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Geometric Aspects of Minimal Projections onto Planes

Boris Shekhtman and Lesław Skrzypek

We give the explicit formula for minimal projection from the set of continuous functions on quadrilateral onto affine functions. The most interesting results show how this minimal projection depends on geometry of a given quadrilateral (Theorem 3 relates it to areas, while Theorem 4 and Remark 2 relates it to diagonals). To prove that this projection is minimal we employ Chalmers-Metcalf Theorem. We also give the explicit formula for the Chalmers-Metcalf operator related to the considered minimal projections (equations (3)), the formula itself also involves geometry of a given quadrilateral.

1. Introduction

Let S be a compact subset of the real line $\mathbb{R}$, and let $C(S)$ be the space of continuous function on S, equipped with the uniform norm. Let $E := \operatorname{span}\{1, x\} \subset C(S)$. It is easy to see that there exists a linear projection P from $C(S)$ onto E with $\|P\| = 1$. It suffices to take the Lagrange interpolating projection P that interpolates at two points $s_0 = \inf S$ and $s_1 = \sup S$. Hence the norm of a minimal projection onto E is 1, regardless of the domain S. The situation changes drastically when we consider the bivariate version of this result. That is, let S be a compact subset of $\mathbb{R}^2$ and let E be a three-dimensional subspace of $C(S)$ consisting of affine functions, i.e.,

$$E := \operatorname{span}\{1, x, y\}$$

In this case the norm of a minimal projection onto E depends on the geometry of S. For instance, if S is a non-degenerate triangle, then the Lagrange interpolation at the vertices of S provides an example of a projection onto E with norm 1, since every affine function $f = a + bx + cy$ attains its norm at the extreme points of S. On the other hand, if S is the unit disk, then the Fourier projection on the unite circle is the minimal projection onto E and its norm is greater than 1 (see Remark 1).

In this note we investigate minimal projections from $C(S)$ onto E, where S is a quadrilateral. We give the explicit form of such projections (depending on S) and evaluate their norms (Theorem 3, Theorem 4 and Remark 2). We show that the norm is maximal ($=\frac{3}{2}$) if and only if S is a parallelogram. As a by-product, we obtain an alternative (geometric) proof of the results in [10], regarding the projection constants of three-dimensional subspaces of l^4_∞.

Here are some notations:

A projection $P_0 \in \mathcal{P}(X, V)$ is called *minimal* if

$$\|P_0\| = \lambda(V, X) = \inf\{\|P\| \,:\, P \in \mathcal{P}(X, V)\}.$$

The constant $\lambda(V, X)$ is called the *relative projection constant.*

Absolute projection constant (or simply projection constant) is defined as

$$\lambda(V) := \sup\{\lambda(V', X) \,:\, V' \subset X\},$$

where V' denotes isometric copy of V in X.

For more information about the above notions see the papers [1], [5], [6], [7], [10], [12]. One of the main tools to study minimal projections is the so-called Chalmers-Metcalf operator. We can define it as follows.

Below we assume that X is a normed space and V is its finite-dimensional subspace.

Definition 1. A pair $(x, y) \in S(X^{**}) \times S(X^*)$ will be called an **extremal pair** for $P \in \mathcal{P}(X, V)$ iff $y(P^{**}x) = \|P\|$, where $P^{**} : X^{**} \to V$ is the second adjoint extension of P to X^{**} ($S(X)$ denotes here a unit sphere of X). Let $\mathcal{E}(P)$ be the set of all extremal pairs for P.

To each $(x, y) \in \mathcal{E}(P)$ we associate the rank-one operator $y \otimes x$ from X to X^{**} given by $(y \otimes x)(z) = y(z) \cdot x$ for $z \in X$.

Theorem 1 (Chalmers-Metcalf [3]). *A projection $P \in \mathcal{P}(X, V)$ has a minimal norm if and only if the closed convex hull of $\{y \otimes x\}_{(x,y)\in\mathcal{E}(P)}$ contains an operator E_P for which V is an invariant subspace.*

The operator E_P is called Chalmers-Metcalf operator and is given by the formula

$$E_P = \int_{\mathcal{E}(P)} y \otimes x \, d\mu(x, y) : X \to X^{**},$$

where μ is a probabilistic Borel measure on $\mathcal{E}(P)$.

If X is finite dimensional then $X^{**} = X$ and $P^{**} = P$.

Even though this theorem seems very theoretical it has been successfully applied to finding minimal projections in many cases, see [2], [3], [8], [9], [13], [14].

2. Results

Observe that if X and Y are isometric under the isometry I, then for any subspace V of X we have $\lambda(V,X) = \lambda(I(V),Y)$. Additionally $P : X \to V$ is a minimal projection if and only if $Q = I \circ P \circ I^{-1} : Y \to I(V)$ is a minimal projection. This simple observation will help us to solve the posted problem if S is a unit circle, the corresponding minimal projection is Fourier projection.

Remark 1. Let $S = T$ be a unit disk. Then the following projection $P : S \to E$ is minimal

$$P(f)(x,y) = \int_T f(u,v)\,dudv + \Big(\int_T u \cdot f(u,v)\,dudv\Big)x + \Big(\int_T v \cdot f(u,v)\,dudv\Big)y.$$

In other words, P is the orthogonal projection onto E, i.e., $P = 1 \otimes 1 + x \otimes x + y \otimes y$.

Proof. Consider the following isometry

$$I : C(T) \ni f(x,y) \mapsto f(\cos t, \sin t) \in C[0, 2\pi].$$

Observe that $I(E) = \operatorname{span}\{1, \cos t, \sin t\}$. Now since Fourier projection $F = 1 \otimes 1 + \sin t \otimes \sin t + \cos t \otimes \cos t : C[0,2\pi] \to I(E)$ is minimal, then following the reasoning made before this remark, we obtain that $P = 1 \otimes 1 + x \otimes x + y \otimes y : C(T) \to E$ is also minimal. □

Now we will move to the case when S is quadrilateral. By δ_A we denote the point evaluation functional at A, i.e., $\delta_A(f) = f(A)$. Now consider the following operator

$$L : C(S) \ni f \mapsto (\delta_{A_1}(f), \delta_{A_2}(f), \delta_{A_3}(f), \delta_{A_4}(f)) \in \ell^4_\infty.$$

Observe that

$$\|L\|_{C(S) \to \ell^4_\infty} \le 1,$$

Additionally, if $S = A_1A_2A_3A_4$ is a convex quadrilateral, then since affine functions attain maximum at extreme points of S (points A_1, A_2, A_3, A_4) we see that L is also an isometry on E, i.e.,

$$\|f\|_{C(S)} = \|L(f)\|_{\ell^4_\infty}, \quad \text{for any } f \in E.$$

Theorem 2. *Assume that* $S = A_1A_2A_3A_4$ *is a convex quadrilateral. Then* $P : \ell^4_\infty \to L(E)$ *is a minimal projection if and only if*

$$Q = L^{-1} \circ P \circ L : C(S) \to E$$

is a minimal projection.

Proof. L in general is not invertible, although L is isometry between E and $L(E)$ therefore is invertible on E. As a result, the formula $Q = L^{-1} \circ P \circ L$ makes sense and represents a projection.

We know that for any subspace V the space L_∞ is a maximal overspace. Also, the norm of minimal projection onto V does not depend on the actual position of V in L_∞, i.e., if V and W are isometric, then $\lambda(V, L_\infty) = \lambda(W, L_\infty)$ (see [12]).

As a result,

$$\lambda(E, C(S)) = \lambda(L(E), \ell_\infty^4).$$

Therefore, it is enough to prove that $\|Q\| = \|P\|$. Obviously, by the definition of Q we have

$$\|Q\| \leq \|L^{-1}\| \cdot \|P\| \cdot \|L\| \leq \|P\|.$$

On the other hand, let $x = (x_1, x_2, x_3, x_4)$ be a norming point for P, take $f = x_1\delta_{A_1} + x_2\delta_{A_2} + x_3\delta_{A_3} + x_4\delta_{A_4}$. Observe that $\|x\|_{\ell_\infty^4} = \|f\|_{C(S)}$, and

$$\|Q(f)\| = \|L^{-1}(P(L(f)))\| = \|L^{-1}(P(x))\| = \|P(x)\| = \|P\|,$$

hence,

$$\|Q\| = \|P\| = \lambda(L(E), \ell_\infty^4) = \lambda(E, C(S)) \qquad \square$$

Now we will compute a minimal projection from ℓ_∞^4 onto $L(E)$.

Theorem 3. *Assume that $S = A_1A_2A_3A_4$ is a convex quadrilateral. Put*

$$S_1 = \text{Area}(\triangle A_2A_3A_4), \qquad S_2 = \text{Area}(\triangle A_1A_3A_4),$$
$$S_3 = \text{Area}(\triangle A_1A_2A_4), \qquad S_4 = \text{Area}(\triangle A_2A_3A_4).$$

Then

$$L(E) = \ker(S_1, -S_2, S_3, -S_4)$$

and the following projection is a minimal projection from ℓ_∞^4 onto $L(E)$:

$$P = Id - \frac{1}{M}(S_1, -S_2, S_3, -S_4) \otimes \left(\frac{1}{S_3}, -\frac{1}{S_4}, \frac{1}{S_1}, -\frac{1}{S_2}\right), \tag{1}$$

here $M = \frac{S_1}{S_3} + \frac{S_3}{S_1} + \frac{S_2}{S_4} + \frac{S_4}{S_2}$. Additionally,

$$\|P\| = 1 + \frac{2}{M} \leq \frac{3}{2},$$

with "=" if and only if $A_1A_2A_3A_4$ is a parallelogram.

Proof. Let $A_i = (a_i, b_i)$, we have then

$$L(1) = (1,1,1,1), \quad L(x) = (a_1, a_2, a_3, a_4) \quad \text{and} \quad L(y) = (b_1, b_2, b_3, b_4).$$

As a result $L(E)$ is a kernel of some functional (z_1, z_2, z_3, z_4) which is perpendicular to $L(1), L(x), L(y)$. Consider the determinants

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 & b_4 \\ 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix} = 0.$$

If we expand each determinant with respect to first row, knowing that

$$S_1 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \end{vmatrix} \qquad S_2 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \end{vmatrix}$$

$$S_3 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_4 \\ b_2 & b_2 & b_4 \end{vmatrix} \qquad S_4 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

we obtain

$$\begin{aligned} &S_1 - S_2 + S_3 - S_4 = 0 \\ &a_1 S_1 - a_2 S_2 + a_3 S_3 - a_4 S_4 = 0 \\ &b_1 S_1 - b_2 S_2 + b_3 S_3 - b_4 S_4 = 0, \end{aligned}$$

which proves that $L(E)$ is equal to $\ker(S_1, -S_2, S_3, -S_4)$.

Our next step is to find the norm of projection given by (1). Consider functional $e_1 \circ P$, take $x = (x_1, x_2, x_3, x_4) \in S(\ell_\infty^4)$ and compute

$$\begin{aligned} |(e_1 \circ P)(x_1, x_2, x_3, x_4)| &= \frac{1}{M}\left| Mx_1 - \frac{S_1x_1 - S_2x_2 + S_3x_3 - S_4x_4}{S_3}\right| \\ &= \frac{1}{M}\left|\Big(M - \frac{S_1}{S_3}\Big)x_1 + \frac{S_2}{S_3}x_2 - x_3 + \frac{S_4}{S_3}x_4\right| \\ &= \frac{1}{M}\left|\Big(\frac{S_3}{S_1} + \frac{S_2}{S_4} + \frac{S_4}{S_2}\Big)x_1 + \frac{S_2}{S_3}x_2 - x_3 + \frac{S_4}{S_3}x_4\right| \\ &\le \frac{1}{M}\left[\Big(\frac{S_3}{S_1} + \frac{S_2}{S_4} + \frac{S_4}{S_2}\Big) + \frac{S_2}{S_3} + 1 + \frac{S_4}{S_3}\right] \\ &= \frac{1}{M}\Big(\frac{S_3}{S_1} + \frac{S_2}{S_4} + \frac{S_4}{S_2} + 1 + \frac{S_2 + S_4}{S_3}\Big) \\ &= \frac{1}{M}\Big(\frac{S_3}{S_1} + \frac{S_2}{S_4} + \frac{S_4}{S_2} + 1 + \frac{S_1 + S_3}{S_3}\Big) \\ &= \frac{1}{M}(M+2) = 1 + \frac{2}{M}. \end{aligned}$$

Hence, $\|e_1 \circ P\| \le 1 + \frac{2}{M}$ and the norming point for this functional is $(1, 1, -1, 1)$. We can carry on the same computations for every $e_i \circ P$ therefore,

$$\|P\| = 1 + \frac{2}{M}, \tag{2}$$

and the following pairs are norming pairs for this projection (we would like to point it out that these pairs do not depend on numbers $S_1, \ldots, S_4$)

$$
\begin{aligned}
&e_1 \otimes (1, 1, -1, 1)\\
&e_2 \otimes (1, 1, 1, -1)\\
&e_3 \otimes (-1, 1, 1, 1)\\
&e_4 \otimes (1, -1, 1, 1).
\end{aligned}
$$

We will prove that P is a minimal projection by constructing appropriate Chalmers-Metcalf operator (see Theorem 1). Consider the following operator

$$
\begin{aligned}
E_P = \frac{S_1}{S_3} \cdot e_1 \otimes (1, 1, -1, 1) + \frac{S_2}{S_4} \cdot e_2 \otimes (1, 1, 1, -1)&\\
+ \frac{S_3}{S_1} \cdot e_3 \otimes (-1, 1, 1, 1) + \frac{S_4}{S_1} \cdot e_4 \otimes (1, -1, 1, 1).& \quad (3)
\end{aligned}
$$

We have already proved that $L(E) = \ker(S_1, -S_2, S_3, -S_4)$. By Chalmers-Metcalf Theorem (Theorem 1) it is enough to show that

$$E_P(L(E)) \subset L(E). \tag{4}$$

Let $w = (1, 1, 1, 1)$, obviously $w \in L(E)$. We will rearrange the operator given by (3),

$$
\begin{aligned}
E_P =& \frac{S_1}{S_3} \cdot e_1 \otimes (w - 2e_3) + \frac{S_2}{S_4} \cdot e_2 \otimes (w - 2e_4)\\
&+ \frac{S_3}{S_1} \cdot e_3 \otimes (w - 2e_1) + \frac{S_4}{S_1} \cdot e_4 \otimes (w - 2e_2)\\
=& \Big(\frac{S_1}{S_3} \cdot e_1 + \frac{S_2}{S_4} \cdot e_2 + \frac{S_3}{S_1} \cdot e_3 + \frac{S_4}{S_2} \cdot e_4 \Big) \otimes w\\
&- 2\Big(\frac{S_1}{S_3} \cdot e_1 \otimes e_3 + \frac{S_2}{S_4} \cdot e_2 \otimes e_4 + \frac{S_3}{S_1} \cdot e_3 \otimes e_1 + \frac{S_4}{S_1} \cdot e_4 \otimes e_2 \Big).
\end{aligned}
$$

To show (4) take any $x \in L(E) = \ker(S_1, -S_2, S_3, -S_4)$, i.e.,

$$S_1\, x_1 - S_2\, x_2 + S_3\, x_3 - S_4\, x_4 = 0.$$

By the above computations

$$E_P(x) = (\textit{something}) \otimes w - 2\Big(\frac{S_3}{S_1} x_3, \frac{S_4}{S_2} x_4, \frac{S_1}{S_3} x_1, \frac{S_2}{S_4} x_2 \Big), \tag{5}$$

Now, $(\textit{something}) \otimes w$ is always in $L(E)$ (since $w \in L(E)$). We need to take care of the second part of E_P. Observe that

$$
\begin{aligned}
< (S_1, -S_2, S_3, -S_4), \Big(\frac{S_3}{S_1} x_3, \frac{S_4}{S_2} x_4, \frac{S_1}{S_3} x_1, \frac{S_2}{S_4} x_2 \Big) >&\\
= S_1\, x_1 - S_2\, x_2 + S_3\, x_3 - S_4\, x_4 = 0&
\end{aligned}
$$

that is, whenever $(x_1, x_2, x_3, x_4) \in L(E)$, also $(\frac{S_3}{S_1}x_3, \frac{S_4}{S_2}x_4, \frac{S_1}{S_3}x_1, \frac{S_2}{S_4}x_2) \in L(E)$. This with (5) gives (4).

As to the inequality $\|P\| \leq \frac{3}{2}$ by (2), it is enough to observe that by $t+\frac{1}{t} \geq 2$ we have

$$M = \frac{S_1}{S_3} + \frac{S_3}{S_1} + \frac{S_2}{S_4} + \frac{S_4}{S_2} \geq 4.$$

Therefore, $1 + \frac{2}{M} \leq 1 + \frac{2}{4} = \frac{3}{2}$ and "=" holds only when $S_1 = S_3$ and $S_2 = S_4$. That implies that $A_1A_2A_3A_4$ has to be a parallelogram. □

Now we will go back to minimal projection from $C(S)$ onto E by Theorem 2.

Theorem 4. *Assume that $S = A_1A_2A_3A_4$ is a convex quadrilateral. Also assume that its diagonals A_1A_3 and A_2A_4 are perpendicular and intersect at origin. Let $A_1 = (-a_1, 0)$, $A_2 = (0, a_2)$, $A_3 = (a_3, 0)$, and $A_4 = (0, -a_4)$. We have*

$$a_1 = OA_1, \quad b = OA_2, \quad c = OA_3, \quad d = OA_4.$$

Then the following projection Q is a minimal projection from $C(S)$ onto E

$$Q = u \otimes 1 + v \otimes x + w \otimes y,$$

where:

$$\begin{aligned} u = & \frac{\frac{a_3}{a_1+a_3}\left(\frac{a_2}{a_4} + \frac{a_4}{a_2}\right)}{\frac{a_1}{a_3} + \frac{a_3}{a_1} + \frac{a_2}{a_4} + \frac{a_4}{a_2}} \cdot \delta_{A_1} + \frac{\frac{a_4}{a_2+a_4}\left(\frac{a_1}{a_3} + \frac{a_3}{a_1}\right)}{\frac{a_1}{a_3} + \frac{a_3}{a_1} + \frac{a_2}{a_4} + \frac{a_4}{a_2}} \cdot \delta_{A_2} \\ & + \frac{\frac{a_1}{a_1+a_3}\left(\frac{a_2}{a_4} + \frac{a_4}{a_2}\right)}{\frac{a_1}{a_3} + \frac{a_3}{a_1} + \frac{a_2}{a_4} + \frac{a_4}{a_2}} \cdot \delta_{A_3} + \frac{\frac{a_2}{a_2+a_4}\left(\frac{a_1}{a_3} + \frac{a_3}{a_1}\right)}{\frac{a_1}{a_3} + \frac{a_3}{a_1} + \frac{a_2}{a_4} + \frac{a_4}{a_2}} \cdot \delta_{A_4}, \end{aligned} \tag{6}$$

$$\begin{aligned} v = & -\frac{1}{a_1 + a_3} \frac{\frac{a_1+a_3}{a_3} + \frac{a_2}{a_4} + \frac{a_4}{a_2}}{\frac{a_1}{a_3} + \frac{a_3}{a_1} + \frac{a_2}{a_4} + \frac{a_4}{a_2}} \cdot \delta_{A_1} - \frac{\frac{a_4}{a_2+a_4}\left(\frac{1}{a_1} - \frac{1}{a_3}\right)}{\frac{a_1}{a_3} + \frac{a_3}{a_1} + \frac{a_2}{a_4} + \frac{a_4}{a_2}} \cdot \delta_{A_2} \\ & + \frac{1}{a_1 + a_3} \frac{\frac{a_1+a_3}{a_1} + \frac{a_2}{a_4} + \frac{a_4}{a_2}}{\frac{a_1}{a_3} + \frac{a_3}{a_1} + \frac{a_2}{a_4} + \frac{a_4}{a_2}} \cdot \delta_{A_3} - \frac{\frac{a_2}{a_2+a_4}\left(\frac{1}{a_1} - \frac{1}{a_3}\right)}{\frac{a_1}{a_3} + \frac{a_3}{a_1} + \frac{a_2}{a_4} + \frac{a_4}{a_2}} \cdot \delta_{A_4}, \end{aligned} \tag{7}$$

$$\begin{aligned} w = & \frac{\frac{a_3}{a_1+a_3}\left(\frac{1}{a_2} - \frac{1}{a_4}\right)}{\frac{a_1}{a_3} + \frac{a_3}{a_1} + \frac{a_2}{a_4} + \frac{a_4}{a_2}} \cdot \delta_{A_1} + \frac{1}{a_2 + a_4} \frac{\frac{a_2+a_4}{a_4} + \frac{a_2}{a_4} + \frac{a_4}{a_2}}{\frac{a_1}{a_3} + \frac{a_3}{a_1} + \frac{a_2}{a_4} + \frac{a_4}{a_2}} \cdot \delta_{A_2} \\ & + \frac{\frac{a_1}{a_1+a_3}\left(\frac{1}{a_2} - \frac{1}{a_4}\right)}{\frac{a_1}{a_3} + \frac{a_3}{a_1} + \frac{a_2}{a_4} + \frac{a_4}{a_2}} \cdot \delta_{A_3} - \frac{1}{a_2 + a_4} \frac{\frac{a_2+a_4}{a_2} + \frac{a_2}{a_4} + \frac{a_4}{a_2}}{\frac{a_1}{a_3} + \frac{a_3}{a_1} + \frac{a_2}{a_4} + \frac{a_4}{a_2}} \cdot \delta_{A_4}. \end{aligned} \tag{8}$$

Proof. We will give a sketch of a proof. Once you know the formulas (6)–(8) you can make direct computations (extremely tedious). Alternatively, you can argue as below. Using notations from Theorem 3 in this situation we have

$$M = \frac{a_1}{a_3} + \frac{a_3}{a_1} + \frac{a_2}{a_4} + \frac{a_4}{a_2}$$

and

$$S_1 = \frac{1}{2}a_3(a_2 + a_4),$$
$$S_2 = \frac{1}{2}a_4(a_1 + a_3),$$
$$S_3 = \frac{1}{2}a_1(a_2 + a_4),$$
$$S_4 = \frac{1}{2}a_2(a_1 + a_3).$$

Take any $z = (z_1, z_2, z_3, z_4)$ and put $V = S_1x_1 - S_2x_2 + S_3x_3 - S_4x_4$. Comparing $Pz = (f_1, f_2, f_3, f_4)$ (from (1)) and $Qz = u \otimes 1 + v \otimes x + w \otimes y$ we obtain

$$u - a_1v = f_1, \tag{9}$$
$$u + a_2w = f_2, \tag{10}$$
$$u + a_3v = f_3, \tag{11}$$
$$u - a_4w = f_4. \tag{12}$$

The first and third equation give $a_3u + a_1u = a_3f_1 + a_1f_3$ and

$$u = \frac{a_3}{a_1 + a_3}f_1 + \frac{a_1}{a_1 + a_3}f_3,$$

while the second and fourth give $a_4u + a_2u = a_4f_2 + a_2f_4$ and

$$u = \frac{a_4}{a_2 + a_4}f_2 + \frac{a_2}{a_2 + a_4}f_4.$$

As a result,

$$2u = \frac{a_3}{a_1 + a_3}f_1 + \frac{a_4}{a_2 + a_4}f_2 + \frac{a_1}{a_1 + a_3}f_3 + \frac{a_2}{a_2 + a_4}f_4. \tag{13}$$

From (1) we know that

$$f_1 = x_1 - \frac{V}{MS_3}, \quad f_2 = x_2 + \frac{V}{MS_4}, \quad f_3 = x_3 - \frac{V}{MS_1}, \quad f_4 = x_4 + \frac{V}{MS_2}.$$

Plugging these in (13) gives

$$\begin{aligned} 2u =& \frac{a_3}{a_1 + a_3}f_1 + \frac{a_4}{a_2 + a_4}f_2 + \frac{a_1}{a_1 + a_3}f_3 + \frac{a_2}{a_2 + a_4}f_4 \\ &+ \frac{V}{M}\Big(- \frac{a_3}{a_1 + a_3}S_3 + \frac{a_4}{a_2 + a_4}S_4 - \frac{a_1}{a_1 + a_3}S_1 + \frac{a_2}{a_2 + a_4}S_4\Big). \end{aligned}$$

The term in brackets at $\frac{V}{M}$ after substitution for S_1, S_2, S_3, S_4 equals

$$\begin{aligned} - \frac{a_3}{a_1 + a_3}\frac{a_1}{a_2 + a_4} + \frac{a_4}{a_2 + a_4}\frac{a_2}{a_1 + a_3} - \frac{a_1}{a_1 + a_3}\frac{a_3}{a_2 + a_4} + \frac{a_2}{a_2 + a_4}\frac{a_4}{a_1 + a_3} \\ = \Big[\Big(\frac{a_2}{a_4} + \frac{a_4}{a_2}\Big) - \Big(\frac{a_1}{a_3} + \frac{a_3}{a_1}\Big)\Big]\frac{1}{(a_1 + a_3)(a_2 + a_4)}, \end{aligned}$$

therefore,

$$2u = \frac{a_3}{a_1+a_3} f_1 + \frac{a_4}{a_2+a_4} f_2 + \frac{a_1}{a_1+a_3} f_3 + \frac{a_2}{a_2+a_4} f_4 + \frac{V}{M}\Big[\Big(\frac{a_2}{a_4}+\frac{a_4}{a_2}\Big) - \Big(\frac{a_1}{a_3}+\frac{a_3}{a_1}\Big)\Big]\frac{1}{(a_1+a_3)(a_2+a_4)}.$$

Let us compute coefficient u_1 at x_1 on the right side. It equals

$$\begin{aligned} u_1 &= \frac{a_3}{a_1+a_3} - \frac{S_1}{M}\Big[\Big(\frac{a_2}{a_4}+\frac{a_4}{a_2}\Big) - (\frac{a_1}{a_3}+\frac{a_3}{a_1})\Big]\frac{1}{(a_1+a_3)(a_2+a_4)} \\ &= \frac{a_3}{a_1+a_3} - \frac{a_3}{M}\Big[\Big(\frac{a_2}{a_4}+\frac{a_4}{a_2}\Big) - \Big(\frac{a_1}{a_3}+\frac{a_3}{a_1}\Big)\Big]\frac{1}{(a_1+a_3)} \\ &= \frac{a_3}{a_1+a_3}\frac{1}{M}\Big[M - \Big(\frac{a_2}{a_4}+\frac{a_4}{a_2}\Big) + \Big(\frac{a_1}{a_3}+\frac{a_3}{a_1}\Big)\Big] \\ &= \frac{a_3}{a_1+a_3}\frac{1}{M}\,2\Big(\frac{a_1}{a_3}+\frac{a_3}{a_1}\Big). \end{aligned}$$

Similarly we can compute coefficients at x_1, x_2 and x_4. Hence, we obtain the formula for functional u. We can find v using (9) and (11) as follows

$$(a_1+a_3)v = f_3 - f_1 = x_3 - x_1 + \frac{V}{M}\Big(\frac{1}{S_3} - \frac{1}{S_1}\Big) = x_3 - x_1 + \frac{V}{M}\Big(\frac{1}{a_1} - \frac{1}{a_3}\Big)\frac{1}{a_2+a_4},$$

after that, straightforward computation, leads to the formula for v. Similarly, considering (10) and (12) leads to

$$(a_2+a_4)v = f_2 - f_4 = x_2 - x_4 + \frac{V}{M}\Big(\frac{1}{S_4} - \frac{1}{S_2}\Big) = x_2 - x_4 + \frac{V}{M}\Big(\frac{1}{a_2} - \frac{1}{a_4}\Big)\frac{1}{a_1+a_3}$$

and the formula for w. □

Remark 2. Using Theorem 4 we can compute minimal projection from $C(S)$ onto E for any convex quadrilateral S as follows. Let $S = A_1A_2A_3A_4$ and let D be a point of intersection of A_1A_3 and A_2A_4, then we can translate and rotate S to get quadrilateral $S' = B_1B_2B_3B_4$ such that B_1B_3 and B_2B_4 intersects at the origin O and B_1B_3 lies on the x-axis. Now there is a linear transformation which transforms triangle B_2OB_3 to right triangle C_2OC_3, and such that $B_2O = C_2O$ and $B_3O = C_3O$. If C_1, C_4 are respectively images of B_1, B_4 in this transformation then since any linear transformation preserves ratios at parallel lines, we obtain a convex quadrilateral $S'' = C_1C_2C_3C_4$ such that

1. C_1C_3 lies on x-axis;
2. C_2C_4 lies on y-axis;
3. C_1C_3 and C_2C_4 intersects at the origin;
4. $C_1O = A_1D$, $C_2O = A_2D$, $C_3O = A_3D$, $C_4O = A_4D$.

Now let L be this linear transformation from S to S''. It will also generate an isometry between $C(S)$ and $C(S')$ by $I(f) = f \circ L$. Additionally, isometry I does not change the subspace E. Following the reasoning at the beginning of this section we know that minimal projection from $C(S)$ onto E can be computed as $L^{-1} \circ Q \circ L$, where Q is a minimal projection from $C(S'')$ onto E. Looking at the formula of Q in Theorem 4 we see that u, v, w depends only on diagonals. Hence, the minimal projection from $C(S)$ onto E is given by

$$R = u \otimes 1 + v \otimes L(x) + w \otimes L(y),$$

where u, v, w are given by (6)–(8) with $a_1 = A_1 D$, $a_2 = A_2 D$, $a_3 = A_3 D$, $a_4 = A_4 D$.

Remark 3. If S is not convex quadrilateral, then the smallest convex set containing S is a triangle. Now the Lagrange interpolation projection with nods in vertices of this triangle has norm one and as a result is also minimal.

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BORIS SHEKHTMAN
Department of Mathematics
University of South Florida
4202 E Fowler Av, PHY 114
Tampa, FL 33620-5700
USA
E-mail: boris@math.usf.edu

LESŁAW SKRZYPEK
Department of Mathematics
University of South Florida
4202 E Fowler Av, PHY 114
Tampa, FL 33620-5700
USA
E-mail: skrzypek@math.usf.edu