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Old and New Results on the Favard Operator

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In the present paper we consider a generalization F_{n,σ_n} of the Favard operators and study the local rate of convergence for smooth functions. As a main result we present the complete asymptotic expansion for the sequence $(F_{n,\sigma_n} f)(x)$ as n tends to infinity ($x \in \mathbb{R}$). Furthermore, we consider a truncated version of these operators. Finally, all results are obtained for simultaneous approximation.

Keywords and Phrases: Approximation by positive operators, rate of convergence, degree of approximation.

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1. Introduction

In 1944 Favard [6, pp. 229, 239] introduced the operator F_n defined by

$$(F_n f)(x) = \frac{1}{\sqrt{\pi n}} \sum_{\nu=-\infty}^{\infty} f\Big(\frac{\nu}{n}\Big) \exp\Big(-n\Big(\frac{\nu}{n} - x\Big)^2\Big) \tag{1}$$

which is a discrete analogue of the familiar Gauss–Weierstrass singular convolution integral

$$(W_n f)(x) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} f(t) \exp\big(-n(t-x)^2\big) dt.$$

in order to approximate functions defined on the real axis. The operators can be applied to functions f defined on $\mathbb{R}$ and satisfying the growth condition

$$f(t) = O\big(e^{Kt^2}\big) \qquad \text{as } |t| \to \infty, \tag{2}$$

with a constant $K > 0$. Favard proved that $(F_n f)(x)$ converges to $f(x)$ as $n \to \infty$ pointwise for every $x \in \mathbb{R}$, and even uniformly on any compact subinterval of $\mathbb{R}$, for all functions f continuous on $\mathbb{R}$ and satisfying (2).

In the first part we report on results by Becker, Butzer and Nessel [3] on saturation in weighted Banach spaces consisting of functions of polynomial and exponential growth, respectively.

The second part contains further developments with respect to results by Gawronski and Stadtmüller [5]. They considered in 1982, for a sequence of positive reals σ_n, the generalization

$$(F_{n,\sigma_n} f)(x) = \sum_{\nu=-\infty}^{\infty} p_{n,\nu,\sigma_n}(x) f\Big(\frac{\nu}{n}\Big), \tag{3}$$

where

$$p_{n,\nu,\sigma_n}(x) = \frac{1}{\sqrt{2\pi}\, n\sigma_n} \exp\Big(-\frac{1}{2\sigma_n^2}\Big(\frac{\nu}{n} - x\Big)^2\Big).$$

The particular case $\sigma_n^2 = 1/(2n)$ reduces to Favard's classical operator (1).

Under certain conditions on f and $(\sigma_n)_{n\in\mathbb{N}}$ the operators possess the basic property that $(F_{n,\sigma_n} f)(x)$ converges to $f(x)$ in each continuity point x of f. Among other results, Gawronski and Stadtmüller [5, Eq. (0.6)] established the Voronovskaja-type theorem

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2}[(F_{n,\sigma_n} f)(x) - f(x)] = \frac{1}{2} f''(x) \tag{4}$$

uniformly on proper compact subsets of $[a, b]$, for $f \in C^2[a, b]$ $(a, b \in \mathbb{R})$ and $\sigma_n \to 0$ as $n \to \infty$, provided that certain conditions on the first three moments of F_{n,σ_n} are satisfied. Actually, Eq. (4) was proved for a truncated variant of (3) which possesses the same asymptotic properties as (3) [5, cf. Theorem 1 (iii) and Remark (i), p. 393]. For a Voronovskaja-type theorem (cf. [11]) in the particular case $\sigma_n^2 = \gamma/(2n)$ see [3, Theorem 4.3]. Abel and Butzer extended formula (4) by deriving a complete asymptotic expansion of the form

$$F_{n,\sigma_n} f \sim f + \sum_{k=0}^{\infty} c_k(f)\sigma_n^k \qquad (n \to \infty)$$

for f sufficiently smooth. The latter formula means that for all positive integers q there holds

$$F_{n,\sigma_n} f = f + \sum_{k=1}^{q} c_k(f)\sigma_n^k + o(\sigma_n^q) \qquad (n \to \infty).$$

The coefficients c_k, which depend on f but are independent of n, are explicitely determined. It turns out that $c_k(f) = 0$, for all odd integers $k > 0$. Moreover, we also deal with simultaneous approximation by the operators (3) and study a truncated version of them which was defined by Gawronski and Stadtmüller [5, cf. Eq. (0.7)].

Pych-Taberska [9] estimated the rate of convergence of the Favard operator in a Holder-type norm. In 2007 Nowak and Sikorska-Nowak [8] considered Kantorovich and Durrmeyer variants of Favard operators. A further related paper by Pych-Taberska and Nowak is [10].

2. Saturation in Weighted Banach Spaces

In this section we sketch the main results by Becker, Butzer and Nessel [3]. Actually, the authors considered the slightly more general operators F_{n,σ_n} when $\sigma_n^2 = \gamma/(2n)$ with a constant $\gamma > 0$. For the sake of simplicity we restrict our representation to the case $\gamma = 1$ which is Favard's classical operator (1).

2.1. Functions of Polynomial Growth

For $N \in \mathbb{N}$, define the weight function

$$w_N(x) = (1 + x^{2N})^{-1}$$

and consider the Banach space

$$X_N := \left\{ f \in C(\mathbb{R}) : f(x) = o\big(w_N^{-1}(x)\big) \text{ for } |x| \to +\infty \right\}$$

equipped with the norm

$$\|f\|_N := \|w_N f\|_\infty = \sup_{x \in \mathbb{R}} \left| (1 + x^{2N})^{-1} f(x) \right|.$$

The Favard operators $F_n : X_N \to X_N$ are bounded operators, more precisely there holds

Theorem 1 (Boundedness). *For each $N \in \mathbb{N}$, there exists a constant M_N such that*

$$\|F_n f\|_N \le \Big(1 + \frac{M_N}{n}\Big) \|f\|_N \qquad (f \in X_N).$$

Furthermore, the sequence $(F_n)_{n \in \mathbb{N}}$ constitutes an approximation process on each space X_N.

Theorem 2 (Approximation). *For each $N \in \mathbb{N}$, there holds*

$$\lim_{n \to \infty} \|F_n f - f\|_N = 0 \qquad (f \in X_N).$$

On the subspace

$$D_N := \{ f \in X_N : f', f'' \in X_N \}$$

there holds the following Voronovskaja-type formula.

Theorem 3 (Voronovskaja-type result). *For all $f \in D_N$, there holds the asymptotic formula*

$$\lim_{n \to \infty} \left\| n(F_n f - f) - \frac{1}{4} f'' \right\|_N = 0.$$

A certain stability condition which follows from the boundedness and the Voronovskaja-type formula are the main assumptions of a general theorem by Trotter which implies the following saturation result.

Theorem 4 (Saturation). *Let $f \in X_N$. Then the following three statements are equivalent:*

$$\|F_n f - f\|_N = O\big(n^{-1}\big) \qquad (n \to \infty),$$
$$\|f(x+h) - 2f(x) + f(x-h)\|_N = O\big(h^2\big) \qquad (h \to 0),$$
$$f \in \widetilde{D_N} \qquad (\textit{the completion relative to } X_N).$$

2.2. Functions of Exponential Growth

For $\beta > 0$, define the weight function

$$w_\beta(x) = e^{-\beta x^2}$$

and the norm

$$\|f\|_\beta := \|w_\beta f\|_\infty = \sup_{x \in \mathbb{R}} \big| e^{-\beta x^2} f(x) \big|.$$

We consider the space

$$X := \bigcap_{\beta > 0} \big\{ f \in C(\mathbb{R}) : \|f\|_\beta < \infty \big\}.$$

Analogously to the preceding subsection there hold the following theorems.

Theorem 5 (Approximation). *For $\beta > 0$, there holds*

$$\lim_{n \to \infty} \|F_n f - f\|_\beta = 0 \qquad (f \in X).$$

Theorem 6 (Voronovskaja-type result). *For all $\beta > 0$ and $f \in D'$, $D' := \{f \in X : f', f'' \in X\}$, there holds*

$$\lim_{n \to \infty} \Big\| n(F_n f - f) - \frac{1}{4} f'' \|_\beta = 0.$$

Theorem 7 (Saturation). *Let $f \in X$. Then the following two statements*

$$\|F_n f - f\|_\beta = O\big(n^{-1}\big) \qquad (n \to \infty)$$

and

$$\|f(x+h) - 2f(x) + f(x-h)\|_\beta = O\big(h^2\big) \qquad (h \to 0)$$

are equivalent.

The paper [3] does not contain a result on the saturation class with regard to the space X.

3. Asymptotic Expansions

Throughout this section we assume that

$$\sigma_n > 0, \quad \sigma_n \to 0, \quad \sigma_n^{-1} = O\big(n^{1-\eta}\big) \qquad (n \to \infty) \tag{5}$$

with (an arbitrarily small) constant $\eta > 0$. Note that the latter condition implies that $n\sigma_n \to \infty$ as $n \to \infty$.

The following theorem presents the complete asymptotic expansion for the sequence $(F_{n,\sigma_n})(x)$ as $n \to \infty$. For $r \in \mathbb{N}$ and $x \in \mathbb{R}$ let $W[r;x]$ be the class of functions on $\mathbb{R}$ satisfying growth condition (2), which admits a derivative of order r at the point x.

Theorem 8. *Let $q \in \mathbb{N}$ and $x \in \mathbb{R}$. Suppose that the real sequence (σ_n) satisfies condition* (5). *For each function $f \in W[2q;x]$, the generalized Favard operators* (3) *possess the complete asymptotic expansion*

$$(F_{n,\sigma_n} f)(x) = f(x) + \sum_{k=1}^{q} \frac{f^{(2k)}(x)}{(2k)!!} \sigma_n^{2k} + o\big(\sigma_n^{2q}\big) \qquad (n \to \infty). \tag{6}$$

Here $m!!$ denote the double factorial numbers defined by $0!! = 1!! = 1$ and $m!! = m \times (m-2)!!$ for integers $m \geq 2$.

The crucial tool for the proof of Theorem 8 is a classical transformation formula for the elliptic theta function (see, e.g., [4, p. 126]). An immediate consequence is the Voronosvkaja-type theorem of [5, Eq. (0.6), Theorem 1 (iii) and Remark (i)].

Corollary 1. *Let $x \in \mathbb{R}$. Suppose that the real sequence (σ_n) satisfies condition* (5). *For each function $f \in W[2;x]$, there holds the asymptotic relation*

$$\lim_{n\to\infty} \sigma_n^{-2} \big((F_{n,\sigma_n} f)(x) - f(x)\big) = \frac{1}{2} f''(x).$$

Concerning simultaneous approximation, it turns out that the complete asymptotic expansion can be obtained by differentiating (6) term-by-term.

Theorem 9. *Let $r \in \mathbb{N}_0$, $q \in \mathbb{N}$ and $x \in \mathbb{R}$. Suppose that the real sequence (σ_n) satisfies condition* (5). *For each function $f \in W[2(r+q);x]$, there holds the complete asymptotic expansion*

$$(F_{n,\sigma_n} f)^{(r)}(x) = f^{(r)}(x) + \sum_{k=1}^{q} \frac{f^{(2k+r)}(x)}{(2k)!!} \sigma_n^{2k} + o\big(\sigma_n^{2q}\big) \qquad (n \to \infty). \tag{7}$$

The proof of Theorem 9 essentially uses a representation of $(F_{n,\sigma_n} f)^{(r)}$ in terms of F_{n,σ_n} which follows by a certain identity of Hermite polynomials.

Remark 1. Formula (7) can be written in the equivalent form

$$\lim_{n\to\infty} \sigma_n^{-2q}\Big((F_{n,\sigma_n} f)^{(r)}(x) - f^{(r)}(x) - \sum_{k=1}^{q} \frac{f^{(2k+r)}(x)}{(2k)!!}\,\sigma_n^{2k}\Big) = 0.$$

Assuming smoothness of f on intervals $I = (a, b)$, $a, b \in \mathbb{R}$, it can be shown that the expansions (6) and (7) hold uniformly on compact subsets of I.

The proofs are based on a localization result which is interesting in itself.

Proposition 1. *Fix $x \in \mathbb{R}$ and let $\delta > 0$. Assume that the function $f : \mathbb{R} \to \mathbb{R}$ vanishes in $(x - \delta, x + \delta)$ and satisfies, for positive constants M_x, K_x, the growth condition*

$$|f(t)| \le M_x e^{K_x (t-x)^2} \qquad (t \in \mathbb{R}). \tag{8}$$

Then, for positive $\sigma < 1/\sqrt{2K_x}$, there holds the estimate

$$|(F_{n,\sigma} f)(x)| \le \sqrt{\frac{2}{\pi}} \cdot \frac{M_x \sigma/\delta}{1 - 2K_x\sigma^2} \exp\Big(-\frac{1 - 2K_x\sigma^2}{2}\Big(\frac{\delta}{\sigma}\Big)^2\Big).$$

Consequently, under the general assumption (5) there exists a positive constant A such that the sequence $\big((F_{n,\sigma_n} f)(x)\big)$ can be estimated by

$$(F_{n,\sigma_n} f)(x) = o\Big(\exp\Big(-A\,\frac{\delta_n^2}{\sigma_n^2}\Big)\Big)$$

as $n \to \infty$.

Remark 2. Note that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies condition (2) if and only if condition (8) is valid. The elementary inequality $(t - x)^2 \le 2(t^2 + x^2)$ implies that

$$M_x e^{K_x(t-x)^2} \le M e^{Kt^2} \qquad (t, x \in \mathbb{R})$$

with constants $M = M_x e^{2Kx^2}$ and $K = 2K_x$.

Gawronski and Stadtmüller [5, Eq. (0.7)] also considered a truncated version of F_{n,σ_n}, namely

$$\big(F^*_{n,\sigma_n,\delta_n} f\big)(x) = \frac{1}{\sqrt{2\pi}\, n\sigma_n} \sum_{\substack{\nu \\ |\nu/n - x| \le \delta_n}} f\Big(\frac{\nu}{n}\Big) \exp\Big(-\frac{1}{2\sigma_n^2}\Big(\frac{\nu}{n} - x\Big)^2\Big) \tag{9}$$

for certain values $\delta_n > 0$. Note that in [5, Eq. (0.7)] $c_n = n\delta_n$. A direct consequence of Proposition 1 is the following

Corollary 2. *Fix $x \in \mathbb{R}$ and let $\delta > 0$. If f satisfies condition (8), then, for each positive $\sigma < 1/\sqrt{2K_x}$, there holds the estimate*

$$|(F_{n,\sigma} f)(x) - (F^*_{n,\sigma,\delta} f)(x)| \le \sqrt{\frac{2}{\pi}} \cdot \frac{M_x\sigma/\delta}{1 - 2K_x\sigma^2} \exp\Big(-\frac{1 - 2K_x\sigma^2}{2}\Big(\frac{\delta}{\sigma}\Big)^2\Big).$$

Consequently, under the general assumption (5) both sequences of operators (F_{n,σ_n}) and $\big(F^*_{n,\sigma_n,\delta_n}\big)$ are asymptotically equivalent if $\sigma_n = o(\delta_n)$ as $n \to \infty$. This means that under these conditions there is a positive constant A (depending on f and x) such that

$$(F_{n,\sigma_n} f)(x) - \big(F^*_{n,\sigma_n,\delta_n} f\big)(x) = o\Big(\frac{\sigma_n}{\delta_n} \exp\Big(-A\frac{\delta_n^2}{\sigma_n^2}\Big)\Big)$$

as $n \to \infty$. For the derivatives of all orders a similar estimate is valid.

A direct consequence is the fact that, under the condition

$$\log\frac{\delta_n}{\sigma_n} + A\frac{\delta_n^2}{\sigma_n^2} - 2q|\log\sigma_n| \to \infty \qquad (n \to \infty),$$

the asymptotic expansions (6) and (7) are valid also for the sequences $\big(F^*_{n,\sigma_n,\delta_n} f\big)_{n\in\mathbb{N}}$ and $\big(\big(F^*_{n,\sigma_n,\delta_n} f\big)^{(r)}\big)_{n\in\mathbb{N}}$, respectively, of the truncated version (9) of the Favard operators.

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