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A Late Report on Interlacing of Zeros of Polynomials*

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In this short paper I try to answer questions raised by my teacher Borislav Bojanov which concern interlacing of zeros of real polynomials and consider two specific topics. The first one concerns one of his favorite results, a theorem due to Vladimir Markov which states that the derivatives of two polynomials with real interlacing zeros posses zeros which also interlace. The second is a problem about monotonicity of zeros of classical orthogonal polynomials and Sturm's comparison theorem for solutions of Sturm-Liouville differential equations.

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1. Introduction and Markov's Interlacing Property

Discussing mathematics with Professor Bojanov was a rare experience for the author of this note. It was pleasure and fun where ideas, challenge and jokes were composing an amalgama that I shall never forget and I shall miss. As I miss its main ingredient: Bojanov himself, his personality, rigor and smile. Though he had the ability of a theory builder, what he really adored was to be a problem solver. He used to appreciate very much papers containing a piece, a nice, clever and ingenious idea that one remembers forever. I remember when he saw for the first time the Collected Papers of Szegő [18], edited by Richerd Askey, while he was visiting our department in Brazil in 1997. He was reading exhaustively Szegő's papers and, some days before he left, Bojanov told me why he thought Szegő was a great mathematician. The reason, in Bojanov's opinion, was that not only Gabor Szegő saw important problems

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and solved them long before others, but also because every single paper of the great Hungarian master contained a piece, there was a nice little trick that catches one's thoughts, and even feelings, and that mixture made Szegő's work an art.

One of the favorite nice pieces Bojanov used to adore and comment frequently was a result of Vladimir Markov [10]. In order to formulate it, we introduce the notion of interlacing. In what follows we denote by π_n the space of algebraic polynomials of degree not exceeding n. Let $p(x)$ and $q(x)$ be real algebraic polynomials with only real distinct zeros and $p(x) = (x-x_1)(x-x_2)\cdots(x-x_n)$ and either $q(x) = q_n(x) = (x-y_1)(x-y_2)\cdots(x-y_n)$ or $q(x) = q_{n-1}(x) = (x-y_1)(x-y_2)\cdots(x-y_{n-1})$. Sometimes the real algebraic polynomials with real zeros are called hyperbolic ones. Also, the zeros of the first derivative of a polynomial are called its critical points. We say the the zeros of p and q interlace and write $p \prec q$ if

$$x_1 < y_1 < x_2 < \cdots < y_{n-1} < x_n < y_n$$

when $q(x) = q_n(x)$ or

$$x_1 < y_1 < x_2 < \cdots y_{n-1} < x_n$$

when $q(x) = q_{n-1}(x)$.

Theorem A (Vladimir Markov). *If $p \prec q$ then $p' \prec q'$.*

It is clear that this nice result states that the operator of differentiation preserves the property of interlacing of zeros of two polynomials. I was still in the beginning of my studies as MSc student when Bojanov showed me this result and made various comments on it. The first one was if one could find a proof different from the original one which is reproduced in Rivlin's book on Chebyshev polynomials [15]. That proof uses the Lagrange interpolation formula and somehow "hides" the nature, and even the beauty, of the statement. I remember I came up with a proof for the case when the polynomials p and q are both of degree n and proudly presented it to my teacher as I do now, without the proud of twenty-five years ago. The idea is rather simple. One thinks what would happen with the critical points if one "pushes a bit" only one of the zeros of p to the right. It turns out that all zeros of $p'(x)$ also go to the right so that we formulate the following:

Lemma 1. *If $p(x) = (x-x_1)(x-x_2)\cdots(x-x_n)$ has distinct and real zeros and $p'(x) = n(x-\xi_1)(x-\xi_2)\cdots(x-\xi_{n-1})$, then all the critical points $\xi_1, \xi_2, \ldots, \xi_{n-1}$ of p are increasing functions of each of its zeros x_k. More precisely, if*

$$p_\varepsilon(x) = (x-x_1)\cdots(x-x_{k-1})(x-x_k-\varepsilon)(x-x_{k+1})\cdots(x-x_n),$$

where ε is a sufficiently small positive number, and

$$p'_\varepsilon(x) = (x-\xi_1(\varepsilon))(x-\xi_2(\varepsilon))\cdots(x-\xi_{n-1}(\varepsilon)),$$

then $\xi_j < \xi_j(\varepsilon)$ for every $j = 1, \ldots, n-1$.

Similarly, if ε is a sufficiently small negative number, then $\xi_j(\varepsilon) < \xi_j$ for every $j = 1, \ldots, n-1$.

Proof. The proof is based on the simple technique of counting the sign changes. We consider only the case when $\varepsilon > 0$ and count the signs of the "new" polynomial $p_\varepsilon(x)$ at the critical points ξ_j of the "old" one. Since

$$p_\varepsilon(x) = p(x) - \varepsilon \tilde{p}(x), \quad \text{where} \quad \tilde{p}(x) = p(x)/(x - x_k),$$

$p'(x) = (x - x_k)\tilde{p}'(x) + \tilde{p}(x)$, and $p'(\xi_j) = 0$, then

$$p'_\varepsilon(\xi_j) = -\varepsilon \tilde{p}'(\xi_j) = \varepsilon \frac{\tilde{p}(\xi_j)}{(\xi_j - x_k)}.$$

Recalling that $\varepsilon > 0$, we obtain

$$\operatorname{sign} p'_\varepsilon(\xi_j) = \operatorname{sign}\,(\xi_j - x_k)\,\tilde{p}(\xi_j) = \operatorname{sign} p(\xi_j).$$

Therefore

$$\operatorname{sign} p'_\varepsilon(\xi_j) = (-1)^{n-j} = \operatorname{sign} p'(x_j)$$

Since $p'_\varepsilon(x)$ has a unique zero in (x_j, x_{j+1}), which we denote by $\xi_j(\varepsilon)$, and the sign of this polynomial at ξ_j is still the same is in the left end point x_j, it changes sign after ξ_j. Thus, $x_j < \xi_j(\varepsilon)$. □

Since my enthusiasm was not shared completely by Professor Bojanov and I saw only his curious smile when I showed him this simple argument, I realized he knew this proof. Nevertheless, I needed to prove a similar fact about symmetric polynomials some years later, when I wrote my first paper on monotonicity of zeros of orthogonal polynomials [4] and I only modified slightly the above proof (see [4, Lemma 1]).

It is worth mentioning that Shadrin [16] provided a proof of the lemma with arguments identical with the above ones. Another proof of Lemma 1 was given by Nikolov [13]. Much earlier, in 1951, Videnskii [19] established some sufficient conditions for interlacing of zeros of generalized polynomials, and applied his result to establish some Markov-type inequalities with curved majorants. Until 2007 or so, Bojanov was unaware about Videnskii's result, it was communicated to him by A. Shadrin. Professor Bojanov himself obtained various results on V. Markov's zero interlacing property for perfect splines and splines in [1, 3] (see also Theorem 5.7 in [2]), which do not follow from Videnskii's sufficient conditions. Recently Milev and Naidenov [11, 12] derived zero-interlacing properties of exponential polynomials and alike. I do thank my colleague and friend Geno Nikolov for the information about all these contributions.

Back to my discussions with Bojanov in the late eighties, I remember that he emphasized another important question: which linear operators, except for differentiation, preserve the interlacing property. Needles to say, he wanted to see another proof, deep enough to reveal the role of the differential operator and which would allow the desired extensions. Some years later I realized the tight connection between interlacing of zeros of real hyperbolic polynomials and stability, which is given by the classical Hermite-Biehler theorem and which, together with the Gauss-Lucas theorem, implies immediately Markov's result.

It turns out Bojanov's questions were deep and an answer, though still not complete, can be given with the help of the very recent progress on operators which preserve stability, due to Julius Borcea (who, sadly enough, passed away at the very same day, April 8, 2009, as Professor Bojanov) and Peter Branden.

One of the versions of Hermite-Biehler's (see, for instance [14]) theorem is as follows:

Theorem B. *Let $p(x)$ and $q(x)$ be real polynomials whose degrees are equal or consecutive integers. Then the zeros of the polynomial*

$$f(x) = p(x) + iq(x)$$

belong to one side of the real axes if and only if p and q are hyperbolic polynomials whose zeros interlace.

Recall that the Gauss-Lucas theorem states that the critical points of a complex polynomial belong to the convex hull of its zeros. The the Hermite-Biehler and Gauss-Lucas theorems immediately yield Markov's one. Moreover, it is clear that one can characterize the operators which preserve the interlacing of zeros of hyperbolic polynomials if finds all operators which preserve the property that a polynomial possesses zeros only on one side of the real axis. Since a simple rotation, that is, multiplication by $\pm i$, takes the upper or the lower half-planes to the left, one, we may consider the polynomials whose zeros belong to the half-plane $\Re z < 0$. The real polynomials with this property are called Hurwitz or stable ones. In what follows we recall the Routh-Hurwitz criterion for stability. In order to this, and for other purposes, suppose that $P(x) = a_0 + a_1x + \cdots a_nx^n$ and $Q(x) = b_0 + b_1x + \cdots + b_{n-1}x^{n-1} + b_nx^n$ are real algebraic polynomial with $a_n > 0$ and either $b_n = 0$, $b_{n-1} > 0$ or $b_n > 0$, which means that P is always of degree n and Q is either of degree $n-1$ or n, and both polynomials posses positive leading coefficients. Then we form the polynomial

$$\begin{aligned} G(x) &= P(x^2) + xQ(x^2) \\ &= a_0 + b_0x + a_1x^2 + b_1x^3 + \cdots + b_{n-1}x^{2n-1} + a_nx^{2n} + b_nx^{2n+1}. \end{aligned}$$

It is either of degree $2n$, when $b_n = 0$, or $2n+1$ otherwise, and its leading coefficient is also positive. Vice versa, given an algebraic polynomial $G(x)$ with positive leading coefficients, it can be represented in the above form and thus define the polynomials $P(x)$ and $Q(x)$. Then the Hermite-Biehler theorem and above mentioned rotation of the upper or lower half-planes to the left one imply:

Theorem C. *Let the polynomials $P(x)$ and $Q(x)$ and $G(x)$ be defined as above. Then the zeros of the polynomial G belong to the left half plane $\Re z < 0$ if and only if P and Q are hyperbolic polynomials whose zeros are negative and interlace.*

This result appears as Theorem 13 on p. 228 in [5]. With the above polynomial G, or equivalently with the P and Q, we associate its Hurwitz matrix

$$H(G) = H(P,Q) = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_n & 0 & \cdots & \cdots & 0 \\ a_0 & a_1 & a_2 & \cdots & a_n & 0 & \cdots & \cdots & 0 \\ 0 & b_0 & b_1 & b_2 & \cdots & b_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & a_2 & \cdots & a_n & 0 & \cdots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & b_n \end{pmatrix}.$$

The the Routh-Hurwitz theorem states:

Theorem D. *Let the polynomial $G(x)$ be defined as above. Then it is stable if and only if all principal minors of $H(G)$ are positive.*

Thus, we obtain a simple criterion for interlacing of zeros of hyperbolic polynomials provided all their zeros are negative.

Theorem 1. *The zeros of P and Q are real, negative and interlace, if and only if the principal minors of $H(P,Q)$ are positive. Moreover, if $Q \in \pi_{n-1}$, then $P \prec Q$ and, if $Q \in \pi_n$, then $Q \prec P$.*

In the general case, we may consider the Taylor expansion of the polynomials at a sufficiently large real number, larger than the zeros of the two polynomials.

Recall that if $A(x)=A_0+A_1x+\cdots+A_nx^n$ and $B(x)=B_0+B_1x+\cdots+B_nx^n$, then their Hadamrd product is defined by

$$(A * B)(x) = A_0B_0 + A_1B_1x + \cdots + A_nB_nx^n.$$

Very interesting property of stable polynomials was established by Garloff and Wagner [6]. They proved that if A and B are stable, then their Hadamard product $A * B$ is also stable. Therefore, we can state the following consequence of this interesting fact:

Theorem 2. *Let p, q, P and Q be real hyperbolic polynomials with negative zeros and $p \prec q$ and $P \prec Q$. Then $(p * P) \prec (q * Q)$.*

2. Monotonicity of Zeros Satisfying a Sturm-Liouville Differential Equation

As I have already mentioned above, I became interested in zeros of orthogonal polynomials after I finished my graduate studies. In 1992 I asked Prof. Mourad

Ismail for help concerning zeros of a specific family of orthogonal polynomials which arises in the L_2 Markov inequality when he kindly sent me a bunch of papers. There, in [8], I found a very interesting conjecture on monotonicity of positive zeros of ultraspherical polynomials. Bojanov was not an expert in this topic but I remember his initial interest on the problem and the many discussions we had on this theme after I really got interested and begun contributing on it. It is widely known that the zeros of the orthogonal polynomials are all real, distinct and are located in the convex hull of the support of the measure with respect to which they are orthogonal. These zeros are the nodes of the corresponding Gaussian quadrature formula and this is one of the main reason for the interest in their behaviour and location. The most famous and well known orthogonal polynomial and those of Jacobi, $P_n^{(\alpha,\beta)}(x)$, Gegenbauer, $C_n^{(\lambda)}(x)$, Laguerre, $L_n^{(\alpha)}(x)$, and Hermite, $H_n(x)$. Very interesting questions are related to the behaviour of the zeros of $P_n^{(\alpha,\beta)}(x)$, $C_n^{(\lambda)}(x)$ and $L_n^{(\alpha)}(x)$ when they are considered as functions of the parameters α, β and λ. There is an additional challenge here because the zeros of the classical orthogonal polynomials obey a very interesting electrostatic interpretation. We recall it for the zeros $x_{nk}(\alpha,\beta)$ of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. Consider an electrostatic field generated by two fixed charges at -1 and $+1$ with forces $(\beta+1)/2$ and $(\alpha+1)/2$, respectively, where $\alpha,\beta > -1$, so that these charges are positive, and n free unit charges located in $(-1,1)$. Suppose that they repel each other according to the logarithmic potential law which means that the force is reciprocal to the distance between the charges. Equivalently, we may interpret this situation as if the charges are distributed along infinite wires perpendicular to the real axis. Then the energy of this electrostatic field attains its minimum at a unique location of the free charges, when they coincide with the zeros $x_{nk}(\alpha,\beta)$ of $P_n^{(\alpha,\beta)}(x)$. It is clear from this electrostatic interpretation of $x_{nk}(\alpha,\beta)$ that they are increasing functions of β and decreasing functions of α. Formally this fact was established by Andrei Markov [9] who proved a nice simple criteria for monotonicity of zeros of orthogonal polynomials using the fact that the zeros of the orthogonal polynomials coincide with the nodes of the Gaussian quadrature formula (see also [17]).

Once we discussed with Professor Bojanov the classical Sturm comparison theorem on zeros of solutions of Sturm-Liouville differential equation and its eventual application to results on interlacing and monotonicity of zeros of orthogonal polynomials.

Theorem E (Sturm's comparison theorem). *Let $y(x)$ and $Y(x)$ be solutions of the differential equations*

$$y''(x) + f(x)y(x) = 0 \tag{1}$$

and

$$Y''(x) + F(x)Y(x) = 0, \tag{2}$$

where $f, F \in C(a,b)$ and $f(x) \le F(x)$ in (a,b). Let x_1 and x_2, with $a < \zeta_1 < \zeta_2 < b$ be two consecutive zeros of $y(x)$. Then the function $Y(x)$ has at least *one variation of sign in the interval* (ζ_1, ζ_2) *provided* $f(x) \not\equiv F(x)$ *there. The statement holds also:*

- *for* $\zeta_1 = a$ *if*

$$y(a+0) = 0 \quad \textit{and} \quad \lim_{x \to a+0} \{y'(x)Y(x) - y(x)Y'(x)\} = 0; \tag{3}$$

- *for* $\zeta_2 = b$ *if*

$$y(b-0) = 0 \quad \textit{and} \quad \lim_{x \to b-0} \{y'(x)Y(x) - y(x)Y'(x)\} = 0. \tag{4}$$

We refer to the preliminary chapter of Szegő's book [17] or to Chapter 8 of Hille's book [7].

This theorem is widely used for obtaining sharp limit for the zeros of the above families of orthogonal polynomials which are called the classical families of orthogonal polynomials. Indeed, the Jacobi, Gegenbauer, Laguerre and Hermite polynomials are solutions of such differential equations. Chapter 6 of Szegő classical reading [17] contains various such results. Nevertheless, there was no proof in the literature on the monotonicity of zeros of classical orthogonal polynomials using Sturm's theorem and Bojanov said it was surprising. Some year later I discussed seriously the same question with my colleague and friend Panos Siafarikas who also passed away too early. I remember I had already thought about this matter and that Panos was sceptical about such an application because if one considers $Y(x) - y(x)$ in the case corresponding to the classical orthogonal polynomials, with two different values of the parameter, the difference usually changes sign in the interval of orthogonality. I was sure that this phenomenon was not only necessary but also sufficient for establishing monotonicity and shared my belief with Siafarikas but I never took this idea seriously till very recently when I worked with my colleague and friend Ranga on zeros of certain para-orthogonal polynomials whose zeros are located in the unit circumference of the complex plane. It turns out that Sturm's theorem is very helpful for establishing monotonicity of zeros of polynomial functions which are solutions of Sturm-Liouville differential equation. Here is the theorem we need:

Theorem 3. *Let* $f, F \in C(a,b)$ *and* $y(x)$ *and* $Y(x)$ *be solutions of the differential equations* (1) *and* (2)*, satisfy* (3) *and* (4)*, and both have* n *distinct zeros in* (a,b)*. Let the zeros of* $y(x)$ *in* (a,b) *be* $x_1 < x_2 < \cdots < x_n$ *and those of* $Y(x)$ *be* $X_1 < X_2 < \cdots < X_n$*. If there exists* $\eta \in (a,b)$*, such that* $f(\eta) = F(\eta)$ *and*

- $F(x) - f(x) < 0$ *for* $x \in (a, \eta)$ *and* $F(x) - f(x) > 0$ *for* $x \in (\eta, b)$*, then* $x_k < X_k$ *for every* $k = 1, \ldots, n$*;*
- $F(x) - f(x) > 0$ *for* $x \in (a, \eta)$ *and* $F(x) - f(x) < 0$ *for* $x \in (\eta, b)$*, then* $x_k > X_k$ *for every* $k = 1, \ldots, n$*.*

Before we prove this theorem, it is worth mentioning its simplicity. It says that we may draw conclusion about monotonicity of zeros of solution of Sturm-Liouville differential equations provided the difference $F(x)-f(x)$ changes sign in (a,b). Besides, it is not possible that the difference $F(x)-f(x)$ maintains the sign in (a,b) if both y and Y have exactly n distinct zeros in (a,b). Indeed, if it was so, say $F(x)-f(x)>0$ in (a,b), and if we set $x_0=a$ and $x_{n+1}=b$, then by Sturm's Comparison Theorem E, the solution $Y(x)$ of (2) would have changed sign in every interval (x_k,x_{k+1}), $k=0,1,\ldots,n$, which would produce at least $n+1$ distinct zeros of $Y(x)$ in (a,b), a contradiction.

Proof of Theorem 3. We prove only the statement in the case when $F(x)-f(x)<0$ for $x\in(a,\eta)$ and $F(x)-f(x)>0$ for $x\in(\eta,b)$ because the arguments in the other case are identical.

Let $y(x)$ have m zeros on (a,η) and $n-m$ zeros in $[\eta,b)$, that is

$$a<x_1<x_2<\ldots<x_m<\eta\le x_{m+1}<\ldots<x_n<b.$$

The reader will realize that the conclusion in the cases $m=0$ and $m=n$ are immediate from the arguments provided below, so that we consider the general situation $1<m<n$.

First we shall prove that $X_k>x_k$ for $k=1,\ldots,m$. Assume the contrary, that there is a j with $1\le j\le m$, such that $X_j<x_j$. Since $F(x)<f(x)$ for $x\in(a,x_j)$, then by Theorem E, the function $y(x)$ would change sign at least once in all the intervals $(a,X_1),\ldots(X_{j-1},X_j)$. This means that $x_j<X_j$, a contradiction.

The fact that $X_k>x_k$ for $k=m+1,\ldots,n$ is also a consequence of Sturm's theorem. Since $f(x)<F(x)$ for $x\in(x_{m+1},b)$, $Y(x)$ changes sign at least once in (x_n,b), at least twice in (x_{n-1},b), and so on, at least $n-m$ times in (x_{m+1},b). Hence, $X_n>x_n$, $X_{n-1}>x_{n-1}$, and so on, until $X_{m+1}>x_{m+1}$. □

Now we may consider a family of Sturm-Liouvile differential equations which depends on a parameter τ,

$$y''(x;\tau)+f(x;\tau)\,y(x;\tau)=0, \tag{5}$$

where the differentiation is with respect to the variable x and $\tau\in(c,d)$. Suppose that $f\in C[(a,b)\times(c,d)]$, the solutions $y(x;\tau)$ depend continuously on τ and, for every $\tau\in(c,d)$, $y(x;\tau)$ satisfies

$$\lim_{x\to a+0} y(x,\tau)=0 \quad\text{and}\quad \lim_{x\to b-0} y(x,\tau)=0, \tag{6}$$

and possesses n distinct zeros $x_k(\tau)\in(a,b)$,

$$a<x_1(\tau)<\cdots<x_n(\tau)<b,$$

which also depend continuously on τ. Suppose further that, for some $\tau_1,\tau_2\in(c,d)$,

$$\lim_{x\to a+0}\{y'(x,\tau_1)y(x;\tau_2)-y(x;\tau_1)y'(x;\tau_2)\}=0 \tag{7}$$

and

$$\lim_{x\to b-0}\{y'(x,\tau_1)y(x;\tau_2)-y(x;\tau_1)y'(x;\tau_2)\}=0. \tag{8}$$

Then we may formulate the following useful consequence of Theorem 3:

Theorem 4. *Let the solutions $y(x;\tau)$ of (5) obey the properties described above. If, for some $\tau_1,\tau_2\in(c,d)$ there is $\eta\in(a,b)$, such that $f(\eta;\tau_1)=f(\eta;\tau_2)$ and*

- *$f(x;\tau_2)-f(x;\tau_1)<0$ for $x\in(a,\eta)$ and $f(x;\tau_2)-f(x;\tau_1)>0$ for $x\in(\eta,b)$, then $x_k(\tau_1)<x_k(\tau_2)$ for every $k=1,\dots,n$;*
- *$f(x;\tau_2)-f(x;\tau_1)>0$ for $x\in(a,\eta)$ and $f(x;\tau_2)-f(x;\tau_1)<0$ for $x\in(\eta,b)$, then $x_k(\tau_1)>x_k(\tau_2)$ for every $k=1,\dots,n$.*

It is worth mentioning that in the applications we should use either $\tau_2=\tau_1+\varepsilon$ or $\tau_2=\tau_1-\varepsilon$, with sufficiently small positive ε. In this situation, because of the continuous dependence of the zeros with respect to the parameter τ, we shall have not only monotonicity, but also interlacing of the zeros of $y(x;\tau)$ and $y(x;\tau\pm\varepsilon)$. As an application of these results, we establish the monotonicity of the zeros of the Jacobi polynomials in the case when $\alpha,\beta>0$. It is known that

$$y(x;\alpha,\beta)=(1-x)^{(\alpha+1)/2}\,(1+x)^{(\beta+1)/2}\,P_n^{(\alpha,\beta)}(x)$$

is a solution of the differential equation

$$y''(x;\alpha,\beta)+f(x;\alpha,\beta)\,y(x;\alpha,\beta)=0,$$

where

$$f(x;\alpha,\beta)=\frac{1-\alpha^2}{4(1-x)^2}+\frac{1-\beta^2}{4(1+x)^2}+\frac{n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)/2}{1-x^2}.$$

Fix $\beta>0$ and let us apply the theorem with $\tau=\alpha$ which varies in $(0,\infty)$. Obviously $y(x;\alpha,\beta)$ obeys the requirements (6), (7) and (8). Let $\tau_1=\alpha$ and $\tau_2=\alpha+\varepsilon$. Then

$$f(x;\alpha+\varepsilon,\beta)-f(x;\alpha,\beta)=\frac{\varepsilon\,\{2n+1-\alpha+\beta-\varepsilon-(2n+1+\alpha+\beta+\varepsilon)x\}}{(1-x)^2(1+x)}.$$

Hence, if $\varepsilon>0$, then $f(\eta;\alpha+\varepsilon,\beta)=f(\eta;\alpha,\beta)$, where

$$\eta=\frac{2n+1-\alpha+\beta-\varepsilon}{2n+1+\alpha+\beta+\varepsilon}\in(-1,1),$$

and

$$\begin{aligned} f(x;\alpha+\varepsilon,\beta)-f(x;\alpha,\beta)>0 &\qquad \text{for } x\in(-1,\eta),\\ f(x;\alpha+\varepsilon,\beta)-f(x;\alpha,\beta)<0 &\qquad \text{for } x\in(\eta,1). \end{aligned}$$

Therefore, the zeros $x_{nk}(\alpha, \beta)$ of the Jacobi polynomial of degree n are decreasing functions of α for $\alpha \in (0, \infty)$. Observe that they are degreasing functions of α in the entire range $\alpha \in (-1, \infty)$ but our theorem can not be applied for $\alpha < 0$ because in this case we can not guarantee that (6), (7) and (8) hold. Nevertheless, I wish Professor Bojanov knew these little pieces and Panos Siafarikas knew the above Theorems 3 and 4.

References

[1] B. Bojanov, Markov interlacing property for perfect splines, *J. Approx. Theory* **100** (1999) 183–201.

[2] B. Bojanov, Markov-type inequalities for polynomials and splines, *in* "Approximation Theory X: Abstract and Classical Analysis", (C. K. Chui, L. L. Schumaker and J. Stökler, Eds.), pp. 31–90, Vanderbilt University Press, Nashville, TN, 2002.

[3] B. Bojanov and N. Naidenov, Exact Markov-type inequalities for oscillating perfect splines, *Constr. Approx.* **18** (2002), 37–59.

[4] D. K. Dimitrov, On a conjecture concerning monotonicity of zeros of ultraspherical polynomials, *J. Approx. Theory* **85** (1996), 88–97.

[5] F. R. Gantmacher, "The Theory of Matrices", Vol. 2, Chelsea, New York, 1959.

[6] J. Garloff and D. G. Wagner, Hadamard products of stable polynomials are stable, *J. Math. Anal. Appl.* **202** (1996), 797–809.

[7] E. Hille, "Lectures on Ordinary Differential Equations", Addison-Wesley, Reading, 1969.

[8] M. E. H. Ismail, Monotonicity of zeros of orthogonal polynomials, *in* "q-Series and Partitions" (D. Stanton, Ed.), pp. 177-190, IMA Volumes in Mathematics and Its Applications, Vol. 18, Springer-Verlag, New York, 1989.

[9] A. Markov, Sur les racines de certaines équations, *Math. Ann.* **27** (1896), 177–152

[10] V. Markov, On functions least deviated from zero in a given interval, St. Petersburg, 1892 [in Russian]; German transl.: Über Polynome die in einen gegebenen Intervalle möglichst wegun von Null abweichen, *Math. Ann.* **77** (1916), 213–250.

[11] L. Milev and N. Naidenov, Markov interlacing property for exponential polynomials, *J. Math. Anal. Appl.* **367** (2010), 669–676.

[12] L. Milev and N. Naidenov, Interlacing properties of certain Tchebycheff systems, *in* "Constructive Theory of Functions, Sozopol 2010" (G. Nikolov and R. Uluchev, Eds.), pp. 192–200, Prof. Marin Drinov Academic Publishing House, Sofia, 2012.

[13] G. Nikolov, An extremal property of Hermite polynomials, *J. Math. Anal. Appl.* **290** (2004), 405–413.

[14] N. Obrechkoff, "Zeros of Polynomials", Publ. Bulg. Acad. Sci., Sofia, 1963 [in Bulgarian]; English transl.: Prof. Marin Drinov Academic Publishing House, Sofia, 2003.

[15] T. J. Rivlin, "Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory", 2nd ed., Wiley, New York, 1990.

[16] A. Yu. Shadrin, Interpolation with Lagrange polynomials. A simple proof of Markov inequality and some of its generalizations, *Approx. Theory Appl.* **8** (1992), 51–61.

[17] G. Szegő, "Orthogonal Polynomials", 4th ed., Amer. Math. Soc. Coll. Publ., Providence, RI, 1975.

[18] G. Szegő, "Collected Papers" (R. Askey, Ed.), Birkhäuser, Boston, 1982.

[19] V. S. Videnskii, On the estimates of the derivatives of a polynomial, *Izv. Akad. Nauk SSSR* **15** (1951), 401–420 [in Russian].

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