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Interpolation of Mixed Type Data by Bivariate Polynomials

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We consider the problem of interpolation of bivariate functions on the unit disk by polynomials. The data known for the function consist of Radon projections along chords in multiple directions and function values at points lying on the unit circle. We prove a sufficient condition for a configuration of chords and points to be regular, i.e. the interpolation problem to be poised. Regularity of a particular scheme of chords and points is considered. Numerical experiments are presented.

1. Introduction and Preliminaries

In medicine, biology, materials science, radiology, geophysics, oceanography, archeology, astrophysics, and other sciences, the idea of tomography (imaging by sections or sectioning) is used. Modern methods of tomography involve gathering projection data from multiple directions and applying this data into a tomographic reconstruction software algorithm processed by a computer. Various types of signal acquisition can be used in similar algorithms in order to create a 3D image. However, in the general case the output from these reconstruction procedures appears as 2D slice images.

There exist different reconstruction algorithms: filtered back projection, iterative reconstruction, direct methods, etc. These procedures give inexact results: they represent a compromise between accuracy and computation time required.

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Because of the importance of such methods for applications in science and practice they have been intensively investigated by many mathematicians [2], [6], [13], [14], [15], [16], and others. Besides the algorithms based on the inverse Radon transform (see [14], [15] and the bibliography therein), other direct interpolation and fitting methods have been recently studied (see [1], [4], [7], [8], [9], [11], [13]).

We denote by Π_n^2 the set of all algebraic polynomials in two variables of total degree at most n and real coefficients. Then, Π_n^2 is a linear space of dimension $\binom{n+2}{2}$, and $P \in \Pi_n^2$ if and only if

$$P(x,y) = \sum_{i+j\le n} \alpha_{ij} x^i y^j, \qquad \alpha_{ij} \in \mathbb{R}.$$

Let $\mathbf{B} := \{\mathbf{x} = (x,y) \in \mathbb{R}^2 \,:\, \|\mathbf{x}\| \le 1\}$ be the unit disk in the plane, where $\|\mathbf{x}\| = \sqrt{x^2+y^2}$. Given $t \in [-1,1]$ and an angle of measure $\theta \in [0,\pi)$, the equation $x\cos\theta + y\sin\theta - t = 0$ defines a line ℓ perpendicular to the vector $\langle \cos\theta, \sin\theta \rangle$ and passing through the point $(t\cos\theta, t\sin\theta)$. The set $I(\theta,t) := \ell \cap \mathbf{B}$ is a *chord* of the unit disk $\mathbf{B}$ which can be parameterized in the manner

$$\begin{cases} x = t\cos\theta - s\sin\theta, \\ y = t\sin\theta + s\cos\theta, \end{cases} \qquad s \in [-\sqrt{1-t^2}, \sqrt{1-t^2}],$$

where the quantity θ is the direction of $I(\theta,t)$ and t is the distance of the chord from the origin. Suppose that for a given function $f : \mathbb{R}^2 \to \mathbb{R}$ the integrals of f exist along all line segments on the unit disk $\mathbf{B}$. *Radon projection* (or *X-ray*) of the function f over the segment $I(\theta,t)$ is defined by

$$\mathcal{R}_\theta(f;t) := \int_{I(\theta,t)} f(\mathbf{x})\,d\mathbf{x} = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\,ds.$$

Clearly, $\mathcal{R}_\theta(\,\cdot\,;t)$ is a linear functional. Since $I(\theta,t) \equiv I(\theta+\pi,-t)$ it follows that $\mathcal{R}_\theta(f;t) = \mathcal{R}_{\theta+\pi}(f;-t)$. Thus, the assumption above for the direction of the chords $0 \le \theta < \pi$ incurs no loss of generality.

It is well-known that the set of Radon projections

$$\left\{\mathcal{R}_\theta(f;t) \,:\, -1 \le t \le 1,\ 0 \le \theta < \pi\right\}$$

determines f uniquely (see [12], [16]). According to a more recent result in [17], an arbitrary function $f \in L^1(\mathbb{R}^2)$ with compact support in $\mathbf{B}$ is uniquely determined by any infinite set of X-rays. Since the function $f \equiv 0$ has all its projections equal to zero, it follows that the only function which has the zero Radon transform is the constant zero function. It was shown by Marr [13] that every polynomial $P \in \Pi_n^2$ can be reconstructed uniquely by its projections only on a finite number of directions.

Another important property (see [13], [3]) is the following:

Lemma 1. *If $P \in \Pi_n^2$ then for each fixed θ there exists a univariate polynomial p of degree n such that*

$$\mathcal{R}_\theta(P;t) = \sqrt{1-t^2}\, p(t), \qquad -1 \le t \le 1,$$

and

$$p(-1) = 2P(-\cos\theta, -\sin\theta) \quad \text{and} \quad p(1) = 2P(\cos\theta, \sin\theta).$$

The space Π_n^2 has a standard basis of the power functions $\{x^i y^j\}$. Studying various problems for functions on the unit disk, it is often helpful to use some orthonormal basis. In [2], the following orthonormal basis of Π_n^2 was constructed. Denote the Chebyshev polynomial of second kind of degree m as usual by

$$U_m(t) := \frac{1}{\sqrt{\pi}} \frac{\sin(m+1)\theta}{\sin\theta}, \quad t = \cos\theta$$

and the bivariate *ridge polynomial* in direction θ by

$$U_m(\theta;\mathbf{x}) := U_m(x\cos\theta + y\sin\theta).$$

For $\theta_{mj} := \frac{j\pi}{m+1}$, $m = 0,\dots,n$, $j = 0,\dots,m$, the ridge polynomials

$$U_{mj}(\mathbf{x}) := U_m(\theta_{mj};\mathbf{x}), \qquad m = 0,\dots,n,\ j = 0,\dots,m, \tag{1}$$

form an orthonormal basis of Π_n^2 on the unit disk $\mathbf{B}$.

The following important relation was proved by Marr [13] and we shall call it *Marr's formula.*

Lemma 2. *For each $t \in (-1,1)$, θ and φ, we have*

$$\mathcal{R}_\varphi(U_m(\theta;\cdot);t) = \frac{2}{m+1}\sqrt{1-t^2}\, U_m(t)\, \frac{\sin(m+1)(\varphi-\theta)}{\sin(\varphi-\theta)}.$$

2. Interpolation Problem for Radon Projections Type of Data

For a given scheme of chords I_k, $k = 1,\dots,\binom{n+2}{2}$, of the unit circle $\partial\mathbf{B}$, find a polynomial $P \in \Pi_n^2$ satisfying the conditions:

$$\int_{I_k} P(\mathbf{x})\, d\mathbf{x} = \gamma_k, \qquad k = 1,\dots,\tbinom{n+2}{2}. \tag{2}$$

If (2) has a unique solution for every given set of values $\{\gamma_k\}$, the interpolation problem is called *poised* and the scheme of chords – *regular.*

The first known scheme which is regular for every degree n of the interpolating polynomial was found by Hakopian [11]. Hakopian's scheme consists of all $\binom{n+2}{2}$ chords, connecting given $n+2$ points on the unit circle $\partial\mathbf{B}$. Bojanov and Xu [4] proposed a regular scheme consisting of $2\lfloor\frac{n+1}{2}\rfloor+1$ equally spaced directions with $\lfloor\frac{n}{2}\rfloor+1$ chords, associated with the zeros of the Chebyshev polynomials of certain degree, in each direction.

Another family of regular schemes was provided by Bojanov and Georgieva [1]. There the Radon projections are taken along $n+1$ directions

$$\Theta := \{\theta_0, \theta_1, \dots, \theta_n\}, \qquad 0 \le \theta_0 < \cdots < \theta_n < \pi.$$

To every direction θ_k are associated $n-k+1$ chords with the distances

$$1 > t_{kk} > t_{k,k+1} > \cdots > t_{kn} > -1.$$

This results in $\binom{n+2}{2}$ chords of the unit circle, $\{I(\theta_k, t_{ki})\}_{k=0,}^{n}{}_{i=k}^{n}$. The scheme is thus fully described by (Θ, T), where $T := \{t_{ki}\}$ is the upper triangular matrix of chord distances to the origin.

The following regularity result for schemes of this type was proved by Bojanov and Georgieva [1].

Theorem 1. *For (Θ, T) as above, the interpolation problem*

$$\int_{I(\theta_k,t_{ki})} P(\mathbf{x})\,d\mathbf{x} = \gamma_{ki}, \qquad k=0,\dots,n, \quad i=k,\dots,n, \quad P \in \Pi_n^2, \tag{3}$$

is poised if

$$\det \mathbf{U}_k \ne 0 \quad \text{for} \quad k = 0, \dots, n,$$

where

$$\mathbf{U}_k = \mathbf{U}_k^{(n)} := \begin{pmatrix} U_k(t_{kk}) & U_{k+1}(t_{kk}) & \cdots & U_n(t_{kk}) \\ U_k(t_{k,k+1}) & U_{k+1}(t_{k,k+1}) & \cdots & U_n(t_{k,k+1}) \\ \cdots & \cdots & \cdots & \cdots \\ U_k(t_{kn}) & U_{k+1}(t_{kn}) & \cdots & U_n(t_{kn}) \end{pmatrix}.$$

Several regular schemes of this type were suggested by Georgieva and Ismail [7] and by Georgieva and Uluchev [8]. In particular, we will make use of the following result from [7].

Theorem 2. *Let $t_{ki} = \eta_i = \cos\frac{(i+1)\pi}{n+2}$, $k=0,\dots,n$, $i=k,\dots,n$, be the zeros of the Chebyshev polynomial of second kind U_{n+1}. Then $\det \mathbf{U}_k \ne 0$ for $k=0,\dots,n$, and thus the problem (3) is poised.*

3. Interpolation Problem for Mixed Type of Data

We consider interpolation using mixed type of data – both Radon projections and function values at points lying on the unit circle. Let

- $\Theta := \{\theta_0, \theta_1, \dots, \theta_n\}$, $0 \le \theta_0 < \dots < \theta_n < \pi$;
- $T := \{t_{ki}\}$ be an upper triangular matrix with $1 > t_{kk} > \dots > t_{k,n-1} > -1$, $k = 0, \dots, n-1$;
- $X := \{\mathbf{x}_0, \dots, \mathbf{x}_n\}$, where $\mathbf{x}_k$ are points on the unit circle.

The problem is to find a polynomial $P \in \Pi_n^2$ satisfying the $\binom{n+2}{2}$ interpolation conditions

$$\begin{aligned} &\int_{I(\theta_k, t_{ki})} P(\mathbf{x})\, d\mathbf{x} = \gamma_{ki}, && k = 0, \dots, n-1, \quad i = k, \dots, n-1, \\ &P(\mathbf{x}_k) = f_k, && k = 0, \dots, n. \end{aligned} \tag{4}$$

The difference to problem (3) is that we replace the interpolation condition on the last chord in each direction θ_k with a function value interpolation condition at a point $\mathbf{x}_k$. If (4) has a unique solution for every given set of values $\{\gamma_{ki}\}$ and $\{f_k\}$, the interpolation problem (4) is called *poised* and the scheme of chords and points (Θ, T, X) – *regular.*

In the following we state and prove a condition for the interpolation problem (4) to be poised with a particular choice of X.

Theorem 3. *For a given set of chords and points* (Θ, T, X) *with* $X = \{\mathbf{x}_k = (-\cos\theta_k, -\sin\theta_k)\}_{k=0}^n$*, the interpolation problem* (4) *is poised if*

$$\det \mathbf{U}_k^* \ne 0, \qquad k = 0, \dots, n,$$

where

$$\mathbf{U}_k^* := \begin{pmatrix} U_k(t_{kk}) & U_{k+1}(t_{kk}) & \dots & U_{n-1}(t_{kk}) & U_n(t_{kk}) \\ U_k(t_{k,k+1}) & U_{k+1}(t_{k,k+1}) & \dots & U_{n-1}(t_{k,k+1}) & U_n(t_{k,k+1}) \\ \dots & \dots & \dots & \dots & \dots \\ U_k(t_{k,n-1}) & U_{k+1}(t_{k,n-1}) & \dots & U_{n-1}(t_{k,n-1}) & U_n(t_{k,n-1}) \\ U_k(-1) & U_{k+1}(-1) & \dots & U_{n-1}(-1) & U_n(-1) \end{pmatrix}.$$

Proof. It if sufficient to show that the only bivariate polynomial $P \in \Pi_n^2$ satisfying zero interpolation conditions is the trivial polynomial, $P(\mathbf{x}) \equiv 0$. For $P \in \Pi_n^2$, let $a_{mj}(P)$ denote the coefficients of P in the basis of ridge polynomials, see (1),

$$a_{mj}(P) := \int_{\mathbf{B}} P(\mathbf{x}) U_{mj}(\mathbf{x})\, d\mathbf{x}, \quad P(\mathbf{x}) = \sum_{m=0}^{n} \sum_{j=0}^{m} a_{mj}(P) U_{mj}(\mathbf{x}).$$

By Lemma 1, for each k we can write

$$\mathcal{R}_{\theta_k}(P; t) = \sqrt{1 - t^2}\; p_k(t)$$

with some univariate polynomial $p_k(t)$ of degree at most n. Expanding p_k in Chebyshev-Fourier series, we obtain

$$\mathcal{R}_{\theta_k}(P;t) = \sqrt{1-t^2}\ \sum_{i=0}^{n} b_{ki}(P)U_i(t)$$

where

$$b_{ki}(P) := 2\int_{-1}^{1} \mathcal{R}_{\theta_k}(P;t)U_i(t)\,dt = 2\int_{\mathbf{B}} P(\mathbf{x})U_i(\theta_k;\mathbf{x})\,d\mathbf{x}. \tag{5}$$

On the other hand, using Marr's formula (Lemma 2), we can express $\mathcal{R}_{\theta_k}(P;t)$ in terms of $\{a_{mj} = a_{mj}(P)\}$. Indeed,

$$\begin{aligned}\mathcal{R}_{\theta_k}(P;t) &= \sum_{m=0}^{n}\sum_{j=0}^{m} a_{mj}\mathcal{R}_{\theta_k}(U_{mj};t)\\ &= \sum_{m=0}^{n}\sum_{j=0}^{m} a_{mj}\frac{2}{m+1}\sqrt{1-t^2}\,U_m(t)\frac{\sin(m+1)(\theta_k-\theta_{mj})}{\sin(\theta_k-\theta_{mj})}\\ &= \sqrt{1-t^2}\sum_{m=0}^{n}\Big(\sum_{j=0}^{m} s_{mkj}a_{mj}\Big)U_m(t),\end{aligned}$$

where we have used the notation

$$s_{mkj} := \frac{2}{m+1}\ \frac{\sin(m+1)(\theta_k-\theta_{mj})}{\sin(\theta_k-\theta_{mj})}\ .$$

The last two representations of $\mathcal{R}_{\theta_k}(P;t)$ lead to the equality

$$\sum_{m=0}^{n}\Big(\sum_{j=0}^{m} s_{mkj}\,a_{mj}\Big)U_m(t) = \sum_{i=0}^{n} b_{ki}U_i(t), \tag{6}$$

where $b_{ki} = b_{ki}(P)$. Comparing the coefficients of $U_m(t)$ on the both sides of (6) yields $s_{mk0}\,a_{m0} + \cdots + s_{mkm}\,a_{mm} = b_{km}$. Since k was arbitrary, we obtain the system

$$\begin{array}{cccc} s_{m00}\,a_{m0} & +\cdots+ & s_{m0m}\,a_{mm} & = b_{0m}\\ \vdots & \vdots & \vdots & \vdots\\ s_{mm0}\,a_{m0} & +\cdots+ & s_{mmm}\,a_{mm} & = b_{mm}. \end{array} \tag{7}$$

Consider the matrix $\mathbf{S}_m := \{s_{mkj}\}$ of this system. It is shown in the proof of Theorem 1 in [1] that $\det\mathbf{S}_m \neq 0$ for any $m = 0,\dots,n$. Consequently, given $b_{0m},\dots,b_{mm}$, the coefficients $a_{m0},\dots,a_{mm}$ are uniquely determined by the linear system (7).

We have just proved the following auxiliary proposition:

Given any numbers $\{\beta_{mj}\}_{m=0,\,j=m}^{n\;\;\;n}$*, there exists a unique polynomial* $P\in\Pi_n^2$ *such that*

$$b_{mj}(P)=\beta_{mj}, \qquad m=0,\dots,n, \quad j=m,\dots,n.$$

Note in particular that only the functionals $b_{mj}(P)$ with $j\ge m$ are needed to determine P uniquely, while those with $j<m$ are redundant.

The next task is to show that any of the functionals $b_{mj}(P)$ can be determined uniquely from the functionals in the set

$$\mathcal{M}:=\big\{\mathcal{R}_{\theta_k}(P;t_{ki})\big\}_{k=0,\,i=k}^{n-1\;\,n-1}\cup\big\{P(\mathbf{x}_k)\big\}_{k=0}^{n}.$$

Note that the set $\mathcal{M}$ consists of $\binom{n+2}{2}$ linear functionals on Π_n^2. Then, for a fixed pair of indices (m,j), there exists a representation of the form

$$b_{mj}(P)=\sum_{k=0}^{n-1}\sum_{i=k}^{n-1}c_{ki}\mathcal{R}_{\theta_k}(P;t_{ki})+\sum_{k=0}^{n}d_kP(\mathbf{x}_k) \qquad \text{for all } P\in\Pi_n^2$$

if and only if

$$\begin{cases}\mathcal{R}_{\theta_k}(P;t_{ki})=0, & k=0,\dots,n-1,\ i=k,\dots,n-1,\\ P(\mathbf{x}_k)=0, & k=0,\dots,n\end{cases} \implies b_{mj}(P)=0. \tag{8}$$

This follows from simple linear algebra arguments.

Assume that the left hand side of (8) holds. We shall prove by induction on k that $b_{mj}(P)=0$, $m=0,\dots,n$, for all $j=0,\dots,n$.

First consider the case $k=0$. The assumption

$$\mathcal{R}_{\theta_0}(P;t_{00})=\dots=\mathcal{R}_{\theta_0}(P;t_{0,n-1})=0 \quad \text{with } -1<t_{0,n-1}<\dots<t_{00}<1$$

gives n zeros of the polynomial $p_0(t)$ in $(-1,1)$. Moreover, by Lemma 1, we have the equality $p_0(-1)=2P(\mathbf{x}_0)=2P(-\cos\theta_0,-\sin\theta_0)$. From $P(\mathbf{x}_0)=0$ it follows that $p_0(t)$ has another zero at $t=-1$. Therefore

$$0\equiv p_0(t)=\sum_{i=0}^{n}b_{0i}(P)U_i(t)$$

and hence $b_{0i}(P)=0$, $i=0,\dots,n$, because of the linear independence of the Chebyshev polynomials $\{U_i(t)\}$.

From the definition of a_{00}, from (5) and since $U_0(t)$ is a constant, we get $a_{00}=\frac{1}{2}\,b_{00}(P)$. Therefore, in the first induction step we have shown that

$$P(\mathbf{x})=a_{10}U_{10}(\mathbf{x})+a_{11}U_{11}(\mathbf{x})+\dots+a_{nn}U_{nn}(\mathbf{x}).$$

Assume that after k induction steps, we have proved that $b_{ij}(P)=0$ for $i<k$ and that P reduces to

$$P(\mathbf{x})=\sum_{i=k}^{n}\sum_{j=0}^{i}a_{ij}U_{ij}(\mathbf{x}).$$

In other words, $P(\mathbf{x})$ is a linear combination of U_{ij} with $i \geq k$. Applying Marr's formula (Lemma 2), we see that its Radon projection $\mathcal{R}_{\theta_k}(P;t)$ must therefore be a linear combination of U_i with $i \geq k$ as well. Thus,

$$\mathcal{R}_{\theta_k}(P;t) = \sqrt{1-t^2}\,\big(b_{kk}(P)U_k(t) + \cdots + b_{kn}(P)U_n(t)\big),$$

and it follows that $b_{ki}(P) = 0$, $i = 0, \dots, k-1$. In order to prove that the remaining coefficients are equal to zero we use the assumptions

$$\mathcal{R}_{\theta_k}(P;t_{kk}) = \cdots = \mathcal{R}_{\theta_k}(P;t_{k,n-1}) = 0 \quad \text{and} \quad P(\mathbf{x}_k) = 0.$$

They produce a homogeneous linear system with respect to the coefficients $b_{ki}(P)$:

$$\begin{array}{llll} b_{kk}(P)\,U_k(t_{kk}) & +\cdots+ & b_{kn}(P)\,U_n(t_{kk}) & = 0, \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ b_{kk}(P)\,U_k(t_{k,n-1}) & +\cdots+ & b_{kn}(P)\,U_n(t_{k,n-1}) & = 0, \\ b_{kk}(P)\,U_k(-1) & +\cdots+ & b_{kn}(P)\,U_n(-1) & = 0. \end{array}$$

Using Lemma 1, the last equation follows from

$$0 = 2P(\mathbf{x}_k) = p_k(-1) = \sum_{i=k}^{n} b_{ki}(P)U_i(-1),$$

since $b_{k0}(P) = \cdots = b_{k,k-1}(P) = 0$ was already shown above.

By the assumption $\det \mathbf{U}_k^* \neq 0$, the only solution to the system is the trivial solution, i.e.

$$b_{ki}(P) = 0, \qquad i = k, \dots, n.$$

All in all, we have shown $b_{ki}(P) = 0$ for all $i = 0, \dots, n$. It follows then from (7) that $a_{k0} = \cdots = a_{kk} = 0$, and therefore

$$P(\mathbf{x}) = \sum_{i=k+1}^{n} \sum_{j=0}^{i} a_{ij}U_{ij}(\mathbf{x}).$$

By the induction hypothesis we get $P(\mathbf{x}) \equiv 0$. The proof is complete. □

4. Regular Schemes for Mixed Type of Data

Here we give a regular interpolatory scheme based on mixed type of data. For the sake of completeness we give a new shorter proof.

Theorem 4 ([10]). *Let n be a positive integer, and*

(i) $\Theta = \{\theta_0, \dots, \theta_n\}$, $0 \leq \theta_0 < \cdots < \theta_n < \pi$;

(ii) $t_{ki} = \eta_i = \cos\frac{(i+1)\pi}{n+1}$, $i = k, \dots, n-1$, *be the zeros of Chebyshev polynomials of second kind* $U_n(x)$*;*

(iii) $X = \{\mathbf{x}_k = (-\cos\theta_k, -\sin\theta_k)\}_{k=0}^{n}$.

Then the interpolation problem (4) *is poised, i.e., the scheme* (Θ, T, X) *is regular.*

Proof. According to Theorem 3, it is sufficient to prove that $\det \mathbf{U}_k^* \neq 0$ for all $k = 0, \dots, n$. Recall that

$$\mathbf{U}_k^* := \begin{pmatrix} U_k(t_{kk}) & U_{k+1}(t_{kk}) & \dots & U_{n-1}(t_{kk}) & U_n(t_{kk}) \\ U_k(t_{k,k+1}) & U_{k+1}(t_{k,k+1}) & \dots & U_{n-1}(t_{k,k+1}) & U_n(t_{k,k+1}) \\ \dots & \dots & \dots & \dots & \dots \\ U_k(t_{k,n-1}) & U_{k+1}(t_{k,n-1}) & \dots & U_{n-1}(t_{k,n-1}) & U_n(t_{k,n-1}) \\ U_k(-1) & U_{k+1}(-1) & \dots & U_{n-1}(-1) & U_n(-1) \end{pmatrix}.$$

We now fix some $k \in \{0, \dots, n\}$.By definition, $(t_{ki})_i$ are the zeros of U_n. Thus, the last column has exactly one nonzero entry, $U_n(-1) = (n+1)(-1)^n$, and the determinant of $\mathbf{U}_k^*$ can be expanded as

$$\det \mathbf{U}_k^* = (n+1)(-1)^n \det \mathbf{U}_k^{(n-1)}$$

with

$$\mathbf{U}_k^{(n-1)} = \begin{pmatrix} U_k(t_{kk}) & U_{k+1}(t_{kk}) & \dots & U_{n-1}(t_{kk}) \\ U_k(t_{k,k+1}) & U_{k+1}(t_{k,k+1}) & \dots & U_{n-1}(t_{k,k+1}) \\ \dots & \dots & \dots & \dots \\ U_k(t_{k,n-1}) & U_{k+1}(t_{k,n-1}) & \dots & U_{n-1}(t_{k,n-1}) \end{pmatrix}$$

as in Theorem 1. By Theorem 2, the determinants of all $\mathbf{U}_k^{(n-1)}$ are nonzero, which finishes the proof. □

5. Numerical Experiments

For simplicity, we have implemented our interpolation scheme using the monomial basis $\{x^i y^j\}$. For integrating a basis function along the chord $I(\theta, t)$, we use the binomial theorem to obtain the formula

$$\begin{aligned} \int_{I(\theta,t)} x^i y^j \, d\mathbf{x} &= \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} (t\cos\theta - s\sin\theta)^i (t\sin\theta + s\cos\theta)^j \, ds \\ &= \sum_{p=0}^{i} \sum_{q=0}^{j} \binom{i}{p} \binom{j}{q} t^{p+q} (\cos\theta)^{j+p-q} (\sin\theta)^{i-(p-q)} \times \\ &\quad \times \frac{(-1)^{i-p}}{i+j-p-q+1} (1-t^2)^{\frac{1}{2}(i+j-p-q+1)} \left(1 - (-1)^{i+j-p-q+1}\right). \end{aligned}$$

In the following, we present interpolation results for two different functions on the unit disk.

Example 1. We approximate the mexican hat function

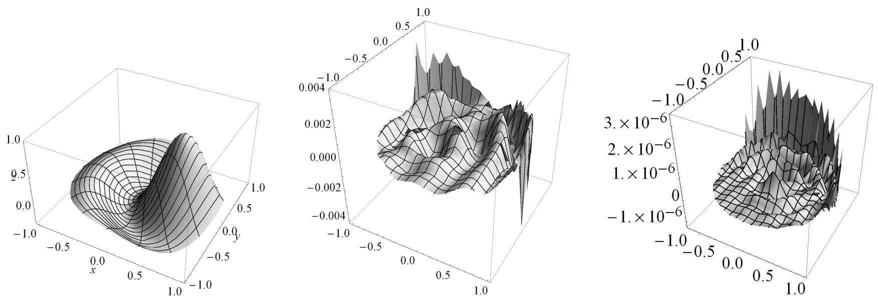
$$f(x,y) = \frac{\sin(2\pi((x-0.2)^2 + y^2 + 10^{-18}))}{2\pi((x-0.2)^2 + y^2 + 10^{-18})}$$

using the mixed interpolatory scheme (Θ, T, X) from Theorem 4 with the choice of directions

$$\Theta = \left\{\theta_k = \frac{k\pi}{n+1}\right\}_{k=0}^{n}.$$

In Figure 1, we show the original function $f(x,y)$ as well as the errors obtained from this scheme with $n = 10$ and $n = 15$.

The relative L_2-errors on the unit disk are $\|f - P_{10}\|_2/\|f\|_2 = 0.00217349$ and $\|f - P_{15}\|_2/\|f\|_2 = 1.08932 \times 10^{-6}$.



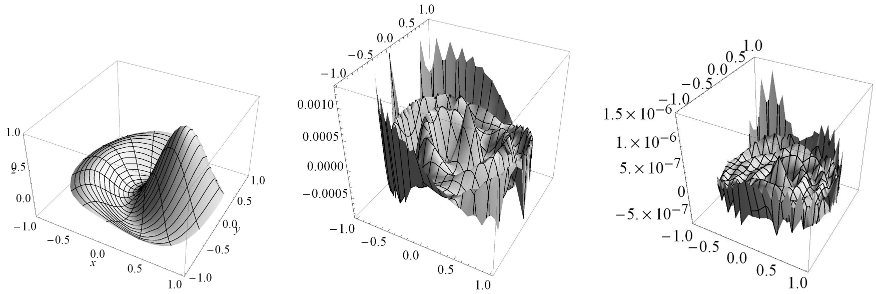
$f(x,y)$ $\qquad$ $f(x,y) - P_{10}(x,y)$ $\qquad$ $f(x,y) - P_{15}(x,y)$

Figure 1. The mexican hat function and errors resulting from the mixed scheme with $n = 10$ and $n = 15$.

For comparison, we perform interpolation using the scheme (Θ, T) from Theorem 2 using only Radon projections. The angles Θ are as above. Figure 2 displays the function and the errors for $n = 10$ and $n = 15$ using this scheme. The relative L_2-errors in this case are $\|f - P_{10}\|_2/\|f\|_2 = 0.000980462$ and $\|f - P_{15}\|_2/\|f\|_2 = 5.14322 \times 10^{-7}$.

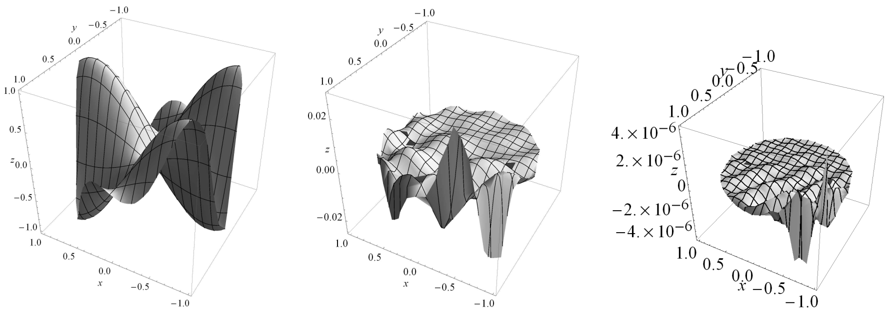
Example 2. We interpolate the function $f(x,y) = \sin(2x)\cos(5y)$ using the mixed type scheme as in Example 1. The graph of the function $f(x,y)$ and the error functions for $n = 10$ and $n = 15$ are presented in Figure 3. Here, the relative L_2-errors on the unit disk are $\|f - P_{10}\|_2/\|f\|_2 = 0.00482072$ and $\|f - P_{15}\|_2/\|f\|_2 = 5.85709 \times 10^{-7}$.

In Section 4, we have presented a regular interpolation scheme based on mixed input data, namely, Radon projections and point-wise function values



$f(x,y)$ $f(x,y)-P_{10}(x,y)$ $f(x,y)-P_{15}(x,y)$

Figure 2. The mexican hat function and errors resulting from the scheme using only Radon projections with $n = 10$ and $n = 15$.



$f(x,y)$ $f(x,y)-P_{10}(x,y)$ $f(x,y)-P_{15}(x,y)$

Figure 3. The function $f(x,y) = \sin(2x)\cos(5y)$ and errors resulting from the mixed scheme with $n = 10$ and $n = 15$.

on the boundary of the unit disk. The scheme's property of reproducing certain function values on the boundary of the computational domain exactly may be advantageous in applications. In Figure 4, we show the condition numbers of the matrices obtained from the mixed scheme and the scheme using only Radon projections. The x-axis corresponds to the degree n of the interpolation polynomial. Our numerical experiments indicate that, for a given polynomial degree, the new scheme from Theorem 4 results in roughly twice the interpolation error of the scheme from Theorem 2, while the condition number is slightly lower.

6. A Scheme for Bivariate Quadratic Spline Interpolation

Suppose that $X = \{\mathbf{x}_k = (-\cos\theta_k, -\sin\theta_k)\}_{k=0}^{n}$ are points on the unit circle, $\theta_0 < \theta_1 < \cdots < \theta_n$. Denote by I_{ij} the chords $\overline{\mathbf{x}_i\mathbf{x}_j}$ and by G_i the sectors

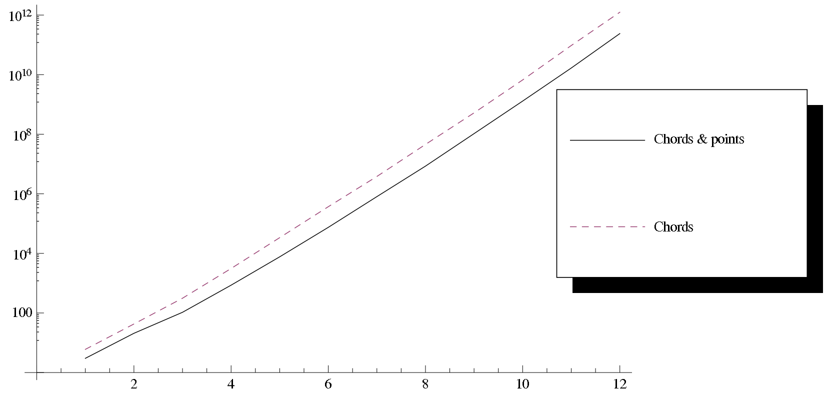


Figure 4. Comparison of condition numbers. x: polynomial degree n, y: condition number

of the unit disk bounded by the chords I_{0i}, $I_{0,i+1}$ and the arc with end points $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$.

Theorem 5. *The interpolation problem*

$$\begin{aligned} &\int_{I_{0i}} P_i(\mathbf{x})\,d\mathbf{x} = \gamma_{0i}, \qquad \int_{I_{0,i+1}} P_i(\mathbf{x})\,d\mathbf{x} = \gamma_{0,i+1}, \qquad \int_{I_{i,i+1}} P_i(\mathbf{x})\,d\mathbf{x} = \gamma_{i,i+1}, \\ &P_i(\mathbf{x}_0) = f_0, \qquad P_i(\mathbf{x}_i) = f_i, \qquad P_i(\mathbf{x}_{i+1}) = f_{i+1} \end{aligned} \tag{9}$$

has a unique solution $P_i(\mathbf{x}) \in \Pi_2^2$. *Moreover, the function* $S(\mathbf{x})$, *such that*

$$S(\mathbf{x})\Big|_{G_i} = P_i(\mathbf{x}), \qquad i = 1, \ldots, n-1,$$

is a continuous quadratic bivariate spline interpolant of the data X, $\{f_i\}_{i=0}^n$, $\{\gamma_{0i}\}_{i=0}^n$, *and* $\{\gamma_{i,i+1}\}_{i=1}^{n-1}$.

Proof. Let us fix i, $i \in \{1, \ldots, n-1\}$. We try to find a polynomial

$$P_i(\mathbf{x}) = P_i(x, y) = ax^2 + bxy + cx^2 + px + qy + r$$

satisfying conditions (9). The problem (9) is a linear system with respect to the coefficients a, b, c, p, q, r of the polynomial $P(x, y)$ and (9) has a unique solution if and only if the corresponding homogeneous linear system has the trivial zero solution only. It is sufficient to prove that there is no other quadratic bivariate polynomial rather than zero polynomial satisfying (9) with zero data in the right hand sides therein. Suppose there exists a polynomial $Q(\mathbf{x}) = Q(x, y) \neq 0$, satisfying the homogeneous linear system (9). It is well-known (see [5]) that since $Q(\mathbf{x})$ vanishes at the points $\mathbf{x}_0$ and $\mathbf{x}_i$, $Q(\mathbf{x})$ must vanish along the line passing through this two point. For example, if $A_jx + B_jy + C_j = 0$ is an equation of the line through $\mathbf{x}_0$ and $\mathbf{x}_j$, for $j = i, i+1$ both, it follows that $Q(x, y)$ has two linear multipliers $A_j x + B_j y + C_j$, $j = i, i+1$, and therefore Q has the form

$$Q(x, y) = \mu(A_i x + B_i y + C_i)(A_{i+1} x + B_{i+1} y + C_{i+1}) \tag{10}$$

for some constant μ. Analogously $Q(\mathbf{x})$ vanishes along the line through $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$. E.g., Q vanishes at the midpoint of the segment $\overline{\mathbf{x}_i \mathbf{x}_j}$ and none of the two linear multipliers in (10) vanishes. Therefore $\mu = 0$ must hold in (10) and $Q(x, y) = 0$ identically. So we proved that the interpolation problem (9) is poised.

Patching the polynomials $P_i(\mathbf{x})$ defined on curved triangles G_i we obtain a bivariate piece-wise polynomial quadratic function $S(\mathbf{x})$ with $S(\mathbf{x})\big|_{G_i} = P_i(\mathbf{x})$, $i = 1, \dots, n-1$. It remains to prove that the pieces fit continuously on the borders I_{0i}, $i = 2, \dots, n-1$ of the domains $\{G_i\}$. Let us consider the difference

$$R(x, y) := P_{i+1}(x, y) - P_i(x, y)$$

on the segment $I_{0,i+1}$ for a fixed i. Clearly $R(x, y)$ is a quadratic bivariate polynomial. By a linear parameterization of the segment $I_{0,i+1}$ by a parameter $t \in [0, 1]$ it follows that $R(x, y)$ coincides with a quadratic univariate polynomial $\rho(t)$ along $I_{0,i+1}$. Note that $\rho(0) = \rho(1) = 0$ because of the interpolation conditions (9), hence $\rho(t) = \lambda t(t-1)$ for a constant λ. But the Radon projections on $I_{0,i+1}$ of both $P_i(x, y)$ and $P_{i+1}(x, y)$ are equal by (9). Therefore $\int_0^1 \lambda t(t-1)\, dt = 0$ provided $\lambda = 0$ and hence $\rho(t) = 0$ identically in $[0, 1]$. Therefore $R(x, y) = 0$ identically on the border $I_{0,i+1}$ of the domain G_i and G_{i+1}. This completes the proof. □

It is clear that applying bivariate quadratic spline interpolation of this type will not give high accuracy approximation in general. However in some cases, e.g. for convex functions, we can expect better results.

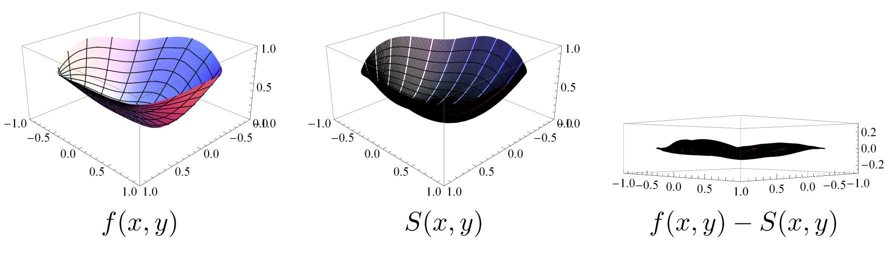


Figure 5. $f(x, y) = \sin\sqrt{x^2 + y^4}$

Example 3. We interpolate the function $f(x, y) = \sin\sqrt{x^2 + y^4}$ by bivariate quadratic polynomial spline $S(x, y)$ based on interpolataion conditions in 20 equidistant points $X = \{\mathbf{x}_k\}_{k=0}^{19}$ on the unit circle and the corresponding chords $\{I_{0i}\}_{i=1}^{19}$, $\{I_{i,i+1}\}_{i=1}^{18}$. The graphs of the function $f(x,y)$, of the spline function $S(x,y)$ and of the error function $f(x,y)-S(x,y)$ are shown in Figure 5. Here, the relative L_2-error on the unit disk is $\|f-S\|_2/\|f\|_2 = 0.126859$.

References

[1] B. Bojanov and I. Georgieva, Interpolation by bivariate polynomials based on Radon projections, *Studia Math.* **162** (2004), 141–160.

[2] B. Bojanov and G. Petrova, Numerical integration over a disc. A new Gaussian cubature formula, *Numer. Math.* **80** (1998), 39–59.

[3] B. Bojanov and G. Petrova, Uniqueness of the Gaussian cubature for a ball. *J. Approx. Theory* **104** (2000), 21–44

[4] B. Bojanov and Y. Xu, Reconstruction of a bivariate polynomials from its Radon projections, *SIAM J. Math. Anal.* **37** (2005), 238–250.

[5] K. C. Chung and T. H. Yao, On the lattices admitting unique Lagrange interpolations, *SIAM J. Numer. Anal.* **14** (1977), 735–743.

[6] M. E. Davison and F. A. Grunbaum, Tomographic reconstruction with arbitrary directions, *Comm. Pure Appl. Math.* **34** (1981), 77–120.

[7] I. Georgieva and S. Ismail, On recovering of a bivariate polynomial from its Radon projections, *in* "Constructive Theory of Functions, Varna 2005" (B. Bojanov, Ed.), pp. 127–134. Prof. Marin Drinov Academic Publishing House, Sofia, 2006.

[8] I. Georgieva and R. Uluchev, Smoothing of Radon projections type of data by bivariate polynomials, *J. Comput. Appl. Math.* **215** (2008), no. 1, 167–181.

[9] I. Georgieva and R. Uluchev, Surface reconstruction and Lagrange basis polynomials, *in* "Large-Scale Scientific Computing 2007" (I. Lirkov, S. Margenov, J. Wasniewski, Eds.), LNCS, vol. 4818, pp. 670–678, Springer-Verlag, Berlin Heidelberg, 2008.

[10] I. Georgieva and R. Uluchev, On interpolation in the unit disk based on both Radon projections and function values, *in* "Large-Scale Scientific Computing 2009" (I. Lirkov, S. Margenov, J. Wasniewski, Eds.), LNCS, vol. 5910, pp. 747–755, Springer-Verlag, Berlin Heidelberg, 2010.

[11] H. Hakopian, Multivariate divided differences and multivariate interpolation of Lagrange and Hermite type, *J. Approx. Theory* **34** (1982), 286–305.

[12] F. John, Abhängigkeiten zwischen den Flächenintegralen einer stetigen Funktion, *Math. Anal.* **111** (1935), 541–559.

[13] R. Marr, On the reconstruction of a function on a circular domain from a sampling of its line integrals, *J. Math. Anal. Appl.* **45** (1974), 357–374.

[14] F. Natterer, "The Mathematics of Computerized Tomography", Classics in Applied Mathematics, vol. 32, SIAM, 2001.

[15] V. Pickalov and T. Melnikova, "Plasma Tomography", Nauka, Novosibirsk, 1995 [in Russian].

[16] J. Radon, Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten, *Ber. Verch. Sächs. Akad.* **69** (1917), 262–277.

[17] D. C. Solmon, The X-ray transform, *J. Math. Anal. Appl.* **56** (1976), no. 1, 61–83.

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