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Behaviour of Trigonometric Polynomials with Only Real Zeros Near a Critical Point

M. A. Qazi and Q. I. Rahman

Let $\xi_0 < \cdots < \xi_{2n-1} < \xi_{2n} = \xi_0 + 2\pi$ be any set of $2n+1$ consecutive critical points of a trigonometric polynomial t of degree n having only real zeros, all simple. Besides, let $m := \min_{0 \le \nu \le 2n-1} |t(\xi_\nu)|$ and $M := \max_{0 \le \nu \le 2n-1} |t(\xi_\nu)|$. Supposing that $m = |t(\xi_k)|$, we study the behaviour of t in the neighbourhood of ξ_k, and decide how far away the closest of its zeros can be.

1. Introduction and Statement of the Main Result

Let h be a polynomial of degree n, and let $y_1, \ldots, y_r$ be the roots of $h(y) = 0$ in $(-1, 1)$. Besides, let $\theta_i \in (0, \pi)$ be such that $\cos\theta_i = y_i$. In paragraph 1 of [2, p. 61], P. Erdős states, and we quote: "if $\max_{y_i \le x \le y_{i+1}} h(x)$ assumes its smallest value for $i = k$, then $\theta_{k+1} - \theta_k \le \pi/n$". This 'result', which he attributes to M. Riesz, plays an important role in his paper. There are some omissions in this statement, which are in some sense harmless. Still, we must mention that the 'result as stated' may not hold if the zeros of h do not all lie in $[-1, 1]$. To see this, let us first take $h(x) := (x^2 + \delta^2)\, T_{n-2}(x)$, where $\delta > 0$ and T_{n-2} is the Chebyshev polynomial of the first kind of degree $n-2$. The roots of $h(y) = 0$ in $(-1, 1)$ are

$$y_i := \cos\frac{(2i-1)\pi}{2r} \qquad (i = 1, \ldots, r := n-2)\,.$$

It is easily seen that the maximum of $|h(x)|$, over the interval having y_i and y_{i+1} as extremities, assumes its smallest value for $i = (n-2)/2$ if n is even, and for $i = (n-3)/2$ and $i = (n-1)/2$ in the case where n is odd. Now, let $\theta_i \in (0, \pi)$ be such that $\cos\theta_i = y_i$. Then, clearly

$$\theta_i = \frac{(2i-1)\pi}{2r} \qquad (i = 1, \ldots, r)\,.$$

Setting

$$\theta_0 := -\frac{\pi}{2r} \quad \text{and} \quad \theta_{r+1} := \frac{(2r+1)\pi}{2r},$$

as is understood from the context, we see that

$$\theta_{i+1} - \theta_i = \frac{\pi}{r} = \frac{\pi}{n-2} > \frac{\pi}{n} \qquad (i = 0, 1, \dots, r),$$

and that there is no i for which $\theta_{i+1} - \theta_i \le \pi/n$. This proves that the statement of the result of Riesz as given by Erdős does not hold true for $r = n-2$. As a matter of fact, it lacks accuracy for any $r \le n-1$ as the example

$$h(x) := (x - A)^{n-r}\, T_r(x) = (x - A)^{n-r} \cos(r \arccos x) \qquad (A > 1)$$

shows. So, we have to forget about r being less than n. This should place in perspective the generalization of the result of Riesz, presented here. Since Erdős did not give any reference whatsoever, and simply called it 'the lemma of Riesz', we can only speculate as to what Riesz had really proved. Notwithstanding the lack of clarity, here is what Erdős really needed; and this is *correct*, and so his own result remains unaffected.

Theorem A. *Let* $p(x) := c \prod_{\nu=1}^{n} (x - x_\nu)$, *where*

$$1 > x_1 > \cdots > x_n > -1.$$

Denote by $\theta_\nu\ (\nu = 1, \dots, n)$ *the unique root of the equation* $\cos\theta = x_\nu$ *in* $(0, \pi)$. *Furthermore, let* $\theta_0 := -\theta_1$ *and* $\theta_{n+1} := 2\pi - \theta_n$. *Set*

$$m_\nu := \max_{x_{\nu+1} \le x \le x_\nu} |p(x)| \qquad (\nu = 1, \dots, n-1).$$

In addition, let

$$m_0 := \max_{x_1 \le x \le 1} |p(x)| \quad \text{and} \quad m_n := \max_{-1 \le x \le x_n} |p(x)|.$$

Finally, suppose that $m_k \le m_\nu$ *for* $\nu = 0, \dots, n$. *Then* $\theta_{k+1} - \theta_k \le \pi/n$.

Remark 1. If p is as in Theorem A then $t(\theta) := p(\cos\theta)$ is a cosine polynomial of degree n, whose zeros are all real. More precisely, it has n zeros in $(0, \pi)$ and n zeros in $(-\pi, 0)$. Let $\theta_\nu, \nu = 1, \dots, n$, be the zeros of t in $(0, \pi)$ arranged in increasing order. In addition, let $\theta_0 := -\theta_1$ and $\theta_{n+1} := 2\pi - \theta_n$. Set $m_\nu := \max_{\theta_\nu \le \theta \le \theta_{\nu+1}} |t(\theta)|$ and suppose that $m_k \le m_\nu$ for $\nu = 0, 1, \dots, n$. Then Theorem A says that $\theta_{k+1} - \theta_k \le \pi/n$. The example $t(\theta) := \cos n\theta$ shows that this upper estimate for $\theta_{k+1} - \theta_k$ is sharp.

In this paper we prove an inequality for trigonometric polynomials, which extends Theorem A in several ways. First, we shall formulate the underlying question.

Let t be a trigonometric polynomial of degree $n\,(\geq 2)$ with only real zeros all simple, and let $\xi_0 < \xi_1 < \cdots < \xi_{2n-1}$ be any of its $2n$ consecutive critical points. Furthermore, let k be such that

$$|t(\xi_k)| = \min_{0\leq \nu \leq 2n-1} |t(\xi_\nu)| = M\,\cos\alpha\,, \quad \text{where } 0 \leq \alpha < \frac{\pi}{2}\,,$$

and suppose that $|t(\xi_\nu)| \geq M$ for any $\nu \in \{0,\ldots,2n-1\}$ other than k. Then the question is: *how far apart can the two (consecutive) zeros of t, which are separated by ξ_k, lie?*

As regards the question just asked we may, without loss of generality, assume that $\xi_k = \xi_0 = 0$. The answer to this question involves the function

$$\tau_n(\theta) = \tau_{n,\alpha}(\theta) := M\,T_{2n}\left(\left(\cos\frac{\alpha}{2n}\right)\cos\frac{\theta}{2}\right), \tag{1}$$

where T_{2n} is the Chebyshev polynomial of the first kind of degree $2n$. In order to be helpful to the reader who may have difficulty seeing what exactly τ_n represents we shall provide some details.

It may be noted that if $u = u(\theta) := \cos(\alpha/2n)\cos(\theta/2)$, then for any $q \in \{1,\ldots,n\}$, we have

$$u^{2q} = \left(\cos\frac{\alpha}{2n}\right)^{2q}\left(\frac{\mathrm{e}^{\mathrm{i}\theta}+2+\mathrm{e}^{-\mathrm{i}\theta}}{4}\right)^{q} = \left(\cos\frac{\alpha}{2n}\right)^{2q}\frac{1}{4^q}\sum_{\kappa=0}^{q} a_{q,\kappa}\cos(\kappa\theta)\,,$$

where the coefficients $a_{q,\kappa}$ are all integers. Thus, u^{2q} is a cosine polynomial of degree q. Since $T_{2n}(u)$ is a linear combination of $1, u^2, \ldots, u^{2n}$, we see that the function τ_n is a real cosine polynomial of degree n.

Clearly,

$$\tau_n(0) = M\,\cos\left(2n\arccos\left(\cos\frac{\alpha}{2n}\right)\right) = M\,\cos\alpha\,. \tag{2}$$

Now set

$$\eta_\nu^* := \frac{\cos(\nu\pi/2n)}{\cos(\alpha/2n)} \qquad (\nu = 1,\ldots,2n-1)\,, \tag{3}$$

and note that $1 > \eta_1^* > \eta_2^* > \cdots > \eta_{2n-1}^* > -1$. Hence, there exists one and only one number ξ_ν^* in $(0\,,\,2\pi)$ such that

$$\cos\frac{\xi_\nu^*}{2} = \eta_\nu^* \qquad (\nu = 1,\ldots,2n-1)\,.$$

In fact,

$$\xi_0^* := 0 < \xi_1^* < \xi_2^* < \cdots < \xi_{2n-1}^* < \xi_{2n}^* := 2\pi\,.$$

From (2) and (3) it follows that for $\nu = 1,\ldots,2n-1$, we have

$$\tau_n(\xi_\nu^*) = M\,T_{2n}\left(\left(\cos\frac{\alpha}{2n}\right)\eta_\nu^*\right) = M\,T_{2n}\left(\cos\frac{\nu\pi}{2n}\right) = (-1)^\nu M\,,$$

that is

$$(-1)^\nu \, \tau_\nu(\xi_\nu^*) = M = \max_{\theta\in\mathbb{R}} |\tau(\theta)| \qquad (\nu = 1, \dots , 2n-1)\,.$$

It is easily seen that the cosine polynomial τ_n has a local maximum at $\xi_0^* := 0$, and that $\tau_n(\theta)$ has the following property: it decreases monotonically from $M\cos\alpha$ to $-M$ as θ increases from ξ_0^* to ξ_1^*; it increases from $-M$ to M on each of the intervals $[\xi_{2\nu-1}^*\,,\, \xi_{2\nu}^*]$ for $\nu = 1, \dots , n-1$; it decreases from M to $-M$ on $[\xi_{2\nu}^*\,,\, \xi_{2\nu+1}^*]$ for $\nu = 1, \dots , n-1$; and increases from $-M$ to $M\cos\alpha$ on $[\xi_{2n-1}^*\,,\, \xi_{2n}^*]$. We conclude in particular that τ_n has exactly one (simple) zero in each of the intervals

$$(\xi_0^*\,,\, \xi_1^*)\,,\ (\xi_1^*\,,\, \xi_2^*)\,, \dots ,\ (\xi_{2n-1}^*\,,\, \xi_{2n}^*)\,.$$

Let us denote them by $\theta_1^*, \theta_2^*, \dots, \theta_{2n}^*$, respectively. It may be noted that θ_1^* *is the smallest positive root of the equation*

$$\left(\cos\frac{\alpha}{2n}\right)\cos\frac{\theta}{2} = \cos\frac{\pi}{4n}\,.$$

The two numbers θ_1^* and ξ_1^* are of special significance. Let us recall that ξ_1^* is the smallest positive critical point of τ_n. In other words, it is the smallest positive root of the equation

$$\left(\cos\frac{\alpha}{2n}\right)\cos\frac{\xi}{2} = \cos\frac{\pi}{2n}\,.$$

It is helpful to observe that

$$\tau_n(\theta) > 0 \qquad (-\theta_1^* < \theta < \theta_1^*)\,,$$

and that

$$\tau_n'(\theta) < 0 \qquad (0 < \theta < \xi_1^*)\,.$$

The following result provides an answer to the question asked above. In fact, it says considerably more.

Theorem 1. *Let t be a trigonometric polynomial of degree $n\,(\geq 2)$ with only real zeros all simple, and let $\xi_0 := 0 < \xi_1 < \cdots < \xi_{2n-1}$ be its critical points in $[0\,,\, 2\pi)$. Suppose that $|t(\xi_\nu)| \geq M$ for $\nu = 1, 2, \dots , 2n-1$, and that $t(0) = M\,\cos\alpha$ for some $\alpha \in [0\,,\, \pi/2)$. In addition, let $\tau_n(\theta) = \tau_{n,\alpha}(\theta)$ be as in* (1)*. Then*

$$t(\theta) \leq \tau_n(\theta) \qquad (0 \leq \theta \leq \xi_1)\,, \tag{4}$$

where $t(\theta) = \tau_n(\theta)$ for some $\theta \in (0\,,\, \xi_1)$ only if $t(\theta) \equiv \tau_n(\theta)$.

2. Proof of Theorem 1

By hypothesis

$$t(\xi_\nu) := (-1)^\nu M_\nu \qquad (\nu = 1, \ldots, 2n-1)\,,$$

where $M_\nu \geq M$ for $\nu = 1, \ldots, 2n-1$. We wish to show that $t(\theta) < \tau_n(\theta)$ for all $\theta \in (0\,,\,\xi_1)$ unless $t(\theta) \equiv \tau_n(\theta)$.

Let $t(\theta) \not\equiv \tau_n(\theta)$. For a proof by contradiction, let us first assume that $t(\theta') > \tau_n(\theta')$ for some $\theta' \in (0\,,\,\xi_1)$. Let $\theta_1, \theta_2, \cdots, \theta_{2n}$ be the zeros of t in $(0\,,\,2\pi)$ arranged in increasing order, and consider the trigonometric polynomial

$$S_1(\theta) := \frac{\sin^2\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta-\theta_1}{2}\right)\,\sin\left(\frac{\theta-(\theta_{2n}-2\pi)}{2}\right)}\,t(\theta)\,.$$

Note that $S_1(0) = S_1'(0) = 0$, $S_1(\theta) < 0$ for $0 < \theta < \theta_2$ and that $S_1(\theta)\,t(\theta) > 0$ in $(\theta_\nu\,,\,\theta_{\nu+1})$ for $\nu = 2, \ldots, 2n-2$. Thus, setting $t_{\varepsilon,1}(\theta) := t(\theta) + \varepsilon S_1(\theta)$, we see that for all sufficiently small $\varepsilon > 0$, we have

$$t_{\varepsilon,1}(0) = \tau_n(0) = M\cos\alpha\ \ ,\ \ t'_{\varepsilon,1}(0) = \tau'_n(0) = 0\ \ ,\ \ t_{\varepsilon,1}(\theta') > \tau_n(\theta')$$

and

$$(-1)^\nu t_{\varepsilon,1}(\xi_\nu) > M_\nu \geq M \qquad (\nu = 1, \ldots, 2n-1)\,.$$

Hence, for any such ε, the trigonometric polynomial $t_{\varepsilon,1} - \tau_n$ has a zero of multiplicity at least 2 at the origin, at least one zero in $(\theta'\,,\,\xi_1)$ and at least one zero in each of the intervals $(\xi_\nu\,,\,\xi_{\nu+1})$ for $\nu = 1, \ldots, 2n-2$. This adds up to at least $2n+1$ zeros in $[0\,,\,2\pi)$. Since $t_{\varepsilon,1} - \tau_n$ is a trigonometric polynomial of degree at most n, this is not possible unless $t_{\varepsilon,1}(\theta) \equiv \tau_n(\theta)$, and this for *all* small positive ε. This is a contradiction. Hence, $t(\theta) \leq \tau_n(\theta)$ for all $\theta \in (0\,,\,\xi_1)$.

Let $I_1 := \{\theta\,:\,0 < \theta < \xi_1\}$. Next we shall show that $t(\theta)$ is strictly less than $\tau_n(\theta)$ for all θ in I_1. Suppose not; then $t(\theta) \leq \tau_n(\theta)$ for all $\theta \in I_1$ with equality for at least one $\theta \in I_1$. Let θ'' be any $\theta \in I_1$ with this property. Since $t(\theta) \not\equiv \tau_n(\theta)$, the point θ'' must be isolated in the sense that $t(\theta) < \tau_n(\theta)$ in $\mathcal{N}_\delta := (\theta'' - \delta\,,\,\theta'') \cup (\theta''\,,\,\theta'' + \delta)$ for some $\delta > 0$. Thus $t(\theta) - \tau_n(\theta) \leq 0$ for all $\theta \in I_1$ and $t - \tau_n$ has a zero of multiplicity at least 2 at the point $\theta'' \in I_1$. Now consider the trigonometric polynomial

$$S_2(\theta) := \frac{\sin^2\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta-\theta_1}{2}\right)\,\sin\left(\frac{\theta-\theta_2}{2}\right)}\,t(\theta)\,.$$

Note that $S_2(0) = S_2'(0) = 0$, $S_2(\theta) > 0$ for $0 < \theta < \theta_3$ and that $S_2(\theta)\,t(\theta) > 0$ in $(\theta_\nu\,,\,\theta_{\nu+1})$ for $\nu = 3, \ldots, 2n-2$. Thus, setting $t_{\varepsilon,2}(\theta) := t(\theta) + \varepsilon S_2(\theta)$, we see that for all sufficiently small $\varepsilon > 0$, we have

$$t_{\varepsilon,2}(0) = \tau_n(0) = M\cos\alpha\ \ ,\ \ t'_{\varepsilon,2}(0) = \tau'_n(0) = 0\,,$$

$$t_{\varepsilon,2}\Big(\theta'' - \frac{\delta}{2}\Big) < \tau_n\Big(\theta'' - \frac{\delta}{2}\Big),\ t_{\varepsilon,2}(\theta'') > \tau_n(\theta''),\ t_{\varepsilon,2}\Big(\theta'' + \frac{\delta}{2}\Big) < \tau_n\Big(\theta'' + \frac{\delta}{2}\Big)$$

and

$$(-1)^\nu t_{\varepsilon,2}(\xi_\nu) > M_\nu \geq M \qquad (\nu = 2, \ldots, 2n-1)\,.$$

Hence, for any such ε, the trigonometric polynomial $t_{\varepsilon,2} - \tau_n$ has a zero of multiplicity at least 2 at the origin, at least one zero in $(\theta'' - \frac{\delta}{2}\,,\,\theta'')$, at least one zero in $(\theta''\,,\,\theta'' + \frac{\delta}{2})$ and at least one zero in each of the intervals $(\xi_\nu\,,\,\xi_{\nu+1})$ for $\nu = 2, \ldots, 2n-2$. This adds up to at least $2n+1$ zeros in $[0\,,\,2\pi)$. Since $t_{\varepsilon,2} - \tau_n$ is a trigonometric polynomial of degree at most n, this is not possible unless $t_{\varepsilon,2}(\theta) \equiv \tau_n(\theta)$, and this for all small positive ε. This contradiction proves that $t(\theta) < \tau_n(\theta)$ for $\theta \in (0\,,\,\xi_1)$ unless $t(\theta) \equiv \tau_n(\theta)$. □

Note. Since $t'(\xi_1) = 0$ and $t(\xi_1) \leq -M$, the above proof also shows that if $t(\theta)$ is not identically equal to $\tau_n(\theta)$ then $t(\xi_1)$ cannot be equal to $\tau_n(\xi_1)$ *unless* $\xi_1 = \xi_1^*$, $t(\theta) - \tau_n(\theta)$ *is negative on* $(0\,,\,\xi_1)$ *and changes sign at* $\theta = \xi_1$.

3. Some Consequences of Theorem 1

Since $t(0) = \tau_n(0) = M\,\cos\alpha$ and $t'(0) = \tau_n'(0) = 0$, it follows from (4) that for small positive values of θ,

$$\frac{t(\theta) - t(0) - \frac{1}{1!}\,t'(0)\,\theta}{\theta^2} \leq \frac{\tau_n(\theta) - \tau_n(0) - \frac{1}{1!}\,\tau_n'(0)\,\theta}{\theta^2}\,,$$

and so the following result holds.

Corollary 1. *Let t and τ_n be as in Theorem 1. Then*

$$t''(0) \leq \tau_n''(0)\,, \quad \text{i.e.,} \quad |t''(0)| \geq |\tau_n''(0)|\,.$$

Remark 2. Inequality (4) implicitly says that t takes the value $-M$ at least once in $(0\,,\,\xi_1^*]$, where ξ_1^* is the smallest positive critical point of τ_n. Let ϑ_1 be the smallest positive number such that $t(\vartheta_1) = -M$. Then, for any $\theta \in (0\,,\,\vartheta_1]$, we have

$$\int_0^\theta |t'(\varphi)|\,\mathrm{d}\varphi = -\int_0^\theta t'(\varphi)\,\mathrm{d}\varphi = M\,\cos\alpha - t(\theta)$$

and

$$\int_0^\theta |\tau_n'(\varphi)|\,\mathrm{d}\varphi = -\int_0^\theta \tau_n'(\varphi)\,\mathrm{d}\varphi = M\,\cos\alpha - \tau_n(\theta)\,.$$

Hence, the following result is contained in Theorem 1.

Corollary 2. *Let t and τ_n be as in Theorem 1, and let ϑ_1 denote the smallest positive number such that $t(\vartheta_1) = -M$. Then*

$$\int_0^\theta |t'(\varphi)|\,\mathrm{d}\varphi \;\geq\; \int_0^\theta |\tau_n'(\varphi)|\,\mathrm{d}\varphi \qquad (0 < \theta \leq \vartheta_1)\,.$$

Remark 3. Let $t(\theta) \not\equiv \tau_n(\theta)$, and let θ_1^* be the smallest positive zero of τ_n. Theorem 1 implies that as θ increases from 0 to ξ_1 the trigonometric polynomial $\eta = t(\theta)$ takes any value $\eta_0 \in (-M\,,\, M\cos\alpha)$ before $\tau_n(\theta)$ does. Since $\tau_n(\theta)$ takes the value $\eta_0 := 0$ at $\theta = \theta_1^*$, the trigonometric polynomial $t(\theta)$ must do so at a point $\theta_1 \in (0\,,\, \theta_1^*)$. In fact, as we shall explain, the following result holds.

Corollary 3. *Let t be a trigonometric polynomial of degree $n\,(\geq 2)$ with only real zeros all simple, and $t(0) \neq 0$. Let $\theta_1 < \ldots < \theta_{2n}$ be the zeros of t in $(0\,,\, 2\pi)$, arranged in increasing order, and let $\xi_0 := 0 < \xi_1 < \cdots < \xi_{2n-1}$ be its critical points in $[0\,,\, 2\pi)$. Suppose that $|t(\xi_\nu)| \geq M$ for $\nu = 1, 2, \ldots, 2n-1$, and that $t(0) = M\,\cos\alpha$ for some $\alpha \in [0\,,\, \pi/2)$. In addition, let $\tau_n(\theta) = \tau_{n,\alpha}(\theta)$ be as in (1), and θ_1^* its smallest positive zero. Finally, let $t(\theta) \not\equiv \tau_n(\theta)$. Then $\theta_1 < \theta_1^*$; more precisely, we either have*

$$t(\theta) < \tau_n(\theta) \qquad (0 < \theta < \theta_1^*)\,,$$

or else $\theta_2 < \theta_1^*$. In particular,

$$\int_0^{\theta_1} \{t(\theta)\}^p \,\mathrm{d}\theta < \int_0^{\theta_1} \{\tau_n(\theta)\}^p \,\mathrm{d}\theta \qquad (0 < p < \infty)\,.$$

Proof. As before let ξ_1^* denote the smallest positive number such that $\tau_n'(\xi_1^*) = 0$. We suppose that $t(\theta)$ is not identically equal to $\tau_n(\theta)$ and then distinguish three different cases.

Case I. $\xi_1 \geq \xi_1^*$

By Theorem 1, $t(\theta) < \tau_n(\theta)$ for all $\theta \in (0\,,\, \xi_1]$ if $\xi_1 > \xi_1^*$ and for all $\theta \in (0\,,\, \xi_1^*)$ if $\xi_1 = \xi_1^*$. Thus, in this case, we certainly have

$$t(\theta) < \tau_n(\theta) \qquad (0 < \theta < \xi_1^*)\,;$$

in particular, $t(\theta_1^*) < \tau_n(\theta_1^*) = 0$. Since $t(0) = M\cos\alpha > 0$, it follows from the intermediate value property that $t(\theta)$ must vanish at least once in $(0\,,\, \theta_1^*)$.

Case II. $\theta_1^* \leq \xi_1 < \xi_1^*$

We shall show that in this case,

$$t(\theta) < \tau_n(\theta) \qquad (0 < \theta \leq \theta_1^*)\,. \tag{5}$$

By (4), the strict inequality $t(\theta) < \tau_n(\theta)$ holds for all $\theta \in (0\,,\, \theta_1^*]$ if $\theta_1^* < \xi_1$ and for all $\theta \in (0\,,\, \theta_1^*)$ if $\theta_1^* = \xi_1$. However, if $\theta_1^* = \xi_1$ then $t(\theta_1^*) = t(\xi_1) \leq -M$ and $\tau_n(\theta_1^*) = 0$, i.e., $t(\theta_1^*) < \tau_n(\theta_1^*)$, which proves (5). In particular, $t(\theta_1^*)$ is negative; and since $t(0)$ is positive, $t(\theta)$ must vanish at least once in $(0\,,\, \theta_1^*)$.

Case III. $0 < \xi_1 < \theta_1^*$

This time $t(\theta)$ may not be strictly less than $\tau_n(\theta)$ throughout the interval $(0\,,\, \theta_1^*)$. Since $t(\xi_1) \leq -M < 0$ and $t(0) = M\,\cos\alpha > 0$, the intermediate value property implies that $t(\theta)$ must vanish at least once in $(0\,,\,\xi_1)$ and so at least once in $(0\,,\,\theta_1^*)$. Now, two things can happen: either (i) $t(\theta) < \tau_n(\theta)$ for all $\theta \in (0\,,\,\theta_1^*)$ or else (ii) t has *at least two* zeros in $(0\,,\,\theta_1^*)$, and in this latter situation not only $\theta_1 < \theta_1^*$ but also $\theta_2 < \theta_1^*$. □

In the following remark, which should be read in conjunction with Remark 1, we focus on the case $\alpha = 0$ of Corollary 3. It is meant to explain how Corollary 3 covers Theorem A, and in fact says quite a bit more.

Remark 4. Let t be a trigonometric polynomial of degree $n\,(\geq 2)$ with only real zeros all simple. Let $\theta_1 < \cdots < \theta_{2n}$ be the zeros of t in $(0\,,\,2\pi)$, arranged in increasing order, and let $\xi_0 := 0 < \xi_1 < \cdots < \xi_{2n-1}$ be its critical points in $[0\,,\,2\pi)$. Suppose that $t(0) = M$ and that $|t(\xi_\nu)| \geq M$ for $\nu = 1, 2, \ldots, 2n-1$. Now, note that $\tau_{n,0}(\theta)$ is simply $M\cos n\theta$. Then, by Corollary 3, either $\theta_2 < \pi/2n$ or else

$$t(\theta) < M\,\cos n\theta \quad \text{for} \quad 0 < \theta < \frac{\pi}{2n}, \quad \text{and} \quad \theta_1 < \frac{\pi}{2n}\,.$$

Applying these considerations to the trigonometric polynomial $t(-\theta)$, we readily conclude that either t has at least two zeros in $(-\pi/2n\,,\,0)$ or else

$$t(\theta) < M\,\cos n\theta \quad \text{for} \quad -\frac{\pi}{2n} < \theta < 0, \quad \text{and} \quad -\frac{\pi}{2n} < \theta_{2n} - 2\pi < 0\,.$$

There is one additional observation to be made. Theorem A says that

$$\theta_1 - (\theta_{2n} - 2\pi) \leq \frac{\pi}{n}\,,$$

which allows the possibility that θ_1 may be larger than $\pi/2n$ and that $\theta_{2n} - 2\pi$ may be smaller than $-\pi/2n$. Our result is more precise in this respect. It says that $t(\theta)$ must vanish at least once in each of the two intervals $(0\,,\,\pi/2n]$ and $[-\pi/2n\,,\,0)$.

We mention the following result as an addendum to Theorem A.

Corollary 4. *As in Theorem A, let* $p(x) := c\prod_{\nu=1}^{n}(x - x_\nu)$, *where*

$$x_0 := 1 > x_1 > \cdots > x_n > x_{n+1} := -1\,.$$

In addition, let

$$m_\nu := \max_{x_{\nu+1} \leq x \leq x_\nu} |p(x)| \qquad (\nu = 0, \ldots, n)\,.$$

Finally, suppose that $m_0 = 1$ *and that* $m_\nu \geq 1$ *for* $\nu = 1, \ldots, n$. *Then*

$$|p(x)| < T_n(x) \qquad \left(\cos\frac{\pi}{2n} < x < 1\right)$$

unless $p(x) \equiv T_n(x)$, *where* T_n *is the Chebyshev polynomial of the first kind of degree* n. *In particular,* $x_1 > \cos(\pi/2n)$ *unless* $p(x) \equiv T_n(x)$.

For the proof one may assume without loss of generality that c is positive. By Rolle's theorem, the critical points of p must lie in $(x_n\,,\,x_1)$. Hence, $p(x)$ must decrease from 1 to 0 as x decreases from 1 to x_1. Nothing more needs to be added towards the proof of Corollary 4.

4. A Counter-Part of Theorem 1

The result that follows can be seen as a counter-part of Theorem 1. It may be noted that here t may be of degree less than n. Also, we only need the inequality $|t(\theta)| \leq M$ to be true at certain $2n-1$ special points in $(0\,,\,2\pi)$, namely the critical points $\xi_1^*, \ldots, \xi_{2n-1}^*$ of τ_n in $(0\,,\,2\pi)$, arranged in increasing order.

Theorem 2. *Let $\tau_n(\theta) = \tau_{n,\alpha}(\theta)$, $\alpha \in [0\,,\,\pi/2)$ be as in (1) and, as before, let $\xi_1^* < \xi_2^* < \cdots < \xi_{2n-1}^*$ be its critical points in $(0\,,\,2\pi)$. In addition, let t be a real trigonometric polynomial of degree at most n such that $t(0) = M\,\cos\alpha$, $t'(0) = 0$ and $|t(\xi_\nu^*)| \leq |\tau_n(\xi_\nu^*)|$ for $\nu = 1, \ldots, 2n-1$. Then $t(\theta) > \tau_n(\theta)$ for $0 < \theta < \xi_1^*$ except in the case where $t(\theta) \equiv \tau_n(\theta)$.*

Proof. For a proof by contradiction let $t(\theta) \not\equiv \tau_n(\theta)$ and suppose that $t(\theta) \leq \tau_n(\theta)$ for some $\theta \in (0\,,\,\xi_1^*)$, say for $\theta = \theta^\dagger$. First let $t(\theta^\dagger)$ be strictly less than $\tau_n(\theta^\dagger)$. As before, let $\theta_1^* < \theta_2^* < \cdots < \theta_{2n}^*$ be the zeros of τ_n in $(0\,,\,2\pi)$. Then, for all sufficiently small positive ε, the trigonometric polynomial

$$t(\theta) - \left\{1 + \varepsilon\,\frac{\sin^2\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta-\theta_1^*}{2}\right)\,\sin\left(\frac{\theta+\theta_1^*}{2}\right)}\right\}\tau_n(\theta)$$

must have at least $2n-1$ zeros in $(\theta^\dagger\,,\,2\pi)$, and so, together with the multiple zero at 0, it has at least $2n+1$ zeros in $[0\,,\,2\pi)$. This contradiction excludes the possibility that $t(\theta)$ be less that $\tau_n(\theta)$ for any θ in $(0\,,\,\xi_1^*)$.

Now let us suppose that $t(\theta^\dagger)$ is equal to $\tau_n(\theta^\dagger)$. Arguing as in the proof of Theorem 1, we conclude that for all sufficiently small positive ε, the trigonometric polynomial

$$t(\theta) - \left\{1 + \varepsilon\,\frac{\sin^2\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta-\theta_1^*}{2}\right)\,\sin\left(\frac{\theta-\theta_2^*}{2}\right)}\right\}\tau_n(\theta)$$

must then have at least $2n-1$ zeros in $(0\,,\,2\pi)$, and so, at least $2n+1$ zeros in $[0\,,\,2\pi)$. Hence, $t(\theta^\dagger)$ cannot be equal to $\tau_n(\theta^\dagger)$. □

The following result can be seen as a corollary and also as a generalization of Theorem 2.

Corollary 5. *Let $\tau_n(\theta) = \tau_{n,\alpha}(\theta)$, $\alpha \in [0\,,\,\pi/2)$ be as in (1). As before, let $\xi_1^* < \xi_2^* < \cdots < \xi_{2n-1}^*$ be its critical points in $(0\,,\,2\pi)$ and θ_1^* its smallest positive zero. In addition, let t be a trigonometric polynomial of degree at most n, real or not, such that $|t(0)| = M\,\cos\alpha$, $t'(0) = 0$ and $|t(\xi_\nu^*)| \leq |\tau_n(\xi_\nu^*)|$ for $\nu = 1, \ldots, 2n-1$. Then $|t(\theta)| > \tau_n(\theta)$ for $0 < \theta < \theta_1^*$ except in the case where $t(\theta) \equiv \mathrm{e}^{\mathrm{i}\gamma}\,\tau_n(\theta)$, $\gamma \in \mathbb{R}$.*

Proof. Let $t(0) = \mathrm{e}^{\mathrm{i}\gamma}\,|t(0)|$, $\gamma \in \mathbb{R}$. Then $s(\theta) := \Re\{\mathrm{e}^{-\mathrm{i}\gamma}\,t(\theta)\}$ is a real trigonometric polynomial satisfying all the conditions of Theorem 2, and so

$s(\theta) > \tau_n(\theta)$ for all $\theta \in (0\,,\,\xi_1^*)$ unless $\Re\{e^{-i\gamma}\,t(\theta)\} \equiv \tau_n(\theta)$. But in the latter situation, we would have

$$|e^{-i\gamma}\,t(\xi_\nu^*)| = |t(\xi_\nu^*)| \le |\tau_n(\xi_\nu^*)| = |\Re\{e^{-i\gamma}\,t(\xi_\nu^*)\}| \qquad (\nu = 1, \dots, 2n-1)\,,$$

which is possible only if $\Im\{e^{-i\gamma}\,t(\xi_\nu^*)\} = 0$ for $\nu = 1, \dots, 2n-1$, i.e. only if the trigonometric polynomial $v(\theta) := \Im\{e^{-i\gamma}\,t(\theta)\}$ vanishes at $\xi_1^*, \dots, \xi_{2n-1}^*$. Looking at the Maclaurin development of t we see that v has a multiple zero at the origin, and so has at least $2n+1$ zeros in $[0\,,\,2\pi)$. Since, its degree cannot be larger than n, it must be identically zero, and so $t(\theta)$ must be of the form $e^{i\gamma}\tau_n(\theta)$. We conclude that

$$|t(\theta)| \ge s(\theta) > \tau_n(\theta) \qquad (0 < \theta < \theta_1^*)$$

unless $t(\theta) \equiv e^{i\gamma}\,\tau_n(\theta)$, $\gamma \in \mathbb{R}$. □

The following list of references contains not only the paper of P. Erdős that is cited in the text but also some others which deal with problems of a similar nature.

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M. A. Qazi
Department of Mathematics
Tuskegee University
Tuskegee, Alabama 36088
U.S.A.
E-mail: qazima@aol.com

Q. I. Rahman
Département de Mathématiques et de Statistique
Université de Montréal
Montréal H3C 3J7
CANADA
E-mail: rahmanqi@dms.umontreal.ca