CONSTRUCTIVE THEORY OF FUNCTIONS, Sozopol 2010:
In memory of Borislav Bojanov
(G. Nikolov and R. Uluchev, Eds.), pp. 9-20
Prof. Marin Drinov Academic Publishing House, Sofia, 2012

An Inequality of Duffin-Schaeffer Type for Hermite Polynomials*

Alexander Alexandrov and Geno Nikolov

Let H_n be the n-th Hermite polynomial and a_m be the rightmost zero of H_m, $m \in \mathbb{N}$. Let π_n^r be the set of all algebraic polynomials of degree not exceeding n and having only real coefficients. We prove that if $f \in \pi_n^r$ satisfies $|f| \le |H_n|$ at the zeros of H_{n+1}, then for $k = 1, \dots, n$,

$$|f^{(k)}(x+iy)| \le |H_n^{(k)}(a_{n+1}+iy)| \quad \text{for every} \quad (x,y) \in [-a_{n+1}, a_{n+1}] \times \mathbb{R},$$

and the equality occurs if and only if $f = \pm H_n$.

This result may be viewed as an analog of the famous extension of the classical inequality of Markov, found by Duffin and Schaeffer.

Keywords and Phrases: Hermite polynomials, Jensen inequalities, Markov inequality, Duffin and Schaeffer type inequalities.

1. Introduction

We begin with a list of notations that we shall use throughout the paper. By π_n we denote the set of all algebraic polynomials of degree not exceeding n. The subset of π_n of polynomials with real coefficients will be denoted by π_n^r. The notation $\|\cdot\|$ stands for the uniform norm in $[-1,1]$, i.e.,

$$\|f\| := \sup\{|f(x)| \, : \, x \in [-1,1]\}.$$

As usual, the n-th Chebyshev polynomial of the first kind is denoted by $T_n(x)$, where, for $x \in [-1,1]$, $T_n(x) := \cos n \arccos x$. The zeros and the points of local extrema of $T_n(x)$ are denoted by $\{\xi_\nu\}_{\nu=1}^n$ and $\{\eta_\nu\}_{\nu=0}^n$, respectively. We recall that $\xi_\nu := \cos\frac{(2\nu-1)\pi}{n}$ and $\eta_\nu := \cos\frac{\nu\pi}{n}$.

One of the most important polynomial inequalities is the Markov inequality.

*The research on this paper was supported by the National Science Fund of the Bulgarian Ministry of Education, Youth and Science under Grant DDVU-02/30.

Theorem A (A. A. Markov, V. A. Markov). *If $f \in \pi_n$ and $\|f\| \le 1$, then for $k = 1, \dots, n$,*

$$\|f^{(k)}\| \le \|T_n^{(k)}\| \ (= T_n^{(k)}(1)).$$

The equality occurs if and only if $f = \gamma T_n$, where $\gamma \in \mathbb{C}$ and $|\gamma| = 1$.

Remark 1. *The numerical value of $\|T_n^{(k)}\|$ is*

$$\|T_n^{(k)}\| = T_n^{(k)}(1) = \frac{n^2(n^2-1)\cdots(n^2-(k-1)^2)}{(2k-1)!!}.$$

The case $k = 1$ of Theorem A was proved in 1889 by A. A. Markov [3], and the general case was proved in 1892 by his younger brother V. A. Markov [4].

In 1941 Duffin and Schaeffer [5] found the following beautiful refinement of Theorem A.

Theorem B (R. J. Duffin and A. C. Schaeffer). *If $f \in \pi_n$ and*

$$|f(\eta_\nu)| \le 1 \qquad \text{for } \nu = 0, \dots, n, \tag{1}$$

then

$$\|f^{(k)}\| \le \|T_n^{(k)}\| \qquad \text{for } k = 1, \dots, n.$$

The equality occurs if and only if $f = \gamma T_n$, where $\gamma \in \mathbb{C}$ and $|\gamma| = 1$.

For polynomials with real coefficients Duffin and Schaeffer proved even more:

Theorem C (R. J. Duffin and A. C. Schaeffer). *If $f \in \pi_n^r$ satisfies (1), then for $k = 1, \dots, n$,*

$$|f^{(k)}(x+iy)| \le |T_n^{(k)}(1+iy)| \qquad \text{for every } (x,y) \in [-1,1] \times \mathbb{R},$$

and the equality holds if and only if $f = \pm T_n$.

The basic ingredients of the proof of Theorem C are the following two propositions:

Statement 1. *If $f \in \pi_n$ and $|f(\eta_\nu)| \le |T_n(\eta_\nu)|$, $\nu = 0, \dots, n$, then*

$$|f'(\xi_\nu)| \le |T_n'(\xi_\nu)| \qquad \text{for } \nu = 1, \dots, n.$$

Statement 2. *The Chebyshev polynomial $T_n(x)$ possesses the property*

$$|T_n(x+iy)| \le |T_n(1+iy)| \qquad \text{for every } (x,y) \in [-1,1] \times \mathbb{R}.$$

2. The Main Result and Sketch of the Proof

Theorem C may be interpreted as a comparison theorem: the inequalities $|f(\eta_\nu)| \le |T_n(\eta_\nu)|$, $\nu = 0, \ldots, n$, imply inequalities between $|f^{(k)}|$ and $|T_n^{(k)}|$ in a strip in the complex plane. Here, we prove an analogue of Theorem C, where the role of the extremal polynomial T_n is played by H_n, the n-th Hermite polynomial, and the check points $\{\eta_\nu\}_{\nu=0}^n$ are replaced by the zeros of H_{n+1}, the $(n+1)$-st Hermite polynomial. Let a_{n+1} be the largest zero of H_{n+1}. Our main result is

Theorem 1. *If $f \in \pi_n^r$ and $|f| \le |H_n|$ at the zeros of H_{n+1}, then for $k = 1, \ldots, n$,*

$$|f^{(k)}(x+iy)| \le |H_n^{(k)}(a_{n+1}+iy)| \quad \textit{for every} \quad (x,y) \in [-a_{n+1}, a_{n+1}] \times \mathbb{R}. \tag{2}$$

The equality in (2) *occurs if and only if $f = \pm H_n$.*

To prove Theorem 1, we establish the following analogues of Statement 1 and Statement 2:

Statement 1′. *If $f \in \pi_n$ and $|f| \le |H_n|$ at the zeros of H_{n+1}, then $|f'| \le |H_n'|$ at the zeros of H_n. Moreover, the equality $|f'| = |H_n'|$ holds at either zero of H_n if and only if $f = \gamma H_n$, where $\gamma \in \mathbb{C}$ and $|\gamma| = 1$.*

Statement 2′. *The Hermite polynomial $H_n(x)$ possesses the property*

$$|H_n(x+iy)| \le |H_n(a_{n+1}+iy)| \quad \textit{for every} \quad (x,y) \in [-a_{n+1}, a_{n+1}] \times \mathbb{R}.$$

For the proof of Statement 1′ we use an observation of V. A. Markov, which we formulate below. Let $Q(x)$ be an algebraic polynomial of degree n with distinct real zeros $x_1 < x_2 < \cdots < x_n$. Let $\{t_\nu\}_{\nu=0}^n$ separate $\{x_\nu\}_{\nu=1}^n$, i.e.,

$$t_0 \le x_1 \le t_1 \le x_2 \le \cdots \le x_n \le t_n.$$

Set

$$w(u) = \prod_{j=0}^{n} (u - t_j), \; w_\nu(u) = \frac{w(u)}{u - t_\nu}, \; \nu = 0, \ldots, n.$$

With the above notation, the V. A. Markov result is

Theorem D (V. A. Markov). *Let $f \in \pi_n$ and*

$$|f(t_\nu)| \le |Q(t_\nu)| \quad \textit{for} \quad \nu = 0, \ldots, n.$$

Then there exist a set $I_n(w)$ formed by n non-overlapping intervals on $\mathbb{R}$, such that

$$|f'(x)| \le |Q'(x)| \quad \textit{for every} \quad x \in I_n(w). \tag{3}$$

A point $x \in \mathbb{R}$ *belongs to* $I_n(w)$ *if and only if*

$$w_0'(x)w_n'(x) \geq 0. \tag{4}$$

Moreover, if the inequality (4) *is strict, then also inequality* (3) *is strict unless* $f = \gamma Q$*, where* $\gamma \in \mathbb{C}$*,* $|\gamma| = 1$*.*

For the proof of Statement 2′ we exploit the following expansion formula, due to Jensen [2]:

$$|f(x+iy)|^2 = \sum_{k=0}^{n} L_k(f;x)y^{2k}, \quad (x,y) \in \mathbb{R}^2, \tag{5}$$

valid for every $f \in \pi_n$ which has only real zeros. Here, $L_0(f;x) = f^2(x)$, and

$$L_k(f;x) = \sum_{j=0}^{2k} (-1)^{k-j} \frac{f^{(j)}(x)}{j!} \frac{f^{(2k-j)}(x)}{(2k-j)!}, \quad k = 1,2,\dots,n. \tag{6}$$

Formula (5) extends to the functions from the Laguerre-Pólya class, in which case the summation index k ranges from 0 to ∞. We recall that the Laguerre-Pólya class consists of the entire functions that are uniform limits on compact subsets of $\mathbb{C}$ of polynomials with only real zeros.

Let us also mention that if $f(x)$ is even or odd function, then $L_k(f;x)$ are even functions, $k = 0,1,\dots,n$.

If $f(z) = c(z-x_1)(z-x_2)\cdots(z-x_n)$ with $x_1,\dots,x_n$ all real, then the following alternative representation of $L_k(f;x)$ holds true:

$$L_k(f;x) = f^2(x) \cdot \sum \frac{1}{(x-x_{i_1})^2 \cdots (x-x_{i_k})^2}, \tag{7}$$

where the sum is extended over all k-combinations of $\{1,\dots,n\}$. It is seen from (7) that $L_k(f;x) \geq 0$ for every $x \in \mathbb{R}$. Moreover, if f has only simple zeros, then $L_k(f;x)$ is strictly positive on the real line.

An important part of the proof of Statement 2′ is

Theorem 2. *For* $k = 1,\dots,n$*, the function* $L_k(H_n;\cdot)$ *is monotone decreasing in* $(-\infty,0]$ *and monotone increasing in* $[0,\infty)$*.*

3. Proofs

Proof of Statement 1′. We apply Theorem D with $Q = H_n$ and $w = cH_{n+1}$. If $H_n(x) = 0$, then from the identity (see [8, eqn. 5.5.10])

$$H_{n+1}'(x) = (2n+2)H_n(x) \tag{8}$$

we obtain

$$w_0'(x)\cdot w_n'(x)=c^2\Big(\frac{H_{n+1}(u)}{u-a_{n+1}}\Big)'\Big|_{u=x}\cdot\Big(\frac{H_{n+1}(u)}{u+a_{n+1}}\Big)'\Big|_{u=x}=c^2\,\frac{H_{n+1}^2(x)}{(x^2-a_{n+1}^2)^2}>0.$$

So, according to Theorem D, $|f'(x)|\le|H_n'(x)|$, and the inequality is strict unless $f=\gamma H_n$ with $\gamma\in\mathbb{C}$, $|\gamma|=1$. □

Proof of Statement 2′. First, we recall the known fact that if a_{n+1} is the largest zero of the Hermite polynomial $H_{n+1}(x)$, then

$$\max\{|H_n(x)|\,:\,x\in[-a_{n+1},a_{n+1}]\}=|H_n(\pm a_{n+1})|. \tag{9}$$

For the reader convenience, we provide a proof of (9). To this end, we make use of (8) and the second order differential equation satisfied by $y=H_{n+1}$,

$$y''-2xy'+(2n+2)y=0, \tag{10}$$

(see e.g., [8, eqn. 5.5.2]). By (8), the auxiliary function

$$f(x)=\frac{1}{(2n+2)^2}\big[y'(x)^2+(2n+2)y(x)^2\big]$$

satisfies $H_n^2(x)\le f(x)$ for every $x\in\mathbb{R}$, and equality holds if and only if $H_{n+1}(x)=0$. On using (10), we see that f' is representable in the form

$$f'(x)=\frac{1}{(n+1)^2}\,x\big[y'(x)\big]^2,$$

whence $f(x)$ is monotone decreasing on $(-\infty,0)$ and monotone increasing on $(0,\infty)$. Therefore, for every $x\in[-a_{n+1},a_{n+1}]$ we have

$$H_n^2(x)\le f(x)\le f(\pm a_{n+1})=H_n^2(\pm a_{n+1}),$$

which proves (9).

Applying (5) with $f=H_n$, we obtain for every $(x,y)\in[-a_{n+1},a_{n+1}]\times\mathbb{R}$

$$|H_n(a_{n+1}+iy)|^2-|H_n(x+iy)|^2=\sum_{k=0}^{n}[L_k(H_n;a_{n+1})-L_k(H_n;x)]y^{2k}\ge 0,$$

since all the quantities $L_k(H_n;a_{n+1})-L_k(H_n;x)$, $0\le k\le n$, are non-negative. For $1\le k\le n$ this follows from Theorem 2, while for $k=0$ the claim follows from (9), since $L_0(H_n;a_{n+1})-L_0(H_n;x)=H_n^2(a_{n+1})-H_n^2(x)>0$. The proof of Statement 2′ is complete. □

Proof of Theorem 2. We use induction with respect to k. For the case $k=1$, from $L_1(f;x)=[f'(x)]^2-f(x)f''(x)$ we obtain

$$L_1'(f;x)=f'(x)f''(x)-f(x)f'''(x)=[f''(x)]^2\Big(\frac{f(x)}{f''(x)}\Big)'.$$

Hence, it suffices to show that for $y = H_n$

$$\operatorname{sign}\Big(\frac{y(x)}{y''(x)}\Big)' = \operatorname{sign} x. \tag{11}$$

From $y'' - 2xy' + 2ny = 0$ (this is (10) with $n+1$ replaced by n) we obtain

$$2n\,\frac{y}{y''} = 2x\,\frac{y'}{y''} - 1 \qquad \Rightarrow \qquad \Big(\frac{y}{y''}\Big)' = \frac{1}{n}\Big(x\frac{y'}{y''}\Big)',$$

therefore (11) is equivalent to

$$\operatorname{sign}\Big(\frac{xy'(x)}{y''(x)}\Big)' = \operatorname{sign} x. \tag{12}$$

Let $\{x_\nu\}_{\nu=1}^{n-1}$ be the zeros of $y'(x)$. They are located symmetrically with respect to the origin, i.e., $\{x_\nu\}_{\nu=1}^{n-1} \equiv \{-x_\nu\}_{\nu=1}^{n-1}$. Therefore,

$$\frac{y''(x)}{xy'(x)} = \frac{1}{2x}\sum_{\nu=1}^{n-1}\Big(\frac{1}{x-x_\nu} + \frac{1}{x+x_\nu}\Big) = \sum_{\nu=1}^{n-1}\frac{1}{x^2 - x_\nu^2}.$$

Differentiation of the last expression yields

$$\operatorname{sign}\Big(\frac{xy'(x)}{y''(x)}\Big)' = \operatorname{sign}\Big[\Big(\sum_{\nu=1}^{n-1}\frac{1}{x^2-x_\nu^2}\Big)^{-2}\cdot\sum_{\nu=1}^{n-1}\frac{2x}{(x^2-x_\nu^2)^2}\Big] = \operatorname{sign} x,$$

which proves (12). Thus, the case $k = 1$ is settled.

Now assume that the assertion is verified for $L_k(H_n; x)$. The induction step $k \mapsto k+1$ is performed with the help of the following

Identity 1.

$$2n(k+1)L'_{k+1}(H_n; x) = 2(k+1)xL_{k+1}(H'_n; x) + L'_k(H'_n; x).$$

Both summands on the right-hand side of Identity 1 have the sign of x. Indeed, since $H'_n = 2nH_{n-1}$ has only real and simple zeros, we have $L_{k+1}(H'_n; x) > 0$ and $\operatorname{sign} L'_k(H'_n; x) = \operatorname{sign} x$, by the induction hypothesis. It follows from Identity 1 that $\operatorname{sign} L'_{k+1}(H_n; x) = \operatorname{sign} x$, which accomplishes the induction step from k to $k+1$. Theorem 2 is proved. □

The proof of Identity 1 goes through a number of steps. We begin with a technical lemma.

Lemma 1. *The following identities hold true:*

(i) $$\sum_{j=0}^{2k-1}(-1)^{k-j}\frac{f^{(j)}(x)}{j!}\,\frac{f^{(2k-1-j)}(x)}{(2k-1-j)!} = 0;$$

(ii) $$\sum_{j=0}^{2k}(-1)^{k-j}(k-j)\frac{f^{(j)}(x)}{j!}\,\frac{f^{(2k-j)}(x)}{(2k-j)!}=0;$$

(iii) $$\begin{aligned}\sum_{j=0}^{2k}(-1)^{k-j}\frac{f^{(j+1)}(x)}{j!}\,\frac{f^{(2k-j)}(x)}{(2k-j)!}&=\sum_{j=0}^{2k}(-1)^{k-j}\frac{f^{(j)}(x)}{j!}\,\frac{f^{(2k+1-j)}(x)}{(2k-j)!}\\&=\frac{1}{2}L_k'(f;x);\end{aligned}$$

(iv) $$\begin{aligned}\sum_{j=0}^{2k}(-1)^{k-j}\frac{f^{(j+2)}(x)}{j!}\,\frac{f^{(2k-j)}(x)}{(2k-j)!}&=\sum_{j=0}^{2k}(-1)^{k-j}\frac{f^{(j)}(x)}{j!}\,\frac{f^{(2k+2-j)}(x)}{(2k-j)!}\\&=\frac{1}{2}L_k''(f;x)-L_k(f';x).\end{aligned}$$

Proof of Lemma 1. Changing the summation index j to $2k-1-j$ in the sum on the left-hand side of (i), we obtain the same sum but with opposite sign, hence the sum is equal to zero. Identity (ii) is verified in the same way by replacement of j by $2k-j$.

We proceed with the proof of (iii). Since

$$L_k'(f;x)=\sum_{j=0}^{2k}\frac{(-1)^{k-j}}{j!(2k-j)!}[f^{(j+1)}(x)f^{(2k-j)}(x)+f^{(j)}(x)f^{(2k+1-j)}(x)],$$

it suffices to prove only the first identity in (iii). On using (i), we obtain

$$\begin{aligned}\sum_{j=0}^{2k}(-1)^{k-j}\frac{f^{(j+1)}(x)}{j!}\,\frac{f^{(2k-j)}(x)}{(2k-j)!}&=\sum_{j=0}^{2k+1}(-1)^{k+1-j}j\,\frac{f^{(j)}(x)}{j!}\,\frac{f^{(2k+1-j)}(x)}{(2k+1-j)!}\\&=\sum_{j=0}^{2k+1}(-1)^{k-j}(2k+1-j)\frac{f^{(j)}(x)}{j!}\,\frac{f^{(2k+1-j)}(x)}{(2k+1-j)!}\\&=\sum_{j=0}^{2k}(-1)^{k-j}\frac{f^{(j)}(x)}{j!}\,\frac{f^{(2k+1-j)}(x)}{(2k-j)!}.\end{aligned}$$

Now we prove (iv). We have

$$L_k''(f;x)=\sum_{j=0}^{2k}\frac{(-1)^{k-j}}{j!\,(2k-j)!}[f^{(j+2)}(x)f^{(2k-j)}(x)+f^{(j)}(x)f^{(2k+2-j)}(x)]+2L_k(f';x).$$

On using (ii), we prove the first identity in (iv) as follows:

$$\sum_{j=0}^{2k}(-1)^{k-j}\frac{f^{(j+2)}(x)}{j!}\frac{f^{(2k-j)}(x)}{(2k-j)!} = \sum_{j=0}^{2k+2}\frac{(-1)^{k+2-j}j(j-1)}{j!\,(2k+2-j)!}f^{(j)}(x)f^{(2k+2-j)}(x)$$
$$=\sum_{j=0}^{2k+2}\frac{(-1)^{k-j}\left[(2k+1-j)(2k+2-j)-(2k+1)(2k+2-2j)\right]}{j!(2k+2-j)!}f^{(j)}(x)f^{(2k+2-j)}(x)$$
$$=\sum_{j=0}^{2k}(-1)^{k-j}\frac{f^{(j)}(x)}{j!}\frac{f^{(2k+2-j)}(x)}{(2k-j)!}.$$

Now the second identity in (iv) easily holds. Lemma 1 is proved. □

We shall need also another identity, which is true for arbitrary smooth function f:

Identity 2.

$$L_k''(f;x) = 4L_k(f';x) - (2k+1)(2k+2)L_{k+1}(f;x).$$

Proof of Identity 2. By appropriate changes of the summation indices we obtain

$$4L_k(f';x) - L_k''(f;x)$$
$$=\sum_{j=0}^{2k}\frac{(-1)^{k-j}}{j!(2k-j)!}[2f^{(j+1)}(x)f^{(2k+1-j)}(x)-f^{(j+2)}(x)f^{(2k-j)}(x)-f^{(j)}(x)f^{(2k+2-j)}(x)]$$
$$=2\sum_{j=0}^{2k+2}\frac{(-1)^{k+1-j}j(2k+2-j)}{j!\,(2k+2-j)!}f^{(j)}(x)f^{(2k+2-j)}(x)$$
$$-\sum_{j=0}^{2k+2}\frac{(-1)^{k+2-j}j(j-1)}{j!\,(2k+2-j)!}f^{(j)}(x)f^{(2k+2-j)}(x)$$
$$-\sum_{j=0}^{2k+2}\frac{(-1)^{k-j}(2k+1-j)(2k+2-j)}{j!\,(2k+2-j)!}f^{(j)}(x)f^{(2k+2-j)}(x)$$
$$=\sum_{j=0}^{2k+2}\frac{(-1)^{k+1-j}j(2k+2-j+j-1)}{j!\,(2k+2-j)!}f^{(j)}(x)f^{(2k+2-j)}(x)$$
$$+\sum_{j=0}^{2k+2}\frac{(-1)^{k+1-j}(2k+2-j)(j+2k+1-j)}{j!\,(2k+2-j)!}f^{(j)}(x)f^{(2k+2-j)}(x)$$
$$=(2k+1)(2k+2)\sum_{j=0}^{2k+2}\frac{(-1)^{k+1-j}}{j!\,(2k+2-j)!}f^{(j)}(x)f^{(2k+2-j)}(x)$$
$$=(2k+1)(2k+2)L_{k+1}(f;x).$$

The proof of Identity 2 is complete. □

Identity 3. *With $y = H_n$, there holds*

$$L_k(y''; x) - xL'_k(y'; x) + 2(n+k)L_k(y'; x) - 2n(k+1)(2k+1)L_{k+1}(y; x) = 0.$$

Proof of Identity 3. Differentiating j times the identity $y'' - 2xy' + 2ny = 0$, we obtain $y^{(j+2)} - 2xy^{(j+1)} + 2(n-j)y^{(j)} = 0$. Then, using Lemma 1 (iii), we get

$$\begin{aligned} 0 &= \sum_{j=0}^{2k} (-1)^{k-j} \Big[y^{(j+2)} - 2xy^{(j+1)} + 2(n-j)y^{(j)} \Big] \frac{y^{(2k+2-j)}}{j!\,(2k-j)!} \Big] \\ &= L_k(y''; x) - xL'_k(y'; x) + 2\sum_{j=0}^{2k} \frac{(-1)^{k-j}}{j!\,(2k-j)!}(n-j)y^{(j)}y^{(2k+2-j)}. \end{aligned} \tag{13}$$

According to Lemma 1 (iv),

$$n\sum_{j=0}^{2k} \frac{(-1)^{k-j}}{j!\,(2k-j)!}\, y^{(j)}y^{(2k+2-j)} = n[\frac{1}{2}L''_k(y; x) - L_k(y'; x)].$$

Also, from Lemma 1 (ii) we obtain

$$\begin{aligned} \sum_{j=0}^{2k} \frac{(-1)^{k-j}}{j!\,(2k-j)!}\, jy^{(j)}y^{(2k+2-j)} &= \sum_{j=1}^{2k} \frac{(-1)^{k-j}}{(j-1)!\,(2k-j)!}\, y^{(j)}y^{(2k+2-j)} \\ &= \sum_{j=0}^{2k-1} \frac{(-1)^{k-1-j}}{j!\,(2k-1-j)!}\, y^{(j+1)}y^{(2k+1-j)} \\ &= -\sum_{j=0}^{2k} \frac{(-1)^{k-j}}{j!\,(2k-j)!}\, [k + (k-j)]y^{(j+1)}y^{(2k+1-j)} \\ &= -kL_k(y'; x). \end{aligned}$$

Replacement in the last sum in (13) yields

$$\sum_{j=0}^{2k} \frac{(-1)^{k-j}}{j!\,(2k-j)!}\, (n-j)y^{(j)}y^{(2k+2-j)} = n[\frac{1}{2}\, L''_k(y; x) - L_k(y'; x)] + kL_k(y'; x),$$

whence

$$0 = L_k(y''; x) - xL'_k(y'; x) + nL''_k(y; x) + 2(k-n)L_k(y'; x).$$

Finally, replacing $L''_k(y; x)$ by $4L_k(y'; x) - 2(k+1)(2k+1)L_{k+1}(y; x)$, in view of Identity 2, we obtain Identity 3. □

Identity 4. *With $y = H_n$ there holds*

$$L'_{k-1}(y''; x) - 4xL_{k-1}(y''; x) + 2(n-1)L'_{k-1}(y'; x) = 0.$$

Proof of Identity 4. $(j+1)$-fold differentiation of identity $y'' - 2xy' + 2ny = 0$ yields $y^{(j+3)} - 2xy^{(j+2)} + 2(n-j-1)y^{(j+1)} = 0$. Then, using Lemma 1 (iii), we obtain

$$\begin{aligned} 0 &= \sum_{j=0}^{2k-2} (-1)^{k-1-j} [y^{(j+3)} - 2xy^{(j+2)} + 2(n-j-1)y^{(j+1)}] \frac{y^{(2k-j)}}{j!\,(2k-2-j)!} \\ &= \frac{1}{2} L'_{k-1}(y''; x) - 2xL_{k-1}(y''; x) + (n-1)L'_{k-1}(y'; x) \\ &\quad - 2 \sum_{j=0}^{2k-2} (-1)^{k-1-j} j \frac{y^{(j+1)} y^{(2k-j)}}{j!\,(2k-2-j)!}. \end{aligned}$$

The last sum is shown to vanish with the help of Lemma 1 (i):

$$\begin{aligned} \sum_{j=0}^{2k-2} (-1)^{k-1-j} j \frac{y^{(j+1)} y^{(2k-j)}}{j!\,(2k-2-j)!} &= \sum_{j=1}^{2k-2} (-1)^{k-1-j} \frac{y^{(j+1)} y^{(2k-j)}}{(j-1)!\,(2k-2-j)!} \\ &= - \sum_{j=0}^{2k-3} (-1)^{k-1-j} \frac{y^{(j+2)} y^{(2k-1-j)}}{j!\,(2k-3-j)!} = 0. \end{aligned}$$

Identity 4 is proved. □

Proof of Identity 1. We differentiate Identity 3 to obtain

$$L'_k(y''; x) - L'_k(y'; x) - xL''_k(y'; x) + 2(n+k)L'_k(y'; x) - 2n(k+1)(2k+1)L'_{k+1}(y; x) = 0.$$

Identity 4 with $k+1$ instead of k reads as

$$L'_k(y''; x) - 4xL_k(y''; x) + 2(n-1)L'_k(y'; x) = 0.$$

Subtracting pairwise these equalities, we obtain

$$(2k+1)L'_k(y'; x) + 4xL_k(y''; x) - xL''_k(y'; x) - 2n(k+1)(2k+1)L'_{k+1}(y; x) = 0.$$

According to Identity 2, we have

$$L''_k(y'; x) = 4L_k(y''; x) - 2(k+1)(2k+1)L_{k+1}(y'; x).$$

and the replacement implies

$$(2k+1)L'_k(y'; x) + 2x(k+1)(2k+1)L_{k+1}(y'; x) - 2n(k+1)(2k+1)L'_{k+1}(y; x) = 0.$$

The proof of Identity 1 is complete. □

The proof of Theorem 1 is based on the following result from [5]:

Theorem E (R. J. Duffin and A. C. Schaeffer). *Let $g(z)$ be an algebraic polynomial of degree n with n distinct real zeros smaller than real number b and let $|g(x+iy) \le |g(b+iy)|$, $(x,y) \in [a,b] \times \mathbb{R}$. Let $f \in \pi_n^r$ and $|f'(x)| \le |g'(x)|$ at the zeros of g. Then we have that for $k = 1, \dots, n$,*

$$|f^{(k)}(x+iy)| \le |g^{(k)}(b+iy)|, \qquad (x,y) \in [a,b] \times \mathbb{R}.$$

Proof of Theorem 1. The proof is simply application of Theorem E with $g = H_n$ and $a = -a_{n+1}$, $b = a_{n+1}$. Notice that the assumptions of Theorem E are fulfilled due to Statements $1'$ and $2'$ which were already proven. □

Remark 2. Arguing as in [5], one can prove that under the assumptions of Theorem 1 except for the condition $f \in \pi_n^r$ replaced by $f \in \pi_n$, there holds

$$\|f^{(k)}\|_{C[-a_{n+1},a_{n+1}]} \le \|H_n^{(k)}\|_{C[-a_{n+1},a_{n+1}]} \tag{14}$$

for $k = 1, \dots, n$. Moreover, the equality sign occurs in (14) if and only if $f = \gamma H_n$, where $\gamma \in \mathbb{C}$ and $|\gamma| = 1$.

Remark 3. Under the assumptions $f \in \pi_n$ and $|f| \le |H_n|$ at the zeros of H_{n+1}, the following counterpart to (14) was proved in [7]:

$$\int_{-\infty}^{\infty} e^{-x^2} |f^{(k)}(x)|^2 \, dx \le \int_{-\infty}^{\infty} e^{-x^2} |H_n^{(k)}(x)|^2 \, dx \tag{15}$$

for $k = 1, \dots, n$. Equality in (15) holds if and only if $f = \gamma H_n$, where $\gamma \in \mathbb{C}$ and $|\gamma| = 1$.

Remark 4. Another counterpart to Theorem 1 is provided by a remarkable result of Bernstein [1]. Namely, under the assumptions of Theorem 1, outside the disk $D = \{z \in \mathbb{C} : |z| \le a_{n+1}\}$ we have pointwise inequalities between the derivatives of f and H_n, i.e., for $k = 1, \dots, n$,

$$|f^{(k)}(z)| \le |H_n^{(k)}(z)| \qquad \text{for every } z = x+iy, \quad x^2+y^2 \ge a_{n+1}^2.$$

For an extension of Theorem C with T_n replaced by ultraspherical polynomial $P_n^{(\lambda)}$, $\lambda > 0$, the reader is refereed to [6].

References

[1] S. N. Bernstein, Sur une propriété des polynômes, *Comm. Soc. Math. Kharkov Ser. 2* **14** (1930), no. 1–2, 1–6.

[2] J. L. W. V. Jensen, Recherches sur la théorie des équations, *Acta Math.* **36** (1913), 181–195.

[3] A. A. Markov, On a question of D. I. Mendeleev, *Zap. Petersburg Akad. Nauk* **62** (1889), 1–24 [in Russian].

[4] V. A. Markov, On functions least deviated from zero in a given interval, 1892, St. Petersburg [in Russian]; German translation: *Math. Ann.* **77** (1916), 213–258.

[5] R. J. Duffin and A. C. Schaeffer, A refinement of an inequality of brothers Markov, *Trans. Amer. Math. Soc.* **50** (1941), 517–528.

[6] G. Nikolov, An extension of an inequality of Duffin and Schaeffer, *Constr. Approx.* **21** (2005), 181–191.

[7] G. Nikolov, An extremal property of Hermite polynomials, *J. Math. Anal. Appl.* **290** (2004), 405–413.

[8] G. Szegő, “Orthogonal Polynomials”, Amer. Math. Soc. Colloq. Publ., v. 23, Providence, RI, 1975.

Alexander Alexandrov

Department of Mathematics
University of Sofia
5 James Bourchier Blvd.
1164 Sofia
BULGARIA
E-mail: ava@fmi.uni-sofia.bg

Geno Nikolov

Department of Mathematics
Universlty of Sofia
5 James Bourchier Blvd.
1164 Sofia
BULGARIA
E-mail: geno@fmi.uni-sofia.bg