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# Locally Monotone Approximations of Real Functions on Graphs

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Locally monotone approximations appear naturally in signal processing where the input is typically passed via filter for some useful separation, e.g. noise from signal. Considering the local monotonicity as a concept of smoothness some filters particularly aim at extracting signal with prescribed local monotonicity. The LULU operators, well known in the multi-resolution analysis of sequences, are filters of this type. In abstract mathematical setting we consider the approximation of real functions defined on a connected graph by a set of locally monotone functions on the same domain. The LULU operators suitably extended to this general setting have shape preserving properties important for the processing of signals of arbitrary dimension. In addition to that, we prove that they produce locally monotone approximations which are *nearly optimal* in the sense that the error of the approximation in any $\ell_p$ norm, $p \in [1, \infty]$, is bounded by a constant multiple of the error of any other approximation by functions from the same set.

## 1. Introduction

The LULU operators are a kind of morphological filters extracting information of interest from signals. They were initially defined on sequences, that is one dimensional signals, [5], but later extended to multidimensional arrays, [1], so that they are applicable to images and video sequences as well. The LULU operators, although part of Mathematical Morphology, were developed to a large extend within their own theory which focuses on structure preserving properties like: consistent separation (e.g. noise from signal), total variation and shape preservation, and consistent hierarchical decomposition. It was shown in [2] that it is convenient to consider the LULU operators and respective theory in the setting of functions defined on a graph $\mathbb{G}$. In the processing of one-dimensional signals $\mathbb{G} = \mathbb{Z}$, in image analysis $\mathbb{G} = \mathbb{Z}^2$, and in video sequences

$\mathbb{G} = \mathbb{Z}^3$. Our basic assumption is that $\mathbb{G}$ is unoriented connected graph with countable number of vertices and that the degree of the vertices is finite and bounded. More precisely, there exists a constant $\alpha \in \mathbb{N}$ such that $d(v) \leq \alpha$ for all $v \in \mathbb{G}$. For any $v \in \mathbb{G}$ we denote by $\mathcal{N}(v)$ the set of all vertices in $\mathbb{G}$ which share an edge with $v$. For example, on $\mathbb{Z}$ usually we have $\mathcal{N}(n) = \{n-1, n+1\}$, while on $\mathbb{Z}^2$ typically $\mathcal{N}((n,m)) = \{(n \pm 1, m),\ (n, m \pm 1)\}$ under the so-called 4-connectivity or $\mathcal{N}((n,m)) = \{(n \pm 1, m),\ (n, m \pm 1),\ (n \pm 1, m \pm 1)\}$ under the so-called 8 connectivity. Obviously, $\operatorname{card}(\mathcal{N}(v)) = d(v) \leq \alpha$. Further we should note that the graph connectivity defines a morphological connection on the set of vertices. The concept of connection is introduced axiomatically in Mathematical Morphology and plays a central role in the procedure used in [1] for LULU operators on multidimensional arrays. For a graph $\mathbb{G}$, the set $\mathcal{G}$ of all connected subgraphs, is a morphological connection as it satisfies the respective axioms, see e.g. [7]. Further, it satisfies the additional conditions set in [1] so that the theory developed there is applicable. To avoid complicated notations we denote by $\mathbb{G}$ the set of vertices of the graph and we consider on it the connectivity as determined by $\mathcal{G}$.

In practical situations, filters are typically applied to functions on a finite domain, e.g. the pixels of an image in image analysis. In order to keep the discussion as general as possible, we consider the domain infinite, but countable as already mentioned. However, we assume that sum of the moduli of the functional values is finite. More precisely, we consider the set

$$\mathcal{A}(\mathbb{G}) = \Big\{ f : \mathbb{G} \to \mathbb{R} : \|f\|_1 := \sum_{v \in \mathbb{G}} |f(v)| < \infty \Big\}.$$

It is easy to see that $\mathcal{A}(\mathbb{G})$ is a linear space with respect to the usual point-wise defined operations and that $\|\cdot\|_1$ defines a norm on this space. In fact this is exactly the space of the absolutely summable sequences $\ell_1$. In the sequel we keep the notation $\mathcal{A}(\mathbb{G})$ since the definitions of the LULU operators use the connectivity structure on $\mathbb{G}$ and involve neither the linear operations nor the norm. Often filters are defined by requiring proximity in some sense to the original input, e.g. see [8]. In contradistiction, the LULU operators and, in fact, the morphological filters in general, are focused on shape and do not use distance and proximity in their definition. Nevertheless, it turns out that the LULU operators provide in some sense "near best" approximations by functions of certain kind of local monotonicity. This result which is also the main contribution of this paper extends an earlier result in [4] for LULU operators on sequences.

In the next section we define the LULU operators in the setting of $\mathcal{A}(\mathbb{G})$ and consider their structure preserving properties. The theorems in that section combine results from [5] and [1] and are given here without proofs. Section 3 presents the main theorems and their proofs. Some concluding remarks are given at the end of the paper.

## 2. The LULU Operators

Let us denote by $\mathcal{C}_n(v)$ the set of connected subgraphs containing the vertex $v$ and $n$ other vertices, that is

$$\mathcal{C}_n(v) = \{C \in \mathcal{G} : v \in C,\ \mathrm{card}\,(C) = n+1\}.$$

Then for any $n \in \mathbb{N}$ the operators $L_n$, $U_n : \mathcal{A}(\mathbb{G}) \to \mathcal{A}(\mathbb{G})$ are defined as

$$L_n f(v) = \max_{C\in\mathcal{C}_n(v)} \min_{w\in C} f(w), \qquad U_n f(v) = \min_{C\in\mathcal{C}_n(v)} \max_{w\in C} f(w).$$

The smoothing effect of the LULU operators can be described as removing "peaks" and "pits" of sufficiently small support. This is made precise through the definitions below.

**Definition 1.** Let $C \in \mathcal{G}$. A vertex $v \notin C$ is called *adjacent* to $C$ if $C \cup \{v\} \in \mathcal{G}$. The set of all vertices adjacent to $C$ is denoted by $\mathrm{adj}\,(C)$, that is,

$$\mathrm{adj}\,(C) = \{v \notin C : C \cup \{v\} \in \mathcal{G}\}.$$

**Definition 2.** A set $C \in \mathcal{G}$ is called a *local maximum set* of $f \in \mathcal{A}(\mathbb{G})$ if

$$\sup_{w\in \mathrm{adj}\,(C)} f(w) < \inf_{v\in C} f(v).$$

Similarly $C$ is a *local minimum set* if

$$\inf_{w\in \mathrm{adj}\,(C)} f(w) > \sup_{v\in C} f(v).$$

**Definition 3.** We say that $f \in \mathcal{A}(\mathcal{G})$ is *locally $n$-monotone* if every local maximum or local minimum set of $f$ is of size $n+1$ or more. The set of all functions in $\mathcal{A}(\mathbb{G})$ which are $n$-monotone is denoted by $\mathcal{M}_n$.

The operator $L_n$ removes local maximum sets (peaks) of size $n$ or less while $U_n$ removes local minimum sets (pits) of size $n$ or less so that we have the following theorem.

**Theorem 1.** *For any $n \in \mathbb{N}$ and $f \in \mathcal{A}(\mathbb{G})$ we have that $L_nU_nf \in \mathcal{M}_n$ and $U_nL_nf \in \mathcal{M}_n$. Moreover, $f \in \mathcal{M}_n \iff (L_nf = f,\ U_nf = f)$.*

The following structural properties are considered within the LULU theory, see [1], [5].

**Consistent Separation:** A common requirement for a filter $P$, linear or nonlinear, is its idempotence, i.e. $P^2 = P$. For example, a morphological filter is by definition an increasing and idempotent operator. For linear operators the idempotence of $P$ implies the idempotence of the complementary operator

$id - P$, where $id$ denotes the identity operator. For nonlinear filters this implication generally does not hold so the idempotence of $id - P$, also called co-idempotence, can be considered as an essential measure of consistency. It is also equivalent to

$$P(id - P) = 0, \qquad (id - P)P = 0. \tag{1}$$

In the common interpretation of separation of $f$ into a signal $Pf$ and noise $(id - P)f$, the equalities (1) essentially mean that the extracted noise contains no signal and that the extracted signal contains no noise. In this sense the separation is consistent.

**Total Variation Preservation:** The total variation $TV(\cdot)$ is well recognized as a measure for the information in a signal. For a function $f \in \mathcal{A}(\mathbb{G})$ it is defined as

$$TV(f) = \frac{1}{2} \sum_{v \in \mathbb{G}} \sum_{w \in \mathcal{N}(v)} |f(w) - f(v)|.$$

It is easy to see that it is a semi norm on $\mathcal{A}(\mathbb{G})$. Therefore any separation may only increase the total variation. More precisely, for any operator $P : \mathcal{A}(\mathbb{G}) \to \mathcal{A}(\mathbb{G})$ we have

$$TV(f) \leq TV(Pf) + TV((id - P)f).$$

Hence it is natural to expect that a good separator $P$ should not create new variation, that is we have

$$TV(f) = TV(Pf) + TV((id - P)f). \tag{2}$$

An operator $P$ satisfying property (2) is called *total variation preserving.*

**Trend preservation:** An operator $P$ is *neighbour trend preserving* if for any vertices $v, u$, such that $\{v, u\} \in \mathcal{G}$, and any $f \in \mathcal{A}(\mathcal{G})$ we have

$$f(v) \leq f(u) \implies Pf(v) \leq Pf(u).$$

The operator $P$ is *fully trend preserving* if both $P$ and $id - P$ are neighbour trend preserving.

**Theorem 2.** *The operators $L_n$, $U_n$, $n \in \mathbb{N}$ and all their compositions are:*

*(i) idempotent and co-idempotent;*

*(ii) total variation preserving;*

*(iii) fully trend preserving.*

An example of the application of the LULU operators is given in Fig. 1. The figures on the right are the graphs of the luminosity functions of the images on the left. A noisy input is given in Fig. 1(a). It is well known that random noise creates impulses with small support. The operator $L_{30}U_{30}$ is applied to remove such random noise and the smoothed image is presented in Fig. 1(b). The LULU operators can be also used for extracting features of given size. The keys of the calculator are extracted in Fig. 1(c) by using the composition $(id - L_{3368}U_{3368})L_{624}U_{624}$.

## 3. Locally Monotone Approximations

The rationale for locally monotone approximations is given in [3] for one-dimensional signals, but it also applies to higher dimensions as well as the general setting of functions on a graph considered here. It can be described shortly as follows. Suppose it is known that the expected signal has particular kind of local monotonicity, e.g. it belongs to $\mathcal{M}_n$ for some $n \in \mathbb{N}$. If the input $f$ is not in $\mathcal{M}_n$ then clearly it is contaminated with noise. Then we take the best approximation of $f$ in $\mathcal{M}_n$ as signal. We should remark that the concepts of signal and noise are relative. Signal generally refers to required information or feature that needs to be separated from the input. For example, if from the input on Fig. 1(a) we require the keys of the calculator as they have been extracted on Fig. 1(c), then everything else is considered noise, e.g. including the labels on the keys which are indeed removed.

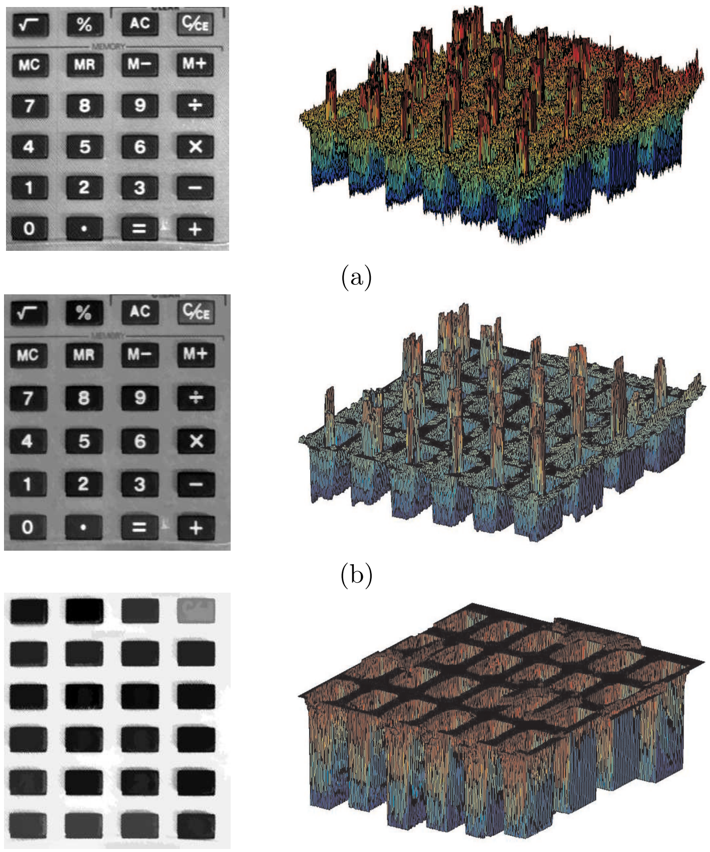


Figure 1. An illustrative example: (a) input; (b) noise removed; (c) features of interest (keys) extracted.

In the stated formulation the problem of signal extraction is an approximation problem. The issue of proximity can be considered in any of the norms $\|\cdot\|_p$, $p\in[1,\infty]$. It is easy to see that $\mathcal{M}_n$ is a closed subset of $\mathcal{A}(\mathbb{G})$ in any one of these norms. Therefore, a best approximation exists. Further analysis of this problem is difficult. On the one hand, uniqueness can not be guaranteed since $\mathcal{M}_n$ is not a convex. On the other hand, constructive algorithms for the best approximation are not available. While future work may resolve this issue we need to note that the best approximation takes into account only proximity and does not necessarily preserve any other essential and/or useful properties of the input. For example the best approximation does not necessarily have the properties mentioned in Theorem 2. Our main result given in Theorem 3 shows that while the LULU operators do not necessarily produce the best approximation, the error of the approximation is bounded by a constant multiple of the error of the best approximation and in this sense it is near best. The involved constant naturally depends on $n$ and on the connectivity of the graph.

We introduce a metric on $\mathbb{G}$ in the usual way. Let $u,v\in\mathbb{G}$. Since $\mathbb{G}$ is connected there exists a path connecting $u$ and $v$. The shortest path is the one with fewest edges. We denote by $\rho(u,v)$ the number of edges in the shortest path connecting $u$ and $v$. Then

$$B(v,n)=\{u\in\mathbb{G}:\rho(u,v)\le n\}$$

can be considered as the ball centered at $v$ with radius $n$. Let

$$\mathcal{K}_n=\sup_{v\in\mathbb{G}}\operatorname{card}(B(v,n)).$$

It is easy to see that $\mathcal{K}_n<\infty$, e.g. we have $\mathcal{K}_n\le\alpha^n$.

**Theorem 3.** *Let $P$ be either $L_n\circ U_n$ or $U_n\circ L_n$. For any $f\in\mathcal{A}(\mathbb{G})$ and any $h\in\mathcal{M}_n$ we have*

$$\begin{aligned}\|Pf-f\|_p &\le \big(1+(\mathcal{K}_n)^{1/p}\big)\|h-f\|_p, \qquad p\in[1,\infty),\\ \|Pf-f\|_\infty &\le 2\|h-f\|_\infty.\end{aligned}$$

The idea of the proof of the inequalities in Theorem 3 comes from the Lebesgue inequality. For a linear, idempotent and bounded operator $P$ on a normed space $X$ for every $f\in X$ and $h\in P(X)$ we have

$$\|Pf-f\|\le(1+\|P\|)\,\|f-h\|. \tag{3}$$

The LULU operators are not linear so that the inequality (3) is not directly applicable. We proceed by establishing the Lipschitz property for these operators.

**Theorem 4.** *For any $f,g\in\mathcal{A}(\mathbb{G})$ we have*

$$\begin{aligned}\|L_nf-L_ng\|_p &\le \mathcal{K}_n^{1/p}\|f-g\|_p,\\ \|U_nf-U_ng\|_p &\le \mathcal{K}_n^{1/p}\|f-g\|_p.\end{aligned}$$

*Proof.* Let $v \in \mathbb{G}$. Without loss of generality we assume $L_n f(v) \geq L_n g(v)$. From the definition of $L_n$

$$L_n f(v) = \max_{C \in \mathcal{C}_n(v)} \min_{w \in C} f(w) = \min_{w \in C_v} f(w)$$

for some $C_v \in \mathcal{C}_n(v)$. Hence $L_n g(v) \geq \min_{w \in C_v} f(w)$. We also have

$$L_n g(v) = \max_{C \in \mathcal{C}_n(v)} \min_{w \in C} g(w) \geq \min_{w \in C_v} g(w) = g(u_v),$$

for some $u_v \in C_v$. Thus

$$|L_n f(v) - L_n g(v)| = L_n f(v) - L_n g(v) \leq \min_{w \in C_v} f(w) - g(u_v) \leq f(u_v) - g(u_v)$$

Using that $u_v \in C_v \in \mathcal{C}_n(v)$ it is easy to see that $\rho(v, u_v) \leq n$. Therefore

$$|L_n f(v) - L_n g(v)|^p \leq |f(u_v) - g(u_v)|^p \leq \sum_{w \in B(v,n)} |f(w) - g(w)|^p. \quad (4)$$

Using the inequality (4) for every $v \in \mathbb{G}$ we obtain

$$\begin{aligned} \|L_n f - L_n g\|_p^p &= \sum_{v \in \mathcal{G}} |L_n f(v) - L_n g(v)|^p \\ &\leq \sum_{v \in \mathcal{G}} \sum_{w \in B(v,n)} |f(w) - g(w)|^p \leq \mathcal{K}_n \sum_{w \in \mathcal{G}} |f(w) - g(w)|^p, \end{aligned}$$

which proves the Lipschitz property of $L_n$. The Lipschitz property of $U_n$ is proved similarly. □

It is easy to obtain from Theorem 4 that the compositions $L_n U_n$ and $U_n L_n$ are also Lipschitz with a constant $\mathcal{K}^{2/p}$ for $p \in [1, \infty)$. However, we actually need a Lipschitz inequality when one of the functions is in $\mathcal{M}_n$. In this case the respective constant is smaller as shown in the next theorem.

**Theorem 5.** *For all $f \in \mathcal{A}(\mathbb{G})$ and $g \in \mathcal{M}_n$ we have*

$$\|L_n U_n f - g\|_p \leq \mathcal{K}_n^{1/p} \|f - g\|_p \ , \qquad \|U_n L_n f - g\|_p \leq \mathcal{K}_n^{1/p} \|f - g\|_p \ .$$

*Proof.* Let $v \in \mathbb{G}$. If $L_n U_n f(v) < g(v)$ using that $U_n \geq id$ we obtain

$$|L_n U_n f(v) - g(v)| = g(v) - L_n U_n f(v) \leq g(v) - L_n f(v) = |L_n f(v) - L_n g(v)|.$$

Then it follows from inequality (4) derived in the proof of Theorem 4 that

$$|L_n U_n f(v) - g(v)|^p \leq |L_n f(v) - L_n g(v)|^p \leq \sum_{w \in B(v,n)} |f(w) - g(w)|^p. \quad (5)$$

If $L_nU_nf(v) \geq g(v)$ then similarly using that $L_n \leq id$ the inequality for $U_n$ which is analogical to (4) we have

$$|L_nU_nf(v) - g(v)|^p \leq |U_nf(v) - U_ng(v)|^p \leq \sum_{w\in B(v,n)} |f(w) - g(w)|. \quad (6)$$

The combined application of (5) and (6) for every $v \in \mathbb{G}$ yields

$$\|L_nU_nf - g\|_p^p = \sum_{v\in\mathbb{G}} |L_nU_nf(v) - g(v)|^p \leq \sum_{v\in\mathbb{G}} \sum_{w\in B(v,n)} |f(w) - g(w)|^p$$
$$\leq \mathcal{K}_n \sum_{w\in\mathbb{G}} |f(w) - g(w)|^p = \mathcal{K}_n \|f - g\|_p^p$$

which prove the inequality for $L_nU_n$. The inequality for $U_nL_n$ is proved in a similar manner. □

**Remark 1.** *Letting $p \to \infty$ we obtain from Theorems 4 and 5 that the operators $L_n$, $U_n$ and their compositions all satisfy the Lipschitz property with a constant 1 with respect to the supremum norm.*

*Proof of Theorem 3.* Let $p \in [1, \infty)$. Using Theorem 5 we obtain

$$\begin{aligned}\|Pf - f\|_p &\leq \|Pf - h\|_p + \|h - f\|_p \\ &\leq \mathcal{K}_n^{1/p}\|f - h\|_p + \|h - f\|_p \\ &= (1 + \mathcal{K}^{1/p})\,\|h - f\|.\end{aligned}$$

For $p = \infty$ it follows from Remark 1 that

$$\begin{aligned}\|Pf - f\|_\infty &\leq \|Pf - h\|_\infty + \|h - f\|_\infty \\ &= \|Pf - Ph\|_\infty + \|h - f\|_\infty \\ &\leq 2\|h - f\|_\infty,\end{aligned}$$

which completes the proof. □

## 4. Conclusion

The idea of using monotonicity as a concept of smoothness within the Approximation Theory originates in the works of Sendov and Popov, e.g. [6]. In this paper we consider the situation when a signal or a feature with smoothness defined in terms of its local monotonicity needs to be extracted from a given input. We show that the LULU operators typically considered for their structure preserving properties also provide near best locally monotone approximations. The general setting of functions defined on a graph includes as particular cases both sequences as in [5] and multidimensional arrays as in [1].

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