product

$$\prod_{k=0}^{n} A_n(x_{n,k})$$

in several different ways. One of them is using the standard Silvester formula. This method creates a problem with change of basis of the polynomial representation. The approach showed below provides us with a useful way of computing the mentioned product for small degree n. The not-standard part of it includes representation of the symmetric function of the zeroes of the polynomial as a function of the value of the polynomial $p_n(x)$ and its differences in different points. Using this approach one omits the problem of changing the basis. We have

$$\begin{aligned}
&\prod_{i=1}^{3} \Big(A + \frac{B_0}{x_{3,i}} + \frac{B_1}{x_{3,i}-1}\Big) \\
&= A^3 - \frac{B_0^3\gamma_3}{p_3(0)} - \frac{B_1^3\gamma_3}{p_3(1)} - \frac{A^2B_0\gamma_3}{p_3(0)}\Big(\frac{\Delta p_3(-1)}{\gamma_3} + \frac{\Delta^2 p_3(-1)}{\gamma_3} - 1\Big) \\
&\quad - \frac{A^2B_1\gamma_3}{p_3(1)}\Big(\frac{\Delta p_3(0)}{\gamma_3} + \frac{\Delta^2 p_3(0)}{2\gamma_3} - 1\Big) \\
&\quad + \frac{B_0^2A\gamma_3\Delta^2 p_3(-1)}{p_3(0)2\gamma_3} + \frac{B_0^2B_1\gamma_3^2}{p_3(0)p_3(1)}\Big(-\frac{3p_3(1)}{\gamma_3} + \frac{\Delta p_3(0)}{\gamma_3} + \frac{\Delta^2 p_3(0)}{\gamma_3} - 1\Big) \\
&\quad + \frac{B_1^2A\gamma_3}{p_3(1)}\frac{\Delta^2 p_3(0)}{2\gamma_3} + \frac{B_1^2B_0\gamma_3^2}{p_3(1)p_3(0)}\Big(-\frac{3p_3(0)}{\gamma_3} + \frac{\Delta p_3(-1)}{\gamma_3} + \frac{\Delta^2 p_3(-1)}{2\gamma_3} - 1\Big) \\
&\quad + \frac{AB_0B_1\gamma_3^2}{p_3(0)p_3(1)}\Big[\frac{p_3(0)}{\gamma_3}\Big(\frac{\Delta^2 p_3(0)}{\gamma_3} - 3\Big) + \frac{\Delta^2 p_3(0)}{2\gamma_3}\Big(\frac{\Delta p_3(-1)}{\gamma_3} + \frac{\Delta^2 p_3(-1)}{2\gamma_3} - 2\Big)\Big].
\end{aligned}$$

Similar expression exists for $n = 4$. In conclusion one can say that the problem of finding the discriminants for $n \geq 5$ for polynomial sequences from the Askey scheme is still open.

References

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