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Sharp Integral Inequalities for Trigonometric Polynomials

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Sharp inequalities for linear operators in the set of trigonometric polynomials with respect to integral functionals $\int_0^{2\pi}\varphi(|f(x)|)\,dx$ over the class of all functions φ defined, nonnegative, and nondecreasing on the semiaxis $[0,\infty)$ are discussed.

1. Prehistory

1.1. Let $\mathbb{P}$ be the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$ depending on the situation. Let $C_{2\pi} = C_{2\pi}(\mathbb{P})$ be the space of continuous 2π-periodic functions with values in the field $\mathbb{P}$. The space $C_{2\pi}$ is a Banach space with respect to the uniform norm

$$\|f\|_{C_{2\pi}} = \max\{|f(t)| :\ t \in [0, 2\pi]\}.$$

Let us denote by $\mathscr{F}_n(\mathbb{P})$ the set of trigonometric polynomials

$$f_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kt + b_k \sin kt) \tag{1.1}$$

of order $n \geq 0$ with coefficients from the field $\mathbb{P}$. In the sequel, we also use the exponential notation for polynomials (1.1):

$$f_n(t) = \sum_{k=-n}^{n} c_k e^{ikt}.$$

In the set $\mathscr{F}_n(\mathbb{C})$, the known Bernstein inequality is valid:

$$\|f_n'\|_{C_{2\pi}} \leq n\|f_n\|_{C_{2\pi}}, \qquad f_n \in \mathscr{F}_n(\mathbb{C}). \tag{1.2}$$

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All extremal polynomials in inequality (1.2) have the form $ae^{int}+be^{-int}$, where a and b are arbitrary complex numbers. Bernstein obtained inequality (1.2) for polynomials with real coefficients [15, Subsect. 10]. Moreover, in the original variant [13] of paper [15], he proved this inequality with the constant n for odd and even trigonometric polynomials and, as a consequence, with the constant $2n$ in the class of all polynomials (1.1) from $\mathscr{F}_n(\mathbb{R})$. Bernstein's comments [16, Subsect. 3.4] to paper [15] contain the following phrase: "The conclusion given here and showing that the general inequality is an elementary consequence of the same inequality for the sum of sines, which was announced to me by E. Landau soon after the appearance of dissertation [13], was published for the first time in [14, §10]". Note that paper [13] was published in 1912, and monograph [14] was published in 1926.

In 1914, Riesz [33, 34] (see also, for example, [39, Vol. 2, Ch. 10]) obtained inequality (1.2) with the best constant n (both on the set $\mathscr{F}_n(\mathbb{R})$ and on the set $\mathscr{F}_n(\mathbb{C})$) with the help of the known interpolation formula for a derivative of a trigonometric polynomial. Namely, Riesz proved the following statement.

Theorem 1. *For the derivative of an arbitrary trigonometric polynomial $f_n \in \mathscr{F}_n(\mathbb{C})$ of order $n \geq 1$, the following formula holds:*

$$f_n'(t) = \sum_{k=1}^{2n} (-1)^{k-1} \alpha_k f_n(t+\tau_k), \qquad t \in (-\infty, \infty), \tag{1.3}$$

where

$$\tau_k = \frac{2k-1}{2n}\pi, \quad \alpha_k = \frac{1}{n\Big(2\sin\dfrac{\tau_k}{2}\Big)^2}, \qquad 1 \leq k \leq 2n.$$

The coefficients in formula (1.3) satisfy the equality

$$\sum_{k=1}^{2n} \alpha_k = n;$$

therefore, (1.3) implies (1.2) (with the constant n).

As a consequence of (1.2), the following sharp inequality holds for any natural n and r:

$$\|f_n^{(r)}\|_{C_{2\pi}} \leq n^r \|f_n\|_{C_{2\pi}}, \qquad f_n \in \mathscr{F}_n(\mathbb{C}). \tag{1.4}$$

Later, inequalities (1.2) and (1.4) were generalized in different directions. On the set $\mathscr{F}_n(\mathbb{C})$, let us consider the functional $\|f\|_p$ for $0 \leq p \leq \infty$, which is defined by the following relations depending on p:

$$\|f\|_p = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p\,dt\Big)^{1/p}, \qquad 0 < p < \infty,$$

$$\|f\|_\infty = \lim_{p\to+\infty} \|f\|_p = \max\{|f(t)| : t \in \mathbb{R}\} = \|f\|_{C_{2\pi}},$$

$$\|f\|_0 = \lim_{p\to+0} \|f\|_p = \exp\Big(\frac{1}{2\pi}\int\limits_0^{2\pi} \ln|f(t)|\,dt\Big).$$

In 1933, Zigmund, with the help of Riesz interpolation formula (1.3), proved the following statement (see [39, Vol. 2, Ch. 10, Theorem (3.16)]).

Theorem 2. *For any function φ non-negative, non-decreasing, and convex on the semi-axis $[0,\infty)$, the following inequality holds:*

$$\int\limits_0^{2\pi} \varphi(|f_n'(t)|)\,dt \le \int\limits_0^{2\pi} \varphi(n|f_n(t)|)\,dt, \qquad f_n \in \mathscr{F}_n(\mathbb{R}). \tag{1.5}$$

Inequality (1.5) *is sharp and turns into an equality for functions*

$$f_n(t) = A\cos(nt+\xi), \qquad A,\xi \in \mathbb{R}. \tag{1.6}$$

If the function φ strictly increases on $[0,\infty)$, then the equality in (1.5) *holds only for polynomials* (1.6).

As seen from the proof of this theorem in [39, Vol. 2, Ch. 10], in fact, inequality (1.6) holds on the set $\mathscr{F}_n(\mathbb{C})$.

For $p \ge 1$, the functions $\varphi(u) = u^p$, $u \in [0,\infty)$, satisfy the assumptions of Zigmund's Theorem 2. Therefore, Theorem 2 contains, in particular, the Bernstein inequality

$$\|f_n'\|_{L_p} \le n\|f_n\|_{L_p}, \qquad f_n \in \mathscr{F}_n(\mathbb{C}), \tag{1.7}$$

in the spaces L_p, $p \ge 1$. As a consequence, the following sharp inequality holds for any natural n and r for $1 \le p \le \infty$:

$$\|f_n^{(r)}\|_{L_p} \le n^r\|f_n\|_{L_p}, \qquad f_n \in \mathscr{F}_n(\mathbb{C}). \tag{1.8}$$

In 1975, Ivanov [23] and Storozhenko, Krotov, and Osval'd [37] built the theory of approximation of 2π-periodic functions by trigonometric polynomials in the spaces L_p, $0 < p < 1$. In particular, they studied the Bernstein inequality in L_p and proved that, for any p, $0 < p < 1$, there exists a constant $c(p)$ such that, for $n \ge 1$,

$$\|f_n'\|_{L_p} \le c(p)\,n\,\|f_n\|_{L_p}, \qquad f_n \in \mathscr{F}_n(\mathbb{R}). \tag{1.9}$$

The proof of this result in [23, 37] is based on specially chosen integral representations of trigonometric polynomials. In [29], to justify inequality (1.9), Osval'd applied appropriate quadrature formulas. Papers [23, 37, 29] do not contain explicit expressions for the value $c(p)$ in (1.9). In paper [27] published in 1979, Nevai proved inequality (1.9) with the constant

$$c(p) = \left(\frac{8}{p}\right)^{1/p}, \qquad 0 < p < 1.$$

In fact, it turned out that the best constant in inequality (1.9) is $c(p) = 1$; i.e., inequality (1.7) is valid for all p, $0 \le p < \infty$. This result was obtained by the author in 1978–1981. Statements are given and a method of investigation is described in [1]; complete proofs are given in paper [2]. The results of paper [2] will be discussed more completely below.

In 1980, paper [26] by Mate and Nevai was published, where, in particular, the following statement is proved with the help of refined methods of theory of orthogonal polynomials.

Theorem 3. *Let φ be a non-negative, non-decreasing, and convex function defined on $[0,\infty)$; let $0 < p < 1$. Then, on the set $\mathscr{F}_n(\mathbb{R})$ of trigonometric polynomials f_n of order n, the following inequality is valid:*

$$\int_0^{2\pi} \varphi\Big(\Big|\frac{f_n'(t)}{n}\Big|^p\Big)\,dt \le \int_0^{2\pi} \varphi(4e|f_n(t)|^p)\,dt.$$

This statement, in particular, implies that inequality (1.9) holds with a constant $c(p)$ satisfying the condition

$$c(p) \le (4e)^{1/p}, \qquad 0 < p < 1.$$

Paper [26] was published later than [1]. However, [26] was submitted earlier than [1] was published. It should be said that methods in [1, 2] are essentially different from that in [26].

1.2. In [1, 2], we studied inequalities of type (1.5) for a wider class of functions φ and a wider class of operators in the space of polynomials.

Let Φ^+ be the set of functions φ non-decreasing, locally absolutely continuous on $(0,\infty)$, and such that the function $\varphi(e^v)$ is convex downwards on $(-\infty,\infty)$ or, what comes to the same, the function $u\varphi'(u)$ does not decrease on $(0,\infty)$. Functions φ non-decreasing and convex downwards on $[0,\infty)$ belong to this class as well as the specific functions $\ln u$, $\ln^+ u = \max\{0, \ln u\} = \ln\max(1,u)$, and u^p for all $p > 0$.

In fact, inequalities not for trigonometric polynomials on the torus but for algebraic polynomials on the unit circle of the complex plane were studied in [1, 2]. We used the known fact that the formula

$$f_n(t) = e^{-int} P_{2n}(e^{it}) \tag{1.10}$$

establishes one-to-one correspondence between the set $\mathscr{F}_n(\mathbb{C})$ of trigonometric polynomials of order n and the set $\mathscr{P}_{2n}$ of algebraic polynomials of degree $2n$.

Let $\mathscr{P}_n$ be the set of algebraic polynomials of degree $n \ge 0$ with complex coefficients. For the polynomials

$$\Lambda_n(z) = \sum_{k=0}^{n} \lambda_k \binom{n}{k} z^k, \qquad P_n(z) = \sum_{k=0}^{n} a_k \binom{n}{k} z^k,$$

the polynomial

$$(\Lambda_n P_n)(z) = \sum_{k=0}^{n} \lambda_k a_k \binom{n}{k} z^k \tag{1.11}$$

is called the Szegő composition of the polynomials Λ_n and P_n. Properties of the Szegő composition can be found in [31, Sect. V], [24, Ch. IV], see also papers [20, 19] and the references given therein. For a fixed Λ_n, Szegő composition (1.11) is a linear operator in $\mathscr{P}_n$. Namely such operators were considered in papers [1, 2].

Let Ω_n^+ and Ω_n^- be the sets of operators (1.11) generated by polynomials Λ_n all of whose n zeros lie in the unit disk $|z| \le 1$ or in the domain $|z| \ge 1$, respectively. We set $\Omega_n^0 = \Omega_n^+ \bigcap \Omega_n^-$. Operators from Ω_n^+, Ω_n^-, and $\Omega_n^+ \bigcap \Omega_n^-$ are characterized by the property that polynomials P_n all of whose n zeros lie in the unit disk $|z| \le 1$, in the domain $|z| \ge 1$, and on the unit circle $|z| = 1$, respectively, are mapped according to formula (1.11) to polynomials with the same location of zeros (see [31, Sect. V, Problems 151, 152, 116, 117] and the references given in [2]).

One of the main results in [1, 2] is the following statement.

Theorem 4. *For operators $\Lambda_n \in \Omega_n = \Omega_n^+ \cap \Omega_n^-$ and functions $\varphi \in \Phi^+$, the following inequality holds on the set $\mathscr{P}_n$:*

$$\int_0^{2\pi} \varphi(|(\Lambda_n P_n)(e^{it})|)\, dt \le \int_0^{2\pi} \varphi(c_n |P_n(e^{it})|)\, dt, \qquad P_n \in \mathscr{P}_n, \tag{1.12}$$

$$c_n = c_n(\Lambda_n) = \max\{|\lambda_n|, |\lambda_0|\}. \tag{1.13}$$

Inequality (1.12) *is sharp; it turns into an equality for the polynomials*

$$az^n, \qquad b = \text{const}, \qquad az^n + b \qquad (a, b \in \mathbb{C}) \tag{1.14}$$

if Λ_n belongs to the set Ω_n^+, Ω_n^-, and $\Omega_n^+ \bigcap \Omega_n^-$, respectively.

Under certain restrictions on Λ_n and φ, there are no other extremal polynomials in (1.12) except for (1.14) [2, Theorem 5].

As a specific case of results from paper [2], the following statement is valid.

Theorem 5. *For functions $\varphi \in \Phi^+$, the following sharp inequality holds:*

$$\int_0^{2\pi} \varphi(|f_n'(t)|)\, dt \le \int_0^{2\pi} \varphi(n|f_n(t)|)\, dt, \qquad f_n \in \mathscr{F}_n(\mathbb{C}). \tag{1.15}$$

Inequality (1.15) *turns into an equality for the polynomials $f_n(t) = ae^{-int} + be^{int}$, $a, b \in \mathbb{C}$. If $u\varphi'(u)$ strictly increases on $(0, \infty)$, then there are no other extremal polynomials.*

The method of investigating sharp inequalities (1.12)–(1.13) in [1, 2] can be characterized shortly and schematically by the following three stages.

1. Linear operators in the form of Szegő composition (1.11) are considered. Known properties of the Szegő composition related to the location of zeros of polynomials are used. There are no new properties of the Szegő composition in papers [1, 2].

2. Inequality (1.12)–(1.13) is proved for the function $\varphi(u) = \ln u$, $u \in (0,\infty)$, with the help of theory of subharmonic functions. In addition, the following known statement is used (see, for example, [31, Sect. III, Problem 175], [25, Ch. VI, §2]): If a function h is meromorphic in the disk $|z| \le 1$, analytic, and nonzero at the center of the disk, then *Jensen's formula*

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|h(e^{it})|\,dt = \ln|h(0)| + \sum_{\nu=1}^{r} \ln\frac{1}{|z_\nu|} - \sum_{\nu=1}^{s} \ln\frac{1}{|\zeta_\nu|} \tag{1.16}$$

is valid, where $z_1,\dots,z_r$ are the zeros and $\zeta_1,\dots,\zeta_s$ are the poles of the function h in the disk $|z| \le 1$; here, a zero or a pole of multiplicity m is written m times. For the polynomial

$$P_n(z) = A\prod_{k=1}^{n}(z - z_k), \qquad A \ne 0,$$

of degree $n \ge 1$ with a nonzero leading coefficient A, formula (1.16) can be written in the following form:

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|P_n(e^{it})|\,dt = \ln\Big(|A|\prod_{k=1}^{n}\max(1,|z_k|)\Big). \tag{1.17}$$

3. Inequality (1.12)–(1.13) for the function $\varphi(u) = \ln u$, $u \in (0,\infty)$, is extended to arbitrary functions $\varphi \in \Phi^+$ in two steps. First, with the help of Jensen's formula (1.17), the inequality is extended to the function $\varphi(u) = \ln^+ u = \ln\max(1,u)$, $u \in (0,\infty)$. After that, a specific integral representation of the function $\varphi \in \Phi^+$ in terms of the function $\ln^+ u$ is constructed and the inequality for the function $\ln^+ u$ obtained at the previous step is applied.

Recently, Glazyrina [21] showed that the set of functions Φ^+ in inequality (1.12)–(1.13) is natural. More precisely, she showed that, if inequality (1.12)–(1.13) holds for an operator $\Lambda_n \in \Omega_n$ different from a rotation of the complex plane and for a non-decreasing function φ smooth enough, then $\varphi \in \Phi^+$. In particular, if inequality (1.15) holds on the set $\mathscr{F}_n(\mathbb{C})$ of trigonometric polynomials of order $n \ge 1$ (with complex coefficients) for a non-decreasing function φ smooth enough on the semi-axis $(0,\infty)$, then the function φ belongs to the set Φ^+.

Investigations of [2] were continued in author's papers [3, 4, 5, 6]. In particular, the following statement is proved in [3].

Theorem 6. *For any function $\varphi\in\Phi^+$ and any $n\geq 1$, the Szegő composition (1.11) of arbitrary polynomials Λ_n, $P_n\in\mathscr{P}_n$ satisfies the following inequality:*

$$\int_0^{2\pi}\varphi(|(\Lambda_n P_n)(e^{it})|)\,dt\leq\int_0^{2\pi}\varphi(\|\Lambda_n\|_{H_0}|P_n(e^{it})|)\,dt;\tag{1.18}$$

here,

$$\|\Lambda_n\|_{H_0}=\exp\Big(\frac{1}{2\pi}\int_0^{2\pi}\ln|\Lambda_n(e^{it})|\,dt\Big).\tag{1.19}$$

With the help of Jensen's formula (1.17), it is not hard to verify that, if a polynomial Λ_n belongs to the set $\Omega_n=\Omega_n^+\bigcup\Omega_n^-$ (and only for such polynomials), quantity (1.19) takes the following value:

$$\|\Lambda_n\|_{H_0}=\max\{|\lambda_n|,\,|\lambda_0|\}.$$

Consequently, inequality (1.12)–(1.13) is contained in inequality (1.18).

In 1989, von Golitschek and Lorentz [22] considered an inequality of the type (1.15) for the operator

$$Af_n+\frac{B}{n}\,f_n'\tag{1.20}$$

formally more general than the differentiation operator f_n' in the space of trigonometric polynomials. The following statement is one of the main results of their paper.

Theorem 7. *For a function $\varphi\in\Phi^+$ and any real A and B, the following sharp inequality holds:*

$$\int_0^{2\pi}\varphi\Big(\Big|Af_n(t)+\frac{B}{n}\,f_n'(t)\Big|\Big)\,dt\leq\int_0^{2\pi}\varphi\big(\sqrt{A^2+B^2}\,|f_n(t)|\big)\,dt,\quad f_n\in\mathscr{T}_n(\mathbb{C}).\tag{1.21}$$

This result is not new, as it is contained in [2, Corollary 2]. Indeed, by formula (1.10), for complex A and B, we have

$$Af_n(t)+\frac{B}{n}\,f_n'(t)=e^{-int}\,i\Big(\frac{B}{n}\,\zeta P_{2n}'(\zeta)-(B+iA)P_{2n}(\zeta)\Big),\qquad\zeta=e^{it}.$$

Let us consider the operator

$$(\mathcal{D}_nP_n)(z)=\frac{2B}{n}\,zP_n'(z)-(B+iA)P_n(z)\tag{1.22}$$

on the set $\mathscr{P}_n$ of algebraic polynomials of degree n. Operator (1.22) is Szegő's composition (1.11) constructed with the help of the polynomial

$$\mathcal{D}_n(z)=\sum_{k=0}^{n}\Big(\frac{2B}{n}\,k-(B+iA)\Big)\binom{n}{k}z^k=(1+z)^{n-1}((B-iA)z-(B+iA)).$$

Depending on the conditions

$$|B - iA| > |B + iA|, \qquad |B - iA| < |B + iA|, \qquad |B - iA| = |B + iA|,$$

operator (1.22) belongs to the set Ω_n^+, Ω_n^- or $\Omega_n^0 = \Omega_n^+ \bigcap \Omega_n^-$, respectively (see [31, Sect. V, Problems 116, 117] for details). Therefore, Theorems 4, 5 and so Corollary 2 from [2] are valid for operator (1.22). In this case, quantity (1.13) has the value $c_n = \max\{|B-iA|, |B+iA|\}$. Let us suppose that the coefficients A and B are real. Then, all zeros of the polynomial $\mathcal{D}_n$ lie on the unit circle; i.e., $\mathcal{D}_n \in \Omega_n^+ \bigcap \Omega_n^-$. In this case, $c_n = |B - iA| = |B + iA| = \sqrt{A^2 + B^2}$. For the operator $\mathcal{D}_{2n}$ on the set $\mathscr{P}_{2n}$ of polynomials of degree $2n$, inequality (1.12)–(1.13) turns into inequality (1.21) from Theorem 7. To prove inequality (1.21), authors of paper [22] follow the scheme of reasoning from paper [2]. It should be said that paper [22] is shorter than [2]. There are at least two reasons for it. (i) In [22] in comparison with [2], specific operator (1.20) (with real coefficients A and B) is discussed. (ii) The required property of zeros of polynomials under mapping of operator (1.20) is proved in [22, Lemma 1] directly with the help of rather clear considerations without usage of an equivalent known [31, Sect. V, Problems 116, 117] property of operator (1.22).

1.3. For some classical operators on the set of trigonometric polynomials, the respective operator Λ_n of the Szegő composition does not belong to the set $\Omega_n = \Omega_n^+ \bigcup \Omega_n^-$. Theorem 4 cannot be applied to such operators. Certainly, Theorem 6 can be applied; however, inequality (1.18) will be sharp, generally speaking, only for the function $\varphi(u) = \ln u$, $u \in (0, \infty)$. However, even in the latter case, the problem of investigating the behavior of quantity (1.19) remains open. Let us discuss two classical cases: the Szegő inequality for derivatives of the adjoint trigonometric polynomial and the Stechkin–Nikol'skii inequality between the norm of derivative of a polynomial and the norm of its first difference.

In 1928, Szegő [38] proved that the sharp inequality

$$\|f_n' \cos\alpha - \widetilde{f}_n' \sin\alpha\|_C \le n\|f_n\|_C, \qquad f_n \in \mathscr{F}_n(\mathbb{C}), \tag{1.23}$$

holds for any real α, where

$$\widetilde{f}_n(t) = \sum_{k=1}^{n} (-b_k \cos kt + a_k \sin kt)$$

is the polynomial adjoint with the polynomial f_n. Inequality (1.23) was obtained in [38] by using the following quadrature formula similar to Riesz formula (1.3).

Theorem 8. *For any real α and any trigonometric polynomial $f_n \in \mathscr{F}_n(\mathbb{C})$ of order $n \ge 1$, the following formula holds:*

$$f_n'(t)\cos\alpha - \widetilde{f}_n'(t)\sin\alpha = \sum_{k=1}^{2n} \beta_k f_n(t + \tau_k), \qquad t \in (-\infty, \infty), \tag{1.24}$$

where

$$\tau_k = \tau_k(\alpha) = \frac{2k-1}{2n}\,\pi + \frac{\alpha}{n}, \qquad \beta_k = \frac{(-1)^{k-1} + \sin\alpha}{4n\Big(\sin\dfrac{\tau_k}{2}\Big)^2}.$$

The coefficients of formula (1.24) satisfy the equality $\sum_{k=1}^{2n} |\beta_k| = n$; therefore, (1.24) implies inequality (1.23).

It should be said that, in fact, in [38], sharp inequalities of type (1.23) are proved for essentially wider class of operators in comparison with the operator $f_n'(t)\cos\alpha - \widetilde{f}_n'(t)\sin\alpha$. Bernstein [17] extended these results to a wider class of operators (final in a certain sense). Recently, Parfenenkov [30] obtained an appropriate quadrature formula.

In 1933, Zigmund [39, Vol. II, Ch. X, (3.25)], with the help of formula (1.24), obtained the following statement.

Theorem 9. *If a function φ is convex downwards and non-decreasing on the semi-axis $[0,\infty)$, then the following sharp inequality holds for any real α:*

$$\int_0^{2\pi} \varphi(|f_n'(t)\cos\alpha - \widetilde{f}_n'(t)\sin\alpha|)\,dt \le \int_0^{2\pi} \varphi(n|f_n(t)|)\,dt, \qquad f_n \in \mathscr{F}_n(\mathbb{C}). \quad (1.25)$$

Inequality (1.25) *is sharp and turns into an equality for the polynomials of the form $ae^{int} + be^{-int}$, $a, b \in \mathbb{C}$. In addition, if the function φ strictly increases on $[0,\infty)$, then there are no other extremal polynomials.*

Taking the function $\varphi(u) = u^p$, $p \ge 1$, in (1.25), we obtain the inequality

$$\|f_n'\cos\alpha - \widetilde{f}_n'\sin\alpha\|_{L_p} \le n\|f_n\|_{L_p}, \qquad f_n \in \mathscr{F}_n(\mathbb{C}), \quad (1.26)$$

in the space L_p, $1 \le p < \infty$. It follows from (1.23), (1.26), and (1.7) that, for any natural n, r and $1 \le p \le \infty$, along with inequality (1.8), the following (sharp) inequality is also valid:

$$\|\widetilde{f}_n^{(r)}\|_{L_p} \le n^r \|f_n\|_{L_p}, \qquad f_n \in \mathscr{F}_n(\mathbb{C}). \quad (1.27)$$

Inequality (1.8) is valid for all p, $0 \le p \le \infty$. However, as shown in [6], generally speaking, inequality (1.27) cannot be extended to the case $0 \le p < 1$. If $r \ge n\ln 2n$, then the corresponding operator of the Szegő composition belongs to the class $\Omega_{2n}^0 = \Omega_{2n}^+ \bigcap \Omega_{2n}^-$; as a consequence, (for $r \ge n\ln 2n$) the following sharp inequality holds for any function $\varphi \in \Phi^+$:

$$\int_0^{2\pi} \varphi(|\widetilde{f}_n^{(r)}(t)|)\,dt \le \int_0^{2\pi} \varphi(n^r|f_n(t)|)\,dt, \qquad f_n \in \mathscr{F}_n(\mathbb{C}).$$

In particular, inequality (1.27) is valid for all $p \ge 0$. The author conjectures that the necessary and sufficient condition for (1.27) to hold is $r \ge n-1$. For a fixed r, the best constant $K_0(n,r)$ in the analogous to (1.27) inequality for the space L_0

$$\|\widetilde{f}_n^{(r)}\|_{L_0} \le K_0(n,r)\|f_n\|_{L_0}, \qquad f_n \in \mathscr{F}_n(\mathbb{C}),$$

has the property $K_0(n,r) = 4^{\varepsilon_n}$, $\varepsilon_n = n + o(n)$ as $n \to \infty$. It is seen that the growth of this constant with respect to n is essentially greater than that of the n^r in (1.27) for $1 \le p \le \infty$.

In 1948, Stechkin [35] obtained a fine generalization of Bernstein's inequality (1.4). Namely, he proved that the sharp inequality

$$\|f_n^{(r)}\|_{C_{2\pi}} \le \Big(\frac{n}{2\sin nh/2)}\Big)^r \|\Delta_h^r f_n\|_{C_{2\pi}}, \qquad f_n \in \mathscr{F}_n(\mathbb{R}), \tag{1.28}$$

between the uniform norms of the derivative of order $r \ge 1$ of a polynomial and its r-th difference

$$\Delta_h^r f_n(t) = \sum_{\nu=0}^{r}(-1)^{k+\nu}\binom{r}{\nu} f_n(x+\nu t)$$

with step h holds in $\mathscr{F}_n(\mathbb{R})$ for $0 < h < \frac{2\pi}{n}$. However, already in 1914, Riesz obtained [34, Sect. 4] a special case of this inequality for $r = 1$ and $h = \pi/n$. Nikol'skii established [28] an inequality similar to (1.28) for entire functions of exponential type σ for $r \ge 1$ and $h = \pi/\sigma$ in 1948. Storozhenko found [36] the best constant in an inequality similar to (1.28) in the space L_0 for $r = 1$. It turned out that the behavior of this constant is different from that of the respective constant in (1.28).

1.4. Sharp inequalities for trigonometric polynomials is a wide part of function theory. The review of results given here is far from complete; it only reflects author's interests. Rather complete review can be found in monograph [32] by Rahman and Schmeisser. For a review of the sharp inequalities for algebraic polynomials on a segment the reader is referred to the paper of Boyanov [18].

2. Sharp Inequalities for Trigonometric Polynomials with Respect to Integral Functionals

2.1. On the set $\mathscr{F}_n(\mathbb{R})$ of real trigonometric polynomials (1.1) of order at most $n \ge 1$, we define the functional

$$\mu(f_n) = \operatorname{mes}\{t \in [0, 2\pi] : |f_n(t)| \ge 1\},$$

whose value is the Lebesgue measure of points on the torus $\mathbb{T}$ at which the absolute value of a polynomial f_n is greater than or equal to 1. For a linear operator F_n on the set of polynomials $\mathscr{F}_n(\mathbb{R})$, we denote by $B_n(F_n)$ the least possible constant in the inequality

$$\mu(F_n f_n) \le B_n(F_n)\,\mu(f_n), \qquad f_n \in \mathscr{F}_n(\mathbb{R}). \tag{2.1}$$

Babenko [12] studied such inequalities in 1992 and obtained two-sided estimates for the constant $B_n(F_n)$ for a rather wide class of operators, in particular, for the operator G_n which assigns to polynomial (1.1) its leading harmonic

$$G_n(f_n)(t) = a_n \cos nt + b_n \sin nt$$

and for the differentiation operator

$$D_n f_n = \frac{1}{n}\, f_n'. \tag{2.2}$$

If $F_n \not\equiv 0$, then

$$B_n(F_n) \ge 1. \tag{2.3}$$

Indeed, let us suppose that a polynomial $f_n \in \mathscr{F}_n(\mathbb{R})$ satisfies $F_n f_n \not\equiv 0$. Let us consider the family of polynomials $\{u f_n,\ u > 0\}$. We have

$$\mu(u f_n) = \operatorname{mes}\Big\{t \in \mathbb{T} : |f_n(t)| \ge \frac{1}{u}\Big\} \to 2\pi, \qquad u \to \infty.$$

The value $\mu(F_n(u f_n)) = \mu(u F_n(f_n))$ has the same property. Substituting functions $\{u f_n,\ u > 0\}$ into (2.1), we obtain inequality (2.3).

Note also that, if the norm of the operator F_n in the space $C_{2\pi}$ is grater than 1: $\|F_n\|_{C_{2\pi}\to C_{2\pi}} > 1$, then $B_n(F_n) = \infty$. Indeed, under the assumption made, there exists a polynomial $f_n \in \mathscr{F}_n(\mathbb{R})$ such that $\|f_n\|_{C_{2\pi}} < 1$ and $\|F_n f_n\|_{C_{2\pi}} > 1$. It follows that $B_n(F_n) = \infty$.

2.2. Let Φ be the set of functions φ defined, non-negative, and non-decreasing on the semi-axis $[0, \infty)$. For a linear operator F_n on $\mathscr{F}_n(\mathbb{R})$ and for a function $\varphi \in \Phi$, we denote by $A_n(F_n, \varphi)$ the least possible constant in the inequality

$$\int_0^{2\pi} \varphi(|(F_n f_n)(t)|)\, dt \le A_n(F_n, \varphi) \int_0^{2\pi} \varphi(|f_n(t)|)\, dt, \qquad f_n \in \mathscr{F}_n(\mathbb{R}). \tag{2.4}$$

The investigation of the value $A_n(F_n, \varphi)$ for a specific function $\varphi \in \Phi$ seems to be an unsolvable problem. Let us consider the quantity

$$A_n(F_n) = \sup\{A_n(F_n, \varphi) : \varphi \in \Phi\}. \tag{2.5}$$

The class Φ contains, in particular, the function φ^* defined by the relations

$$\varphi^*(u) = \begin{cases} 0, & u \in [0, 1), \\ 1, & u \in [1, \infty). \end{cases} \tag{2.6}$$

For this function,

$$\int_0^{2\pi} \varphi^*(|f_n(x)|)\,dx = \mu(f_n).$$

Therefore, $A_n(F_n, \varphi^*) = B_n(F_n)$; hence, the least constant in (2.1) and value (2.5) are related by the inequality

$$B_n(F_n) \le A_n(F_n). \tag{2.7}$$

In fact, these values coincide. More precisely, the following statement [9] is valid.

Lemma 1. *For any $n \ge 1$ and any linear operator F_n, the equality holds in $\mathscr{F}_n(\mathbb{R})$:*

$$A_n(F_n) = B_n(F_n). \tag{2.8}$$

2.3. In the set $\mathscr{F}_n(\mathbb{R})$ of trigonometric polynomials of order n, let us consider the linear operator G_n which assigns to a polynomial (1.1) its leading harmonic:

$$G_n(f_n)(t) = a_n \cos nt + b_n \sin nt.$$

It is well-known that the norm of the operator G_n on the set $\mathscr{F}_n(\mathbb{R})$ in the space $C_{2\pi}$ is equal to 1 or, what comes to the same, the harmonic $a_n \cos nt + b_n \sin nt$ is not approximated by smaller harmonics in $C_{2\pi}$. The same fact is valid in the spaces $L^p_{2\pi}$, $0 \le p < \infty$ (see [2] and the references given therein).

In fact, a more general fact is valid. We recall that Φ^+ denotes the set of functions φ defined, non-decreasing, locally absolutely continuous on $(0,\infty)$, and such that the function $u\varphi'(u)$, $u \in (0,\infty)$, is also non-decreasing. As a consequence of more general results from [2], for any function $\varphi \in \Phi^+$, the following inequality is valid:

$$\int_0^{2\pi} \varphi(|a_n \cos nt + b_n \sin nt|)\,dt \le \int_0^{2\pi} \varphi(|f_n(t)|)\,dt, \qquad f_n \in \mathscr{F}_n(\mathbb{R}). \tag{2.9}$$

This result also means that, for any function $\varphi \in \Phi^+$, the harmonic of order $n \ge 1$ is not approximated by smaller harmonics with respect to the functional

$$\int_0^{2\pi} \varphi(|f_n(t)|)\,dt. \tag{2.10}$$

Let us consider the problem on the set Φ of functions φ defined, non-negative, and non-decreasing on the semi-axis $[0,\infty)$. For a function $\varphi \in \Phi$, we denote by $A_n(\varphi) = A_n(G_n, \varphi)$ the least constant in the inequality

$$\int_0^{2\pi} \varphi(|a_n \cos nt + b_n \sin nt|)\,dt \le A_n(\varphi) \int_0^{2\pi} \varphi(|f_n(t)|)\,dt, \qquad f_n \in \mathscr{F}_n(\mathbb{R}).$$

The polynomials $f_n(t) = a \cos nt + b \sin nt$ provide the estimate $A_n(\varphi) \geq 1$. Inequality (2.9) means that, if $\varphi \in \Phi^+$, then $A_n(\varphi) = 1$. There are functions $\varphi \in \Phi$ such that $A_n(\varphi) > 1$; by Babenko's result (2.13), function (2.6) possesses this property.

We are interested in the value

$$A_n^* = \sup\{A_n(\varphi) : \ \varphi \in \Phi\}. \tag{2.11}$$

By Lemma 1, value (2.11) coincides with the best constant $\beta_n = B_n(G_n)$ in the inequality

$$\mu(a_n \cos nt + b_n \sin nt) \leq \beta_n \, \mu(f_n), \qquad f_n \in \mathscr{F}_n(\mathbb{R}). \tag{2.12}$$

Babenko [12] proved that the following inequalities are valid for any $n \geq 1$:

$$\sqrt{2n} \leq \beta_n \leq n\sqrt{2}. \tag{2.13}$$

Sharp value of β_n and so of A_n^* are given in [10, 11]. The following statement is valid.

Theorem 10. *For any $n \geq 1$, the best constant β_n in inequality* (2.12) *satisfies*

$$\beta_n = \sqrt{2n}. \tag{2.14}$$

Statements (2.14) and (2.8) imply the equality

$$A_n^* = \sqrt{2n}.$$

The last result can be formulated as follows.

Theorem 11. *For any function $\varphi \in \Phi$ and any $n \geq 1$, the following inequality holds in the set $\mathscr{F}_n(\mathbb{R})$:*

$$\int_0^{2\pi} \varphi(|f_n(t)|) \, dt \geq \frac{1}{\sqrt{2n}} \int_0^{2\pi} \varphi(|a_n \cos nt + b_n \sin nt|) \, dt, \qquad f_n \in \mathscr{F}_n(\mathbb{R}). \tag{2.15}$$

Inequality (2.15) *is unimprovable over the set of all functions $\varphi \in \Phi$ for any $n \geq 1$.*

This result means that, for functions $\varphi \in \Phi$, the leading harmonic $a_n \cos nt + b_n \sin nt$ already can be approximated by smaller harmonics with respect to functional (2.10) but at most at $\sqrt{2n}$ times in comparison with the value of functional (2.10) for the leading harmonic.

The problems for algebraic polynomials on a segment similar to problems considered in this section were discussed in the author's paper [7].

2.4. For differential operator (2.2), inequalities (2.4) are called Bernstein's inequalities. As was mentioned above, it is proved in [2] that $A_n(D_n, \varphi) = 1$

for any function $\varphi \in \Phi^+$. By now, there is no function $\varphi \in \Phi$ not belonging to the class Φ^+ such that an exact value or at least good estimates for the constant $A_n(D_n, \varphi)$ are known. Certainly, the largest constant $A_n(D_n) = \sup\{A_n(D_n, \varphi) : \varphi \in \Phi\}$ is of interest. For operator (2.2), Babenko [12] obtained the estimates

$$\frac{2}{\pi} \ln(2n+1) \le B_n(D_n) \le 2n.$$

In view of this result and Lemma 1, one can only assert that the following estimates are valid:

$$\frac{2}{\pi} \ln(2n+1) \le A_n(D_n) \le 2n.$$

By now, the value of the quantities $B_n(D_n) = A_n(D_n)$ is unknown. Even the values of corresponding quantities for the differential operators of higher orders are unknown.

Writing this note, we have essentially used the author's paper [9] in which Lemma 1 is proved. The results were partially talked about in the author's report at CTF-2010 [8].

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