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Relative Widths of Classes of Differentiable Functions and Splines

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In 1999–2005 the authors published several papers [7–13] devoted to a problem on relative widths of classes of differentiable functions and to related problems. The aim of the present note is to give a brief review of the obtained results and to attract the attention of specialists in approximation theory to this theme.

The widths of sets were introduced by Kolmogorov [2]. The width of order n of a centrally symmetrical set W belonging to a Banach space X is the quantity

$$d_n(W,X) := \inf_{L_n} \sup_{f\in W} \inf_{g\in L_n} \|f-g\|_X. \tag{1}$$

Here L_n are linear subspaces of dimension at most n in the space X.

These widths are called the Kolmogorov widths.

During subsequent years, many other variants of the notion of width of a set were suggested. Particularly, Konovalov [3] introduced widths which became to be called relative widths.

Let W and V be centrally symmetrical sets in a Banach space X. The width of order n of the set W relatively to the set V is the quantity

$$K_n(W,V,X) := \inf_{L_n} \sup_{f\in W} \inf_{g\in L_n\cap V} \|f-g\|_X.$$

Here, as in (1), L_n are linear subspaces of dimension at most n in the space X.

It is clear that $K_n(W,X,X) = d_n(W,X)$ and for any set V

$$K_n(W,V,X) \geqslant d_n(W,X).$$

Let us define the classes of functions which will be discussed below.

Let r be a natural number and $M > 0$. We denote by MW^r the class of 2π-periodic functions $f(x)$ whose derivative $f^{(r-1)}(x)$ is absolutely continuous

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and derivative $f^{(r)}(x)$ satisfies the conditions $|f^{(r)}(x)| \leqslant M$ at those points where it exists. For $M = 1$ we shall write W^r.

Tikhomirov [14] found the exact values for the Kolmogorov widths $d_n(W^r, C)$. He proved that

$$d_{2m-1}(W^r, C) = d_{2m}(W^r, C) = \frac{\mathcal{K}_r}{n^r},$$

where n is equal to $2m-1$ or $2m$, and $\mathcal{K}_r$ are the Favard constants of order r.

Konovalov [3] studied the widths $K_n(W^r, W^r, C)$. He showed that for $r = 1$ and $r = 2$ the following order inequality is true:

$$K_n(W^r, W^r, C) \sim d_n(W^r, C), \quad n \to \infty,$$

and for $r \geqslant 3$

$$K_n(W^r, W^r, C) \sim O(n^{-2}), \quad n \to \infty.$$

In connection with these results, S. B. Stechkin conjectured that if one takes an absolute constant strictly greater than 1 as M, then for $r \geqslant 3$ the widths $K_n(W^r, MW^r, C)$, as well as $d_n(W^r, C)$, are of order n^{-r} as $n \to \infty$.

This conjecture was justified. Babenko [1] proved its validity.

In what follows we shall consider the question about the coincidence of values of the relative widths $K_n(W^r, MW^r, C)$ and the Kolmogorov widths $d_n(W^r, C)$.

With the use of the theorem by Malozemov [4] about approximation of functions by polygonal lines, it can be proven that for $r = 1$ and n even, the following equality holds:

$$K_n(W^r, W^r, C) = d_n(W^r, C). \tag{2}$$

For $r = 2$ and n even, equality (2) was established by Subbotin [6].

Let now $r \geqslant 3$ and $M(n, r)$ be the least value of the multiplier M for which the equality

$$K_n(W^r, MW^r, C) = d_n(W^r, C) \tag{3}$$

holds.

In [7, Theorem 1] it was proved that there exists an absolute constant A such that

$$M(n, r) \geqslant \frac{4}{\pi^2} \log \min(n, r) + A.$$

The following question remains open: "Does there exist an absolute constant M^* such that $M(n, r) \leqslant M^*$?" If the answer to this question is negative, is $\log \min(n, r)$ the true order of the quantity $M(n, r)$?

In [13], the relative widths $K_n(W^r, MW^j, C)$ for $j < r$ were studied. According to Theorem 4 from [13], for the least value $M(n, r, j)$ of the multiplier M for which equality (3) holds, the following estimate is valid:

$$M(n, r, j) = \mathcal{K}_{r-j} + o(1), \quad n \to \infty.$$

Analogs of some results above were also obtained for the case when X is $L_1[0,2\pi]$ and the classes $MW^r_{L_1}$ are defined correspondingly.

The paper [9] is devoted to the problem about the relative widths in the space $L_2[0,2\pi]$. In this paper the case for which the authors succeeded in exact calculation of the least constant $M(n,r,L_2)$ was considered for the first time. Namely, let $MW^r_{L_2}$ be the classes of functions f such that the norm in L_2 of the derivative $f^{(r)}$ is bounded by M. In [9] it was proved that for the least value $M(n,r,L_2)$ of the multiplier M for which the equalities

$$\begin{aligned} K_{2m-1}(W^r_{L_2}, MW^r_{L_2}, L_2) &= K_{2m}(W^r_{L_2}, MW^r_{L_2}, L_2) \\ &= d_{2m-1}(W^r_{L_2}, L_2) = d_{2m}(W^r_{L_2}, L_2) = m^{-r}, \end{aligned}$$

are valid, for $n = 2m-1$ and $n = 2m$, we have

$$M(n,r,L_2) = 1 - m^{-r}.$$

Notice that in this case the mentioned multiplier turned out to be less than 1.

Some approximative properties of splines were established while the results on relative widths were considered.

For instance, in [8] the asymptotic behavior of the Lebesgue constants of periodic interpolation splines with equidistant nodes was obtained.

On $[0,1]$ we consider splines of degree r with deficiency 1 whose nodes are $x_i = i/n$.

With any continuous periodic on $[0,1]$ function f we associate the spline $s_{n,r}(f,x)$ of the above mentioned form which interpolates $f(x)$ at the points $t_i = x_i + (1+(-1)^r)/4n$. For the norms of the operator $f(x) \to s_{n,r}(f,x)$ as an operator from $C[0,1]$ to $C[0,1]$, the estimate

$$L_{n,r} = \frac{2}{\pi} \log \min(n,r) + O(1) \tag{4}$$

was obtained in [8], which is uniform with respect to all parameters.

The problem about the Lebesgue constant of the interpolating splines was studied earlier on the class of functions continuous on the whole real line for splines of degree r with deficiency 1 whose nodes are integer points. Zhensykbaev [15] and Richards [5] proved that in this case the estimate

$$\frac{2}{\pi} \log r + O(1) \tag{5}$$

is valid for the Lebesgue constants. The problem from [8] can be reformulated in a standard way as a problem about interpolating splines given on the whole real line and with period n.

Comparing (4) and (5), we see that the passage from arbitrary functions in $C(-\infty,\infty)$ to functions of period n results in replacing r in (5) by $\min(n,r)$ in (4).

A similar effect of the change to periodic functions also appeared in some other approximate problems on splines [11, 12].

Finally, notice that in the process of our investigations, we also obtained a refinement of an estimate by B. Sz.-Nagy of the Lebesgue constants for the Zigmund method of normal means of summation of the Fourier series (see [7]).

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