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Surface Approximation by Piece-Wise Harmonic Functions

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We describe here a new method for surface approximation on the basis of given values at a regular grid. The resulting approximant is a continuous piece-wise harmonic function.

1. Introduction

There exist various algorithms for surface approximation. Most of them use polynomial spline functions. We present here another approach, which is based on harmonic functions.

Suppose that G is a given domain in the plane and φ is a function defined on the boundary Γ of G. It is well known that under certain restrictions on Γ and φ, there exists a unique harmonic function $u(x,y)$ on G which coincides with $\varphi(x)$ on Γ. This fact suggests the following quite natural and simple way of approximation. Suppose that (x_i, y_j) is a regular grid in G and $\{D_m\}$ are the rectangular cells of the grid, with boundaries $\{\Gamma_m\}$, respectively. Let $f(x,y)$ be a function defined on G. Assume that the values of f are known or easily available on the lines of the grid, i.e., on each Γ_m. Denote by $u_m(x,y)$ the harmonic continuation of f on D_m. In other words, u_m is the unique solution of the Dirichlet problem

$$\left| \begin{array}{l} \Delta u = 0 \quad \text{on} \quad D_m \\ u|_{\Gamma_m} = f, \end{array} \right. \tag{1}$$

where, as usual,

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

and $u|_\Gamma = f$ means that $u(x,y) = f(x,y)$ for $(x,y) \in \Gamma$.

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Having u_m for each m, one can approximate f on G by the piece-wise harmonic function $S(x, y)$ defined as follows:

$$S(x, y) := u_m(x, y) \quad \text{for} \quad (x, y) \in D_m, \text{ all } m.$$

Clearly S is a continuous function and possesses good approximation properties. There is however a serious reason which stops the people from using this method of approximation in practice. It is the necessity of solving the partial differential equation (1) for each m (the number of cells D_m may be very large for fine grids).

We propose here a simple way of constructing $S(x, y)$ which avoids the solution of (1) in each cell D_m. The numerical experiments show that the method is fast and it produces good approximations in some typical cases.

2. Description of the Algorithm

Let us first describe roughly the main idea and look at the precise details. Suppose that the grid on G is defined by the points $\{x_i, y_j\}$,

$$\begin{aligned} x_i &= x_0 + ih, \quad i = 0, \ldots, N, \\ y_j &= y_0 + jh, \quad i = 0, \ldots, M. \end{aligned}$$

Denote by D_{ij} the elementary square cell with vertices

$$(x_i, y_j), (x_{i+1}, y_j), (x_{i+1}, y_{j+1}), (x_i, y_{j+1}).$$

Let Γ_{ij} be the boundary of D_{ij}. Suppose that the values of $f(x, y)$ are known on Γ_{ij} for every (i, j). Introduce the boundary functions

$$\varphi_{ij}(x, y) := f(x, y) \quad \text{for} \quad (x, y) \in \Gamma_{ij}.$$

In order to construct the piece-wise harmonic approximation $S_h(x, y)$ of $f(x, y)$ (as described in the previous section) we need the solutions of the equations

$$\left| \begin{array}{l} \Delta u = 0 \quad \text{on} \quad D_{ij} \\ u|_{\Gamma_{ij}} = \varphi_{ij}. \end{array} \right. \tag{2}$$

For this purpose we transform D_{ij} into the unit square D^* with vertices $(0, 0), (1, 0), (1, 1), (0, 1)$. Then the boundary function $\varphi_{ij}(x, y)$ goes (under this linear transformation) to a certain function $\psi(x, y)$ on the boundary Γ^* of D^*. Let $\{\psi_0, \psi_1, \ldots, \psi_r\}$ be a bases of appropriate preassigned boundary functions on Γ^*. Assume that we know somehow the solutions of the normalized problems

$$\left| \begin{array}{l} \Delta u = 0 \quad \text{on} \quad D^* \\ u|_{\Gamma^*} = \psi_j \end{array} \right. \tag{3}$$

for $j = 0, \dots, r$. Note that this is a small number of equations, which can be solved previously (once forever) and the solutions u_j, $j = 0, \dots, r$, stored. Let us find an approximation $\tilde{\psi} \in \operatorname{span}\{\psi_0, \psi_1, \dots, \psi_r\}$ to ψ. Suppose that

$$\tilde{\psi} = c_0\psi_0 + c_1\psi_1 + \cdots + c_r\psi_r.$$

Then

$$\tilde{u}(x, y) := \sum_{j=0}^{r} c_j u_j(x, y)$$

is the solution of the Dirichlet problem corresponding to the boundary conditions $\tilde{\psi}$ on Γ^*. Finally, by the reverse linear transformation ($D^* \to D_{ij}$) we find from $\tilde{u}$ the wanted approximate solution of (2) and consequently, the approximation S_h of f on G.

Next we use this idea to construct explicitly a piece-wise harmonic approximation S_h of f on the bases of the values $\{f_{ij}\}$ of f at the grid points (x_i, y_j). We call this method of construction *Algorithm 1.* First, we compute the approximations $\{f_{ij}^x, f_{ij}^y\}$ of the derivatives $\partial f/\partial x$, $\partial f/\partial y$ at (x_i, y_j), using the formulas (see for example [3])

$$f_{0j}^x = \frac{-3f_{0,j} + 4f_{1,j} - f_{2,j}}{2h}, \qquad f_{Nj}^x = \frac{3f_{N,j} - 4f_{N-1,j} + f_{N-2,j}}{2h},$$

$$f_{ij}^x = \frac{f_{i+1,j} - f_{i-1,j}}{2h},$$

for $0 < i < N$ and $j = 0, \dots, N$. Similarly we compute f_{ij}^y.

Then using cubic Hermite interpolation we define the functions φ_{ij} on the boundary Γ_{ij} of D_{ij}. Precisely, for $x_i \le x \le x_{i+1}$ and $y = y_j$ the function $\varphi_{ij}(x, y)$ coincides with the cubic polynomial $p(x)$ satisfying the interpolation conditions

$$p(x_i) = f_{ij}, \qquad p(x_{i+1}) = f_{i+1,j}, \qquad p'(x_i) = f_{ij}^x, \qquad p'(x_{i+1}) = f_{i+1,j}^x.$$

The definition of φ_{ij} on the other edges of D_{ij} is similar.

It is clear that the function φ_{ij} can be presented as a sum of 12 terms, separated in four groups, each group corresponding to one of the vertices of D_{ij}. For example, the group corresponding to the vertex (x_i, y_j) will be

$$f_{ij}\lambda(x, y) + f_{ij}^x\mu(x, y) + f_{ij}^y\nu(x, y),$$

where λ, μ and ν are cubic polynomials on the edges of D_{ij} such that

$$\lambda(x_i, y_j) = 1, \qquad \frac{\partial}{\partial x}\mu(x_i, y_j) = 1, \qquad \frac{\partial}{\partial y}\nu(x_i, y_j) = 1$$

and all other not specified values of λ, μ, ν and their first partial derivatives are equal to 0 at the vertices of D_{ij}. Then the solution u_{ij} of the Dirichlet problem (2) is a linear combination, with coefficients f_{kl}, f_{kl}^x, f_{kl}^y, $(k, l) \in$

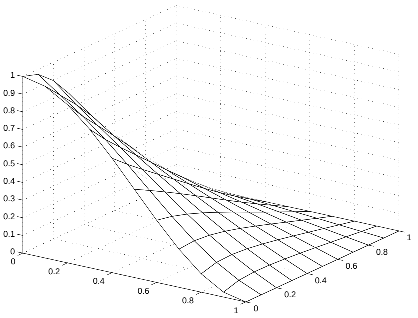

Figure 1. $u^*(x, y)$

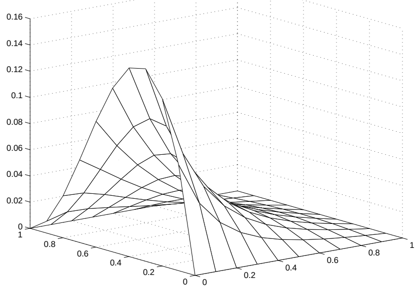

Figure 2. $v^*(x, y)$

$\{(i, j), (i + 1, j), (i + 1, j + 1), (i, j + 1)\}$, respectively, of 12 specific functions (solutions of Dirichlet problem with specific boundary conditions like λ, μ, ν). Because of the symmetry all these 12 functions can be obtained by symmetry and rotation from the solutions $u(x, y)$ and $v(x, y)$ of the following two problems

$$\left| \begin{array}{l} \Delta u = 0 \quad \text{on} \quad D_{ij} \\ u|_{\Gamma_{ij}} = \lambda, \end{array} \right. \qquad \text{and} \qquad \left| \begin{array}{l} \Delta v = 0 \quad \text{on} \quad D_{ij} \\ v|_{\Gamma_{ij}} = \mu. \end{array} \right.$$

Further, these two solutions can be obtained by a linear transformation from the corresponding solutions u^* and v^* on the unit square D^*. Thus all we need is to solve previously the Dirichlet problem on D^* with boundary condition $\lambda^*(x, y)$ and $\mu^*(x, y)$, where

$$\lambda^*(x, y) = \begin{cases} 2x^3 - 3x^2 + 1, & \text{for } 0 \le x \le 1,\ y = 0, \\ 2y^3 - 3y^2 + 1, & \text{for } 0 \le y \le 1,\ x = 0, \\ 0, & \text{if } x = 1 \text{ or } y = 1, \end{cases}$$

$$\mu^*(x, y) = \begin{cases} x(x - 1)^2, & \text{for } 0 \le x \le 1,\ y = 0, \\ 0, & \text{if } x = 0 \text{ or } 1,\ y = 1 \end{cases}$$

(see u^* and v^* on Figure 1 and Figure 2, respectively).

These two particular problems can be solved numerically with a high accuracy using some standard numerical method. The values of u^* and v^* at some finite number of points $\Omega_n := \{(k/n, i/n), k = 0, \dots, n, i = 0, \dots, n\}$ can be stored in the memory. In the examples below we have $n = 5$.

Note that the surface S_h resulting from Algorithm 1 is continuous on G. In addition, it follows from the construction that $\frac{\partial}{\partial x} S_h$ and $\frac{\partial}{\partial y} S_h$ are continuous at the grid points (x_i, y_j). Let us sketch below a modification of Algorithm 1 (we call it Algorithm 2), which produces a surface S_h having first and second derivatives continuous at the grid points.

Algorithm 2. Given $\{f_{ij}\}$, compute the first derivatives $\{s_{ij},\ i = 1, \dots, N - 1\}$ of the cubic natural spline $P_j(x)$ with knots at $\{x_{ij},\ i = 1, \dots, N - 1\}$, which interpolates the values $\{f_{ij},\ i = 0, \ldots, N\}$. As shown in [2], for every fixed j, the quantities $\{s_{ij},\ i = 1, \ldots, N-1\}$ satisfy the linear system of equations

$$s_{i-1,j} + 4s_{i,j} + s_{i+1,j} = 3(f_{i+1,j} - f_{i-1,j})/h, \quad i = 1, \ldots, N-1.$$

Having $f_{i,j}$ and $s_{i,j}$ define the boundary functions $\varphi_{ij}(x,y)$ on $x_{i,j} < x < x_{i+1,j},\ \ y = y_j$ as the unique cubic polynomial p which satisfies the interpolation conditions

$$p(x_i) = f_{ij}, \qquad p(x_{i+1}) = f_{i+1,j}, \qquad p'(x_i) = s_{ij}, \qquad p'(x_{i+1}) = s_{i+1,j}$$

and proceed further as in Algorithm 1.

3. Error Estimation

We give here an estimation of the error

$$R_h(x,y) := f(x,y) - S_h(x,y)$$

under certain restrictions on f, provided the solutions u^* and v^* of the basic Dirichlet problems are known exactly or with a high accuracy.

Denote, as usual, by $\|f\|$ the uniform norm of f on $\bar{G}$.

Theorem 1. *Suppose that $f \in C^2(\bar{G})$ and $S_h(x,y)$ is the piece-wise harmonic approximation given by Algorithm 1. Then there is a constant C such that*

$$\|f - S_h\| \le Ch^2.$$

Proof. Let $\epsilon_{ij}(x,y)$ be the error function in the Hermite -like interpolation of f on Γ_{ij}. Precisely,

$$\epsilon_{ij}(x,y) := f(x,y) - \varphi_{i,j}(x,y) \quad \text{on} \quad \Gamma_{i,j}.$$

First we shall give an estimation of $|\epsilon_{i,j}|$. In order to do this consider ϵ_{ij} on any fixed side of $\Gamma_{i,j}$, say on $\{x_i \le x \le x_{i+1},\ y = y_j\}$. Note that $\varphi_{i,j}$ coincides on this side with the cubic polynomial $p(f;x)$ which interpolates the data $f_{ij}, f_{i+1,j}, f^x_{i,j}, f^x_{i+1,j}$. Thus $p(f;x)$ is a linear operator of f which annihilates the polynomials of first degree. Then, by the Peano kernel theorem,

$$|\epsilon_{ij}| = |f(x,y_j) - p(f;x)| = \left| \int_{x_0}^{x_N} p((x-t)_+;x) \frac{\partial^2}{\partial x^2} f(x,y_j)\, dt \right|.$$

Assume now that $0 < i < N$ (In case $i = 0$ or $i = N$ the reasoning is similar and we shall omit it). Since $(x-t)_+ = x - t$ for $x > t$ and it vanishes for $x < t$, it is clear that $p((x-t)_+;x) = 0$ for t outside $I := (x_{i-1,j}, x_{i+1,j})$. Set

$$M := \max_{(x,y)\in\bar{G}} |\Delta f|.$$

Therefore

$$|\epsilon_{ij}| \le M \int_I |p((x-t)_+;x)|\, dt \le 3Mh \max_{x\in I} |p((x-t)_+;x)|$$

It is not difficult to see that $p((x-t)_+;x)$ is a monotone function of x in $[x_{i+1,j}, x_{i+1,j}]$. Therefore

$$|p((x-t)_+;x)| \le |x_{i+1,j} - t| \le 2h$$

if $t \in I$. So,

$$|\epsilon_{i,j}| \le 6Mh^2. \tag{4}$$

Next part of the proof is standard. Consider the difference $R_h(x,y)$ on Γ_{ij}. ClearlyR_h is a solution of the Dirichlet' problem

$$\left|\begin{array}{l} \Delta R_h = \Delta f \quad \text{on} \quad D_{ij} \\ R_h|_{\Gamma_{ij}} = \epsilon_{ij}. \end{array}\right.$$

It is well-known from the theory of harmonic functions (see [1]) that for each $(x,y) \in D_{ij} \cup \Gamma_{ij}$.

$$|R_h(x,y)| \le \max_{\Gamma_{ij}} |\epsilon_{ij}| + h^2 \max_{D_{ij}} |\Delta f|.$$

Now we apply the estimation (4) and complete the proof. □

The same estimate holds also in case of Algorithm 2. The proof is similar.

4. Numerical Experiments

We applied Algorithm 1 for numerical reconstruction of the surface $f(x,y)$ in the following two cases.

Example 1. $f(x,y) = 1 - x^2 - y^2$.

The domain G is the square $[-5,5] \times [-5,5]$, and $h = 1$. The solutions $u^*(x,y)$ and $v^*(x,y)$ of the Dirichlet problem on the unit square D^* are evaluated with a high precision at 100 points. Figure 3 illustrates the approximation surface $S_h(x,y)$ while the graph of the function $f(x,y)$ is given on Figure 4. □

Example 2. $f(x,y) = \dfrac{\cos(x^2+y^2)}{3+x^2+y^2}$.

Similarly to the Example 1 the domain G is the square $[-5,5] \times [-5,5]$, and $h = 1$. The solutions $u^*(x,y)$ and $v^*(x,y)$ of the Dirichlet problem on the unit square D^* are evaluated with a high precision at 100 points. Figure 5 illustrates the approximation surface $S_h(x,y)$ while the graph of the function $f(x,y)$ is given on Figure 6. □

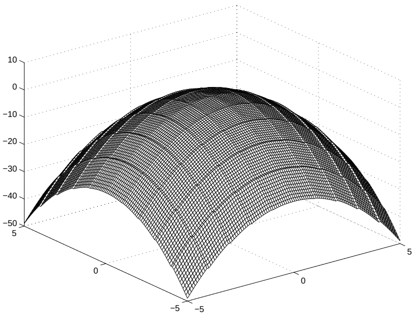

Figure 3. $S_h(x, y)$

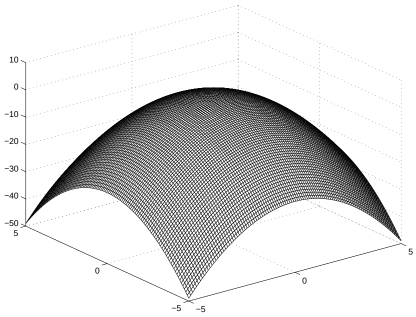

Figure 4. $f(x, y)$

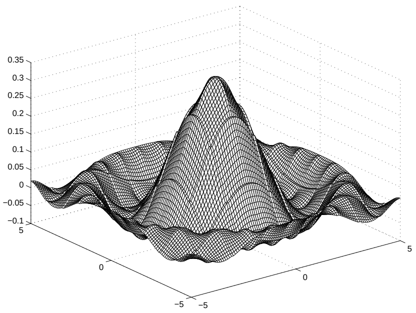

Figure 5. $S_h(x, y)$

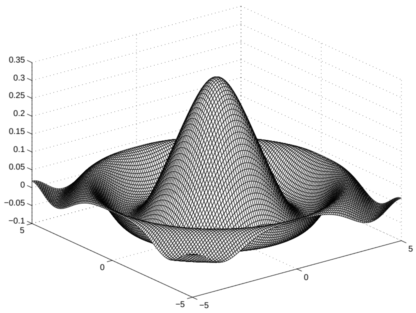

Figure 6. $f(x, y)$

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