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# Smooth Convex Resolution of Unity on General Partitions of Multidimensional Domains and Multivariate Hermite Interpolation on Scattered Point Sets Using Radial Generalized Expo-Rational B-splines

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Consider a finite scattered point set in a multidimensional domain and a general finite partition of the domain in subdomains such that: (i) every subdomain corresponds to one, and only one, element of the scattered point set (henceforward called "its point"); (ii) the subdomains are either disjoint or may overlap in such a way that each subdomain contains "its point" in its interior; (iii) the subdomains are bounded and simply connected. Starting from two families of radial generalized expo-rational B-splines (GERBS), one of which has Hermite interpolation property at the given point set, and the other one forms a smooth convex resolution of unity associated with the given subdomain partition, an explicit algebraic construction of a new family of basis functions is designed which combines the property of Hermite interpolation with that of smooth convex resolution of unity. Multivariate Hermite interpolation at the scattered point set is achieved by a linear combination of the new basis functions where the coefficient of each basis function is a (tensor-product) Taylor polynomial centered at "its point" and including all partial derivatives up to the total order of Hermite interpolation, which order may vary at different elements of the scattered point set. Once the vector field has been Hermite-interpolated, it is very easy to switch from interpolatory to Bézier form of the presentation, by changing the monomial bases in each variable used in the tensor-product Taylor polynomial to tensor-product Bernstein polynomial bases, where the Bernstein polynomials are scaled to a hyper-rectangle containing the support of the original radial GERBS families. The construction is readily parallelized. The assumptions on the partition subdomains are very general and include disjoint convex covers such as Voronoi tilings as well as overlapping star-shaped covers such as the star-1 neighbourhoods of the vertices in a triangulation (simplectification) in dimensions 2, 3 and higher. Replacing the Taylor polynomial coefficient by polynomial coefficients of the same total degree which are optimal with respect to a local least-squares or K-functional criterion, in combination with the resolution of unity, provides high-quality data fitting and multivariate approximation. In the case of Hermite interpolation, the respective generalized Vandermonde matrix is always in Jordan normal form.

*Keywords and Phrases:* Expo-rational B-splines, Euler Beta-function B-splines, Hermite interpolation.

## 1. Introduction

Expo-rational B-splines (ERBS) and their generalization, "generalized ERBS" (GERBS) provide a new mathematical tool for multivariate approximation and iso-geometric representation, with application to geometric modelling in Computer-Aided Geometric Design (CAGD) and Finite Element/Volume Analysis (FEA/FVA). For a detailed introduction to the emerging theory of ERBS and GERBS, see [3], [2].

At the Seventh International Conference on Mathematical Methods for Curves and Surfaces in Tønsberg, Norway, in June 2008, at the lecture of Thomas J. R. Hughes, on which the author of the present work was also present, Tom Hughes informed the geometric modelling community of the world of his vision of a united approach to geometric modelling in CAGD and FEA in the modelling and simulation via boundary-value problems for PDEs. The main common tool which Tom Hughes proposed was the current industrial standard in CAGD: Non-Uniform Rational B-splines (NURBS), and the NURBS-based methods proposed by him gave the start of *Iso-geometric Analysis* (IGA).

On the next day of the afore-mentioned conference, the author of the present paper gave for the first time a communication [1] on the topic of GERBS, with Tom Hughes, Larry Schumaker, Tom Lyche and other well-known spline specialists in the audience. What seemed to impress the audience most was the idea that it is possible to construct GERBS-based smooth convex partitions of unity on triangulations, where each GERBS had the support of the usual piecewise linear/affine B-spline (i.e., the star-1 neighbourhood of "its" vertex in the triangulation) while at the same time GERBS was smooth, and multiplication of each GERBS with a coefficient which was not constant, but a Taylor polynomial "around the vertex of the GERBS" immediately implied Hermite interpolation at this vertex of all derivatives present in the Taylor polynomial. The conversion to "Bézier form" was also done effortlessly by simply changing the monomial basis in the Taylor polynomial around each vertex with respective *tensor-product* Bernstein basis; moreover, this conversion was done independently for every vertex in the triangulation, i.e., the procedure was readily parallelized.

During and after this conference there was a lot of interest in the four multivariate constructions based on GERBS, announced in [1]. All of these four constructions provide Hermite interpolation on scattered point sets in domains in $\mathbb{R}^n$, $n \in \mathbb{N}$, combined with smooth, convex, compactly supported resolution of unity for several general types of partitions of these domains. Despite of the interest shown, the author requested (at least) one year more to work on the development of the theory before starting to publish relevant results. This is why the first results on this topic began to appear in the late 2009 and in 2010 and, in particular, in the present work. Here we shall address the most general of the four constructions announced at the Tønsberg conference. Namely, in this case the only constraints on the sub-domains forming the domain partition are that these sub-domains should be simply connected and bounded; both cases of overlapping and mutually disjoint partitions can be handled, the difference being in the selection of some of the parameters of the constructed partition of unity.

This construction is based on the use of two separate families of basis functions: one which has all the necessary Hermite interpolation properties, and another which has the necessary properties of a smooth convex resolution of unity. The constructions of both of these two bases are well-known; the new part of the construction is the combined use of these bases for the derivation of a new basis which enjoys having all above-said interpolation and unity resolution properties simultaneously. Moreover, when the partitioned domain in $\mathbb{R}^n$ is bounded, the new resolution of unity is also compactly supported.

The purpose of the present paper is to provide the details of this construction as it was announced in [1]. In particular, the two initial functional bases in the construction will both be assumed radial, as in the exposition of [1].

## 2. The Univariate ERBS and Its Generalizations

### 2.1. ERBS - Definition

ERBS were defined in [3], as follows.

Let $\overrightarrow{t}$ be a strictly increasing knot vector, i.e., $\overrightarrow{t} = \{t_0, t_1, \dots, t_{N+1}\}$, $t_k \in \mathbb{R}$, $k = 0, \dots, N+1$, with $t_k < t_{k+1}$, $k = 0, \dots, N$.

**Definition 1.** The expo-rational B-splines (ERBS) associated with the knot vector $\overrightarrow{t}$ is defined by [3]:

$$B_k(t) = \begin{cases} \int_{t_{k-1}}^{t} \varphi_{k-1}(s)\,ds, & t_{k-1} < t \le t_k, \\ 1 - \int_{t_k}^{t} \varphi_k(s)\,ds, & t_k < t < t_{k+1}, \\ 0, & \text{elsewhere on } \mathbb{R}, \end{cases}$$

with

$$\varphi_k(t) = \frac{e^{-1/((t-t_k)(t_{k+1}-t))}}{\int_{t_k}^{t_{k+1}} e^{1/((s-t_k)(t_{k+1}-s))}\,ds} \tag{1}$$

$k = 1, \dots, N$.

### 2.2. Basic Properties of ERBS

**P1.** $B_k(t) \begin{cases} > 0, & t_{k-1} < t < t_{k+1}, \\ = 0, & \text{elsewhere on } \mathbb{R}, \end{cases}$
$k = 1, \dots, N$;

**P2.** $\sum_{k=1}^{N} B_k(t) = 1$, i.e.,
$B_k(t) + B_{k+1}(t) = 1$, $t_k < t \le t_{k+1}$, $k = 1, \dots, N-1$;

**P3.** $B_k(t_k) = 1$, $k = 1, \dots, N$;

**P4.** If $t_{k-1} < t_k < t_{k+1}$, then $\frac{d^j}{dt^j} B_k(t_i) = 0$, $j = 1, 2, \dots,$ $k = 1, \dots, N$;

**P5.** If $t_{k-1} < t_k < t_{k+1}$, then $B_k \in C_0^\infty(\mathbb{R}) \subset C^\infty(\mathbb{R})$ and $B_k$ is analytic on $\mathbb{R} \setminus \{t_{k-1}, t_k, t_{k+1}\}$, $k = 1, \dots, N$.

For a more detailed consideration of the basic properties of ERBS, see [3].

### 2.3. ERBS Linear Combinations with Functional Coefficients

An ERBS (scalar-valued, vector-valued in a vector space, or point-valued in an affine frame) function $f(t)$ is defined [3] on $(t_1, t_N]$ by

$$f(t) = \sum_{k=1}^{N} l_k(t) B_k(t), \qquad t \in (t_1, t_N],$$

where $l_k(t)$ are local (scalar, vector-valued, or point-valued) functions defined on $(t_{k-1}, t_{k+1})$, $k = 1, \dots, N$. Note that, due to the properties of ERBS, Hermite interpolation at $t_k$ of $l_k(t)$ is available up to transfinite order.

### 2.4. Taylor Expansions as Local Functions

The properties of ERBS ensure also that the Hermite interpolant at the knot $t_k$ with multiplicity $r_k \le \infty$, $k = 1, \dots, N$, is obtained via the following lucid formula [3]

$$f(t) = H(g; t) = \sum_{k=1}^{N} \Big[ \sum_{j=0}^{r_k - 1} \frac{(t - t_k)^j}{j!}\, g^{(j)}(t_k) \Big] B_k(t), \qquad t \in [t_1, t_N], \tag{2}$$

When $r_k = +\infty$, $k = 1, \dots, n$, Hermite interpolation by ERBS is exact on all analytic functions on $[t_1, t_N]$.

In the context of the present paper, we shall be considering (2) in multivariate context, when the family $\{B_k\}_{k=1}^N$ consists of $n$-variate functions, $n \in \mathbb{N}$, and $n$-variate Taylor polynomial coefficients of total degree $r_k - 1$ are used in (2), resulting in

$$F(x) = \sum_{i=1}^{N} l_i(x) B_i(x), \qquad x \in \Omega \subset \mathbb{R}^n. \tag{3}$$

In the case of ERBS, the multiplicities can again be infinite, and in this case the Taylor polynomials are again replaced by Taylor series.

In the sequel of this work we shall find an appropriate setting of the Hermite interpolation problem on a scattered-point set $\{x_i\}_{i=1}^N$, $N \in \mathbb{N}$, $x_i \in \mathbb{R}^n$, $i = 1, \dots, N$, and an appropriate construction of the function family $\{B_i\}_{i=1}^N$.

### 2.5. Generalized ERBS

Generalized ERBS (GERBS) were defined in [2]. Here we shall mention only one instance of GERBS, namely, a smooth, but not infinitively smooth, family of polynomial GERBS based on the incomplete Euler Beta-function.

**Definition 2.** Beta-function B-spline (BFBS): the expo-rational density on $[t_k, t_{k+1}]$ in (1) is being replaced by the Bernstein polynomial density

$$\psi_k(t) = \binom{r_k + r_{k+1}}{r_k} \frac{(t - t_k)^{r_k} (t_{k+1} - t)^{r_{k+1}}}{(t_{k+1} - t_k)^{r_k + r_{k+1}}}, \qquad t \in [t_k, t_{k+1}], \tag{4}$$

where $r_k, r_{k+1} \in \mathbb{N}$ are multiplicities at knots $t_k, t_{k+1}$, respectively.

This GERBS construction provides a less smooth resolution of unity. More precisely, it is $C^\mu$-smooth, where

$$\mu = \min_{0 \le k \le N+1} r_k,$$

and the multiplicity of its Hermite interpolation in (2) is limited by the order of its smoothness. Both of these constraints are due to the limitation imposed by the finite values of $r_k$ in the definition of BFBS. On the other hand, BFBS are simpler to compute than ERBS [1], [3].

## 3. Main Results

### 3.1. Problem Setting

Assume that $\Omega \subset \mathbb{R}^n$, $n = 1, 2, \dots$, is a bounded domain (open, simply connected set, possibly together with part or all of its boundary, with its interior

$Int(\Omega)$ being non-empty, and its closure $\overline{\Omega}$ and boundary $\partial\Omega$ being compact), and let $x_i \in Int(\Omega_i) \subset \Omega$, $i = 1, \ldots, N$, be a scattered point set. Here $\Omega_i$, $i = 1, \ldots, N$, are all simply connected subsets of $\mathbb{R}^n$ with non-empty interiors, compact closures and forming a cover of $\overline{\Omega}$:

$$\bigcup_{i=1}^{N} \overline{\Omega_i} \supseteq \overline{\Omega}.$$

**Example 1.** Voronoi diagrams (non-overlapping).

**Example 2.** Star-1 neighbourhoods of the vertices $x_i$, $i = 1, \cdots, N$ in a Delaunay triangulation (overlapping).

**Example 3.** The spectrum of an $M \times M$-square matrix $\{z_j \in \mathbb{C},\ j = 1, \ldots, N\}$, with respective multiplicity $r_j$, $j = 1, \ldots, N$, so that $\sum_{j=1}^{N} r_j = M$ (only scattered-point set, without definition of a domain partition).

**Example 4.** A finite atlas of maps of a compact smooth manifold (only a finite, compactly supported, resolution of unity, without specifying a scattered-point set for Hermite interpolation).

Let $|\cdot|$ stand for the usual Euclidean norm in $\mathbb{R}^n$. This norm belongs to $C^\infty(\mathbb{R}^n \backslash \{0\})$ which ensures that radial (G)ERBS are $C^\infty$-smooth. In the sequel we shall adhere to this norm and the resulting families of radial-basis functions. (For the use of another norm and the resulting function families, see the concluding remarks.) The reference multivariate radial (G)ERBS, centered at the origin, and scaled to have unit hyper-spherical support, is therefore defined on $\mathbb{R}^n$, as follows:

$$B(x) = b(|x|), \tag{5}$$

where $x \in \mathbb{R}^n$, and $b(t)$, $t \in \mathbb{R}$, is a standard (G)ERBS with center at 0 and support scaled to $[-1, 1]$.

Now we proceed to define the two function families used in the new construction. We begin by defining two sets of positive constants, $R_i$ and $\rho_i$, $0 < \rho_i \le R_i < +\infty$, $i = 1, \ldots, N$, as follows:

$$\max_{x \in \partial\Omega_i} |x - x_i| \le R_i < +\infty, \tag{6}$$

$$0 < \rho_i \le \min\Big\{ \min_{x \in \partial\Omega_i} |x - x_i|,\ \min_{j \ne i}\{|x_i - x_j|,\ j = 1, \ldots, N\} \Big\}. \tag{7}$$

Formulae (6), (7) are valid in the most general case, when $\overline{\Omega}_i$, $i = 1, \ldots, N$, form a possibly overlapping cover of $\overline{\Omega}$, and such that it may happen that, for one or more indices $j$, $Int(\overline{\Omega}_j)$ contains some other elements of the scattered point set $\{x_i\}_{i=1}^N$ besides "its point" $x_j$. Here are some special cases when (6) and (7) simplify.

- Consider the special case when every $Int(\overline{\Omega}_i)$, contains only "its point" $x_i$ (although some other elements of the scattered point set may be situated on $\partial\Omega_i$, see, e.g., Example 2). (This includes the particular case of non-overlapping tiles, see, e.g., Example 1.) In this case, (7) simplifies to

$$0 < \rho_i \leq \min_{x \in \partial\Omega_i} |x - x_i|, \tag{8}$$

  for $i = 1, \ldots, N$.

- Consider the case when mesh-free Hermite interpolation at a scattered-point set is considered, see, e.g., Example 3. Here (6), (7) reduce to

$$0 < \rho_i \leq \min_{j \neq i} \{|x_i - x_j|,\ j = 1, \ldots, N\}, \tag{9}$$

$$\rho_i \leq R_i < +\infty, \tag{10}$$

  for $i = 1, \cdots, N$.

- In the case when there is interest only in the smooth convex compactly supported resolution of unity on some domain partition, without the aspects related to Hermite interpolation, it is an appropriate choice for every $i = 1, \ldots, N$ to select the hypersphere $S_i(\rho_i)$ inscribed in $\overline{\Omega}_i$ so that it has the maximal possible radius $\rho_i$. In this case the scattered-point set $\{x_i\}_{i=1}^N$ of the respective centers of $\{S_i(\rho_i)\}_{i=1}^N$ will still have the Hermite interpolation property, with multiplicities $\{r_i\}_{i=1}^N$ depending on the choice of the (G)ERBS in (5). A respective choice of $\{R_i\}_{i=1}^N$ would then be determined via (6). Note that for general partitions $\{\Omega_i\}_{i=1}^N$ it is possible that for some $j = 1, \ldots, N$ the center of a minimal-radius circumscribed hypersphere around $\overline{\Omega}_j$ may not belong to $Int(\overline{\Omega}_j)$.

### 3.2. The Families $\{\varphi_i\}$, $\{\psi_i\}$

Define

$$\varphi_i(x) = B\Big(\frac{x - x_i}{\rho_i}\Big), \tag{11}$$

where $x \in \mathbb{R}^n$, $\rho_i > 0$ and $x_i \in \mathbb{R}^n$ is an element of the given scattered-point set, $i = 1, \ldots, N$, and $b$ in the definition of $B$ in (5) is an ERBS. Then, the following lemma holds true.

**Lemma 1.**

$$\varphi_i \in C^\infty(\mathbb{R}^n), \quad 0 \leq \varphi_i(x) \leq 1, \qquad \forall x \in \mathbb{R}^n. \tag{12}$$

*Proof (outline).* Follows from the definition of $b$ and $B$ in (5), (11) and the $C^\infty$-smoothness of $|\cdot|$ away from the origin. □

**Remark 1.** In the case when $b$ in the definition of $B$ in (5) is a BFBS, the claim about $C^\infty$-smoothness is replaced with a respective one about $C^k$-smoothness for a corresponding $k \in \mathbb{N}$.

It can be seen that:

- If a linear combination is formed of the functions $\varphi_i$, $i = 1, \ldots, N$, with functional coefficients which are Taylor-expanding polynomials, then, the family $\{\varphi_i\}$ provides Hermite interpolation at all $x_j$, $j = 1, \ldots, N$, in such a way, that only one function of the family, namely $\varphi_i$, is "responsible" for the Hermite interpolation value at "its point" $x_i$, $i = 1, \cdots, N$.

- The supports of $\varphi_i$, $\operatorname{supp} \varphi_i$, $i = 1, \cdots, N$, may not provide a cover of $\Omega$:

$$\bigcup_{i=1}^{N} \operatorname{supp} \varphi_i \subsetneqq \Omega. \tag{13}$$

Next, define the family $\{\psi_i\}$, as follows:

$$\psi_i : \psi_i(x) > 0, x \in Int(\operatorname{supp} \psi_i), \tag{14}$$

where

$$\operatorname{supp} \psi_i = \{x : |x - x_i| \leq R_i\}, \qquad i = 1, \ldots, N. \tag{15}$$

Depending on the selection of the Euclidean norm in $\mathbb{R}^n$, every $\psi_i$, $i = 1, \ldots, N$, is also chosen here to be a radial (G)ERBS:

$$\psi_i(x) = B\Big(\frac{x - x_i}{R_i}\Big), \tag{16}$$

where $x$ and $x_i$ are as in the definition of $\varphi_i$ in (11). For the family $\{\psi_i\}$ we have the following properties:

- The union of supports of $\psi_i$, $i = 1, \cdots, N$, forms a cover of $\Omega$:

$$\bigcup_{i=1}^{N} \operatorname{supp} \psi_i \supseteqq \Omega, \qquad \psi \in C^\infty(\mathbb{R}^n). \tag{17}$$

- $\psi_i$ does not necessarily share the Hermite interpolation property of the $\varphi_i$ at $x_j$, $j = 1, \ldots, N$, $i = 1, \ldots, N$, in the sense that when these functions are multiplied with polynomial coefficients, the generalized Vandermonde matrix of the interpolation process with the family $\{\psi_i\}$ can eventually be band-limited but not block-diagonal, as is the case with the family $\{\varphi_i\}$.

- Remark 1 holds for $\{\psi_i\}$, mutatis mutandis.

### 3.3. The Auxiliary Families $\{\Phi_i\}$, $\{\Psi_i\}$

Using $\{\psi_i\}$, we define on $\Omega$ the following auxiliary function family $\{\Psi_i\}$

$$\Psi_i(x) = \frac{\psi_i(x)}{\sum\limits_{i=1}^{N} \psi_i(x)}. \tag{18}$$

**Lemma 2.** *The family $\{\Psi_i\}$ provides smooth, convex, compactly supported resolution of unity on $\Omega$:*

$$\Psi_i \in C^\infty(\mathbb{R}^n),\ 0 \le \Psi_i(x) \le 1,\ i = 1, \dots, N,\ \sum_{i=1}^{N} \Psi_i(x) \equiv 1,\quad \forall x \in \Omega, \tag{19}$$

*Proof (outline).* This modification of a well-known lemma in geometric modelling follows by a direct application of the definitions in (5), (6), (11) and (18). □

**Remark 2.** *The family $\{\Psi_i\}$ does not necessarily have the Hermite interpolation property of $\{\varphi_i\}$, in the sense, specified above for $\{\psi_i\}$.*

Using $\{\varphi_i\}$, we design the auxiliary function family $\{\Phi_i\}_{i=0}^N$, as follows:

$$\Phi_i(x) = \varphi_i(x) \prod_{j=1}^{i-1} (1 - \varphi_j(x)), \qquad i = 1, \dots, N, \tag{20}$$

$$\Phi_0(x) = \prod_{j=1}^{N} (1 - \varphi_j(x)). \tag{21}$$

**Lemma 3.** *The family $\{\Phi_i\}_{i=0}^N$ satisfies*

$$\Phi_i \in C^\infty(\mathbb{R}^n),\ 0 \le \Phi_i(x) \le 1,\quad i = 0, \dots, N,\quad \sum_{i=0}^{N} \Phi_i(x) \equiv 1,\ \forall x \in \Omega. \tag{22}$$

*Proof (outline).* This is a well-known lemma in analysis (in proving Stone's theorem, representing the maps and atlas of a smooth manifold, etc.). □

**Remark 3.** Family $\{\Phi_i\}_{i=0}^N$ has both the needed smooth convex resolution property and the Hermite interpolation property at $x_i$, $i = 1, \dots, N$, similar to the family $\{\varphi_i\}$. However, it still has one important drawback: the resolution of unity and Hermite interpolation via $\{\Phi_i\}_{i=0}^N$ does not imply good approximation properties of the Hermite interpolant, because $\operatorname{supp} \Phi_0$ is not well localized. Namely, $\operatorname{supp} \Phi_0$ is not simply connected and its diameter $\operatorname{diam}(\operatorname{supp} \Phi_0)$ may remain large for any choice of $N$. In fact, it is easy to construct examples where $\operatorname{diam}(\operatorname{supp} \Phi_0) = \operatorname{diam}(\operatorname{supp} \Omega)$.

### 3.4. New Smooth, Convex, Compactly Supported Resolution of Unity with Hermite Interpolation Property

Now we proceed to define a new multivariate B-spline family $\{B_i\}_{i=1}^N$, as follows.

**Definition 3.**

$$B_i(x) := \Phi_i(x) + \Psi_i(x)\Phi_0(x), \qquad x \in \Omega, \quad i = 1, \ldots, N. \tag{23}$$

Definition 3, together with the following Theorem 1, constitutes the main new result of the present study.

**Theorem 1.** *Let the B-spline function family be defined via formula* (23)*. Then:*

1. $\{B_i\}_{i=1}^N$ *provides smooth convex resolution of unity on* $\Omega$.
   (a) *If* $b$ *in* (5) *is an ERBS, then, the partition of unity is* $C^\infty$*-smooth.*
   (b) *If* $b$ *in* (5) *is a* $C^\mu$*-smooth BFBS,* $\mu \in \mathbb{N}$*, then, the partition of unity is* $C^\mu$*-smooth.*
2. *All* $B_i$, $i = 1, \ldots, N$, *have compact support.*
3. *If the functional coefficients* $l_i$, $i = 1, \ldots, N$, *are all chosen to be Taylor expansions, then, the function system* $\{B_i\}_{i=1}^N$ *exhibits the Hermite interpolation property. (See the properties of the families* $\{\varphi_i\}_{i=1}^N$, $\{\Phi_i\}_{i=0}^N$ *and Remark 3.)*
   (a) *If* $b$ *in* (5) *is an ERBS, then the Taylor-expanding polynomial coefficients* $l_i$*, can be of arbitrarily large total degree* $r_i - 1$ *in* (3)*, including the possibility* $r_i - 1 = +\infty$ *(Taylor series). In the latter case, the Hermite interpolation is of transfinite order, and if* $r_i - 1 = +\infty$ *for all* $i = 1, \ldots, N$*, then* (3) *is exact on all functions which are analytic on* $\Omega$.
   (b) *If* $b$ *in* (5) *is chosen to be a* $C^\mu$*-smooth BFBS,* $\mu \in \mathbb{N}$*, then the total degrees* $r_i - 1$ *of the Taylor expanding polynomial coefficients* $l_i$ *in* (3) *are bounded by* $\mu$*, together with the maximal possible order of Hermite interpolation. In this case, the Hermite interpolation is only of finite order bounded by* $\mu$*, and* (3) *is exact on the multivariate polynomials of total degree* $\nu - 1$*, where* $\nu = \min\limits_{1 \le i \le N} r_i$, $k = 1, \ldots, N$.
4. $\{B_i\}_{i=1}^N$ *is a linearly independent function system.*

*Proof (outline).*

*Proof of Item 1.* Subitem (a) follows straightforwardly from the $C^\infty$-smoothness of $|\cdot|$ away from the origin and from (5), (11), (17)–(23). Subitem (b) follows by a modification of the argument for subitem (a) similar to the one in Remark 1. To prove that $\{B_i\}_{i=1}^N$ provides a convex resolution of unity on $\Omega$, we first derive the following identity

$$\sum_{i=1}^N l_i(x)\underbrace{[\Phi_i(x)+\Psi_i(x)\Phi_0(x)]}_{B_i(x)} = \sum_{i=1}^N l_i(x)\Phi_i(x) + \Big[\sum_{i=1}^N l_i(x)\Psi_i(x)\Big]\Phi_0(x), \quad (24)$$

where $x \in \Omega$, and $l_i(x)$, $i = 1,\dots,N$, are functional coefficients. The identity (24) follows by commuting the order of summation. Selecting $l_i(x) \equiv 1$ in (24) for every $x \in \Omega$ and every $i = 1,\dots,N$, (19) and (22) imply that $\{B_i\}_{i=1}^N$ provides a convex resolution of unity on $\Omega$.

*Proof of Item 2.* From (20) it follows that

$$\operatorname{supp}\Phi_i \subset \operatorname{supp}\varphi_i, \qquad i = 1,\dots,N. \quad (25)$$

On the other hand, (18) implies

$$\operatorname{supp}\Psi_i = \operatorname{supp}\psi_i, \qquad i = 1,\dots,N. \quad (26)$$

By (11) and (16), in view of $0 < \rho_i \le R_i < +\infty$, $i = 1,\dots,N$,

$$\operatorname{supp}\varphi_i \subset \operatorname{supp}\psi_i, \qquad i = 1,\dots,N, \quad (27)$$

holds. Therefore, (25)–(27) together with (23) imply

$$\operatorname{supp}\Phi_i \subset \operatorname{supp}B_i \subset \operatorname{supp}\Psi_i = \operatorname{supp}\psi_i, \qquad i = 1,\dots,N, \quad (28)$$

which proves Item 2, since $\psi_i$, $i = 1,\dots,N$, are compactly supported.

*Proof of Item 3.* By construction of the family $\{\Phi_k\}_{k=1}^N$ in (11), (20), (21), the functions $\Phi_j$, $j = 1,\dots,N$, $j \ne i$, are all identically zero in the hyper-ball with radius $\rho_i$ and center at $x_i$; this also implies that $\Phi_0$ is zero at $x_i$, together with all of its partial derivatives, for all $i = 1,\dots,N$. Now (23) implies that $B_i(x_j) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta, and that all partial derivatives of $B_i$ are zero at $x_j$, for all $i = 1,\dots,N$, $j = 1,\dots,N$; hence, $\{B_i\}_{i=1}^N$ has the Hermite interpolation property. (See also Remark 3.)

*Proof of Item 4.* The argument is the same as with piecewise affine B-splines. We omit the details. □

**Corollary 1.** *The (G)ERBS family $\{B_i\}_{i=1}^N$ for the scattered point set $\{x_i\}_{i=0}^N$ and the respective domain partition $\Omega = \bigcup_{i=0}^N \Omega_i$, given by* (23), *admits the following equivalent representation:*

$$B_i(x) = \prod_{j=1}^{i-1}(1-\varphi_j(x))\Big[\varphi_i(x) + \frac{\psi_i(x)}{\sum_{k=1}^N \psi_k(x)}\prod_{j=i}^N(1-\varphi_j(x))\Big], \quad (29)$$

*where $x \in \Omega$ and $i = 2, \ldots, N-1$, with respective modifications in* (29) *for $i = 1$ and $i = N$.*

*If, additionally, the radii $\rho_i$, $i = 1, \ldots, N$, are small enough, so that*

$$Int(\operatorname{supp}\varphi_i) \cap Int(\operatorname{supp}\varphi_j) = \emptyset, \qquad i, j = 1, \ldots, N, \ i \neq j \tag{30}$$

*holds, then* (29) *can be simplified to the following equivalent representations:*

$$B_i(x) = \varphi_i(x) + \frac{\psi_i(x)}{\sum\limits_{k=1}^{N} \psi_k(x)} \Big(1 - \sum_{j=1}^{N} \varphi_j(x)\Big), \tag{31}$$

$$B_i(x) = \frac{1}{\sum\limits_{k=1}^{N} \psi_k(x)} \Big[\Big(\sum_{\substack{j=1 \\ j\neq i}}^{N} \psi_j(x)\Big)\varphi_i(x) + \Big(1 - \sum_{\substack{j=1 \\ j\neq i}}^{N} \varphi_j(x)\Big)\psi_i(x)\Big], \tag{32}$$

*where $x \in \Omega$ and $i = 2, \ldots, N-1$, with respective modifications in* (31) *and* (32) *for $i = 1$ and $i = N$.*

*Proof (outline).* Formula (29) follows by substituting $\Phi_i$, $\Psi_i$ and $\Phi_0$ in (23) by the right-hand sides in (20), (18), (21), respectively.

To obtain (31) from (29), observe that (30) implies (in fact, is equivalent to):

$$\varphi_i(x)\varphi_j(x) \equiv 0, \qquad \forall x \in \Omega, \quad i, j = 1, \ldots, N, \quad i \neq j. \tag{33}$$

Finally, (32) follows from (31) by commuting the order of summation and simple calculations. □

**Remark 4.** It is always possible to select $\rho_i$, $i = 1, \ldots, N$, in (7)–(9) so that (30) holds true. For this purpose, it suffices to decrease the value of the upper bound in the right-hand sides of (7)–(9) to 1/2 of its value as given in (7)–(9).

**Corollary 2.** *Definition 3 is independent (modulo permutation of the indices $i = 1, \ldots, N$) of the ordering of the scattered-point set $\{x_i\}_{i=1}^N$ and the respective ordering of the elements of the domain partition/cover $\{\Omega_i\}_{i=1}^N$, at least, when condition* (30) *is fulfilled.*

*Proof (outline).* The assertion follows from (31). □

## 4. Concluding Remarks

**Remark 5.** (Varying the metric in $\mathbb{R}^n$.) In the present construction, the Euclidean norm in $\mathbb{R}^n$ can be replaced by other smooth norms in $\mathbb{R}^n$ such as, e.g., weighted $l_2$-norms, leading to (hyper-)ellipsoidal supports of the involved function families. In this context, the use of smooth norms in $\mathbb{R}^n$ which are not Hilbert norms (e.g., $l_p$-norms, $1 < p < +\infty$), as well as smooth metrics in $\mathbb{R}^n$ which are not norms (e.g., $d(x,y) = \frac{|x-y|}{1+|x-y|}$, where $|\cdot|$ is the Euclidean norm) is also possible, leading to more general supports with smooth boundary. Quite a remarkable case arises if the Euclidean norm be replaced by the $l_p$-norm in $\mathbb{R}^n$ for $p = \infty$ or $p = 1$. In both of these cases this norm is not smooth and not uniformly convex, and the supports of the function families involved (squares, cubes or hyper-cubes, depending on the value of $n$) have non-smooth boundary. Nevertheless, if in this case the smooth radial-basis function families $\{\varphi_i\}$ and $\{\psi_i\}$ be replaced by smooth tensor-product ones, the construction in Theorem 1 yields smooth B-splines with the same rectangular supports as $\{\psi_i\}$. Each of the radial and tensor-product case has its own advantages, as follows:

- computing curvature and other quantities requiring higher-order derivatives is much more convenient in the tensor-product case;
- the radial-basis construction is computationally more robust with respect to transformations of the domain $\Omega$.

**Remark 6.** (Dependence/independence of the basis in (23), (29) on the ordering the scattered point set.)

- Corollary 2 tells us that if (30), (33) are fulfilled then Definition 3 is independent (modulo permutation of the indices $i = 1, \ldots, N$) of the ordering of the scattered-point set and the respective ordering of the elements of the domain partition/cover.
- It can be shown that in the complementary case, i.e., when (30), (33) are not fulfilled, the basis in (23), (29) does depend on the ordering of the scattered-point set and the corresponding ordering the domain partition/cover. (This important observation was made first by author's Ph.D. student Peter Zanaty.) The variety of possible different bases depends on:
  - the maximal number of elements of $\{\varphi_i\}$ for which the intersection of the interiors of their supports is non-empty;
  - eventual local symmetries in the geometric positioning of the elements of the scattered-point set and/or the respective domain partitioning/cover.

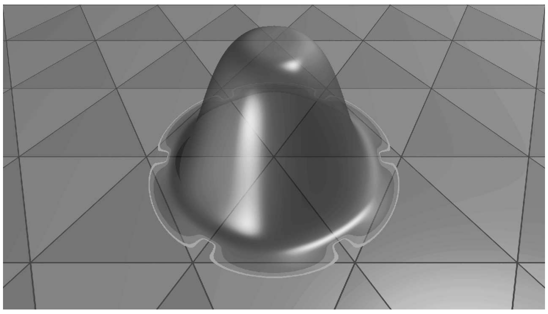
**Example 5.** For instance, consider a polygonal domain in $\mathbb{R}^n$ which can be uniformly triangulated, and let the scattered-point set $\{x_i\}_{i=1}^N$ be the set of vertices in the triangulation, while the domain is covered by $\{\Omega_i\}_{i=1}^N$, where $\Omega_i$ is the star-1 neighbourhood of $x_i$ (i.e., the union of all triangles in the triangulation which share $x_i$ as a vertex). In this case:

- the elements of the cover are all identical uniform hexagons (or parts of such hexagons for those vertices of the triangulation which are on the eventual boundary of $\Omega$);
- the constants $R_i$ can be selected so that the maximal number of elements of $\{\varphi_i\}$ with non-empty intersection of the interiors of their supports is at most 3 for any choice of $\rho_i : 0 < \rho_i \leq R_i$;
- there are many local symmetries.

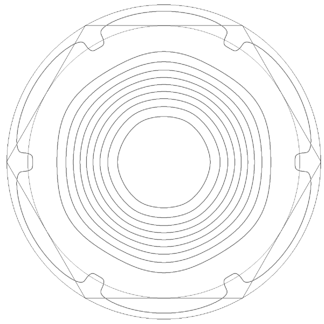
Due to the uniformity and symmetry of this scattered-point set and domain cover, $R_i \geq R$ and $\rho_i \leq \rho$, $i = 1, \ldots, N$, where $R$ and $\rho$ are the radii of the circumscribed and inscribed circle of the uniform hexagon, respectively. If we select $R_i = R$, $\rho_i = \rho$, $i = 1, \ldots, N$, we get a version of (23), (29) with minimally supported $\{\psi_i\}$ and maximally supported $\{\varphi_i\}$, where the number of elements of $\{\varphi_i\}$ with non-empty intersection of the interiors of their supports is equal to 3. In this case there is dependence of the elements of the basis (23) on the ordering of the vertices of the triangulation as elements of the scattered-point set. The author's Ph.D. student Peter Zanaty showed that in this case there are 13 different shapes of the graphs of the B-splines appearing in (23), all of which have the same support, peak with value equal to 1 at the centre of their support, and have all partial derivatives equal to zero at the centre and the boundary of their support. Depending on the index of the vertex $x_i$ in the scattered-point set, the graph of "its" B-spline in (23) takes one of the above 13 shapes. (Here we consider only B-splines which peak at vertices in the interior of the domain; the study of shapes of B-splines peaking at vertices on the boundary requires separate consideration, not included here.) A graphical illustration of this example is given in Figure 1. On sub-figures 1(a) and 1(c) are given 3D-perspective views of two of the 13 graphs, while sub-figures 1(b) and 1(d) contain respective iso-level views of the same graphs. Alternatively, if the values of $\rho_i$ be reduced to $\rho/2$ (or less), then, (30), (33) are fulfilled, and by Corollary 2 the B-spline basis appearing in (23) is independent of the ordering of the scattered-point set. Hence, due to uniformity and symmetry, the graphs of all B-splines in this basis have one and the same shape – see sub-figures 1(e) and 1(f), respectively, for the 3D-perspective and iso-level view of the graph in this case.

**Remark 7.** Some of the potential applications of this construction are given in the following (inexhaustive) list.

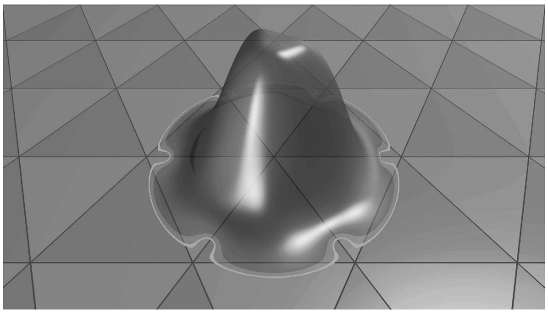
- For interpolation and approximation of functions of one or several real or complex variables.
- For mesh-free methods of solving operator equations, especially PDE's.
- For the generation of finite elements associated with very general domain partitions. In the cases when the supports of some of the new B-splines



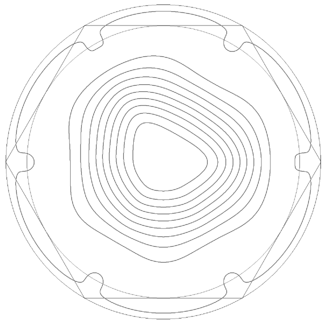
(a) $\rho_i = \rho$, case 1 of 13, 3D-perspective view



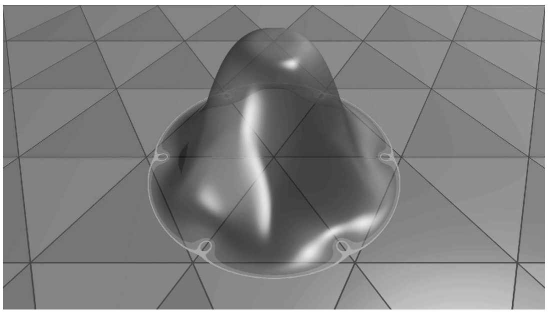
(b) iso-level view of (a)



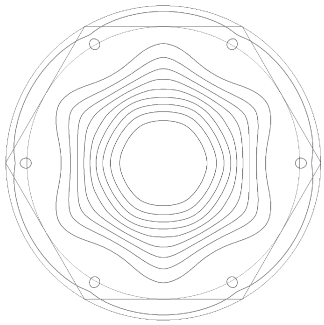
(c) $\rho_i = \rho$, case 10 of 13, 3D-perspective view



(d) iso-level view of (c)



(e) $\rho_i = \rho/2$, unique case, 3D-perspective view



(f) iso-level view of (e)

**Figure 1.** Graphical illustration of Example 5. The 3D-views were generated via ray-tracing. All graphics was designed by the author's Ph.D. student Peter Zanaty.

intersect the boundary of the domain, the restrictions (exact or approximate) of these B-splines onto the domain can be used for transferring information from the boundary, such as initial-value and/or boundary-value conditions.

- In geometric modelling of curves, surfaces and volume deformations. To this end, the tensor-product monomial basis of the Taylor-expanding polynomials of a given total degree $r_k - 1$ is being changed to the tensor-product Bernstein basis of the same total degree, scaled to the Cartesian-product support of $\Psi_k$ and $B_k$, $k = 1, \ldots, N$. Analogously to the upgrading of Bézier curves and surfaces to rational Bézier ones and of polynomial B-spline curves and surfaces to NURBS-based ones, here it is also of considerable interest to develop respective rational forms of this construction.

- For parametrization of very general classes of compact smooth manifolds.
- For computation of analytic function of operators and the respective Cauchy-Riesz-Dunford operator integral.
- For deriving new representation of finite-rank and compact operators, etc.

Because of the big generality of this construction, it can be useful for both theoretical research and applied computational purposes. Together with the other three constructions proposed in [1], this construction has the potential of becoming a new versatile tool of CAGD, FEA and IGA.

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